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## Characterizations of Weakly Approximately Primary Submodules in Some Types of Modules

Khaled Y. Jhad , Bothaynah N. Shahab

Department of Mathematic Ibn- AL-Haitham College of Education , University of Baghdad, Baghdad , Iraq

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#### Corresponding Author:

Name: Khaled Y. Jhad

E-mail:

[khaledyounis1989@gmail.com](mailto:khaledyounis1989@gmail.com)

[dr.buthyna@yahoo.com](mailto:dr.buthyna@yahoo.com)

Tel:

### ABSTRACT

Our aim in this note is to introduce several characterizations of weakly approximately primary submodules in class of multiplication modules. Furthermore, we characterized weakly approximately primary submodules by theirs residule in the class of multiplication modules with the help of some types of modules as non-singular, projective, regular and faithful modules. Also, we characterized weakly approximately primary ideal with some kind of weakly approximately primary submodules in the same previous classes of modules with help of finitely generated modules.

### 1. Introduction

Let  $T$  be a commutative ring with identity and  $S$  be a unital left  $T$ -module. Weakly approximately primary submodule was introduced recently by [1], “where aproper submodule  $E$  of a  $T$ -module  $S$  is called weakly approximately primary if wherever  $0 \neq ah \in E$ , for  $a \in T$ ,  $h \in S$ , implies that  $h \in E + \text{Soc}(S)$  or  $a \in \sqrt{[E + \text{Soc}(S)] :_T S}$ , and an ideal  $A$  of a ring  $T$  is a weakly approximately primary if  $A$  is a weakly approximately primary  $T$ -submodule of a  $T$ -module  $T$ ” [1], “where  $\text{Soc}(S)$  is the socal of modules, defined to be the intersction of all essentail submodule of  $S$ ” [2],

“where a non-zero submodule  $K$  of an  $T$ -module  $S$  is an essentail if  $K \cap L \neq (0)$  for all non-zero submodule  $L$  of  $S$ ” [5]. “An  $T$ -module  $S$  is a multiplication if every submodule  $K$  of  $S$  is of the form  $K = IS$  for some ideal  $I$  of  $T$ , equivalent to  $S$  is multiplication if  $K = [K :_T S]S$  where  $[K :_T S] = \{r \in T : rS \subseteq K\}$ ” [3]. “If  $K, L$  are any submodules of multiplication module  $S$ , then  $K = IS$ ,  $L = JS$  for some ideals  $I, J$  of  $T$  such that  $KL = IJS$  and  $KL = IL$ . Inparticular  $KS = Iw = K$ , also for any  $x \in S$ ,  $K = Ix$ ” [4]. “If  $S$  is a multiplication  $T$ -module, then for every elements  $m, m' \in S$ , by  $m m'$  means the product of two submodules  $T_m, T_{m'}$ , that is  $m m' =$

$T_m \cdot T_{m'}$  is a submodule of  $S$ ” [5]. “In multiplication module  $S$ , the  $S$ -rad( $E$ ) of submodule  $E$  denoted by  $S\text{-rad}(E) = \sqrt{E} = \{m \in S : m^n \in E \text{ for some positive integer } n\}$ ” [6]. “A  $T$ -module  $S$  is regular if for each  $x \in S$  there exists  $f \in \text{Hom}_T(S, T)$  such that  $x = f(x)x$ ” [6]. “Recall that an  $T$ -module  $S$  is faithful if  $\text{ann}_T(S) = (0)$ ” [7], “and a  $T$ -module  $S$  is non-singular if  $Z(S) = \{x \in S : xI = (0) \text{ for some essential ideal } I \text{ of } T\} = (0)$ ” [2]. “If  $E$  is a submodule of a  $T$ -module, and  $I$  is an ideal of  $T$ , then  $[E :_S T] = E$ ,  $[I :_T T] = I$ ” [9].

### 2. Characterizations of weakly approximately primary submodules in class of multiplication modules.

In this section we introduce many characterization of weakly approximately primary submodule in class of multiplication modules.

First we need to recall the following lemma which appear in [1, prop.2.6]

#### Lemma (2.1)

“A proper submodule  $E$  of a  $T$ -module  $S$  is a weakly approximately primary if and only if wherever  $(0) \neq IF \subseteq E$ , where  $I$  is an ideal of  $T$  and  $F$  is a submodule

of  $S$ , implies that  $F \subseteq E + \text{Soc}(S)$  or  $I \subseteq \sqrt{[E + \text{Soc}(S) :_T S]}$ .

**Proposition (2.2)**

Let  $S$  be a multiplication  $T$ -module, and  $E$  is a proper submodule of  $S$ . Then  $E$  is a weakly approximately primary if and only if wherever  $(0) \neq HD \subseteq E$ , where  $H, D$  are submodules of  $S$ , implies that  $D \subseteq E + \text{Soc}(S)$  or  $H^n \subseteq E + \text{Soc}(S)$  for some positive integer  $n$ .

**Proof**

( $\Rightarrow$ ) Let  $HD \subseteq E$ , where  $H, D$  are submodules of  $S$ . Since  $S$  is multiplication, then  $H = IS, D = JS$  for some ideals  $I, J$  of  $T$ , it follows that  $(0) \neq I(JS) \subseteq E$ . But  $E$  is a weakly approximately primary, then by lemma (2.1) we have  $JS \subseteq E + \text{Soc}(S)$  or  $I^n S \subseteq E + \text{Soc}(S)$  for some positive integer  $n$ , that is  $D \subseteq E + \text{Soc}(S)$  or  $H^n \subseteq E + \text{Soc}(S)$ .

( $\Leftarrow$ ) Let  $(0) \neq IK \subseteq E$  for  $I$  is an ideal of  $T$  and  $K$  is a submodule of  $S$ . But  $S$  is a multiplication, then  $K = JS$  for some ideal  $J$  of  $T$ . Thus  $(0) \neq I(JS) \subseteq E$ , it follows that  $(0) \neq LK \subseteq E$  where  $L = IS$ , so by hypothesis we have  $K \subseteq E + \text{Soc}(S)$  or  $L^n \subseteq E + \text{Soc}(S)$ , that is  $I^n S \subseteq E + \text{Soc}(S)$ . Thus  $K \subseteq E + \text{Soc}(S)$  or  $I^n \subseteq \sqrt{[E + \text{Soc}(S) :_T S]}$ . Hence by lemma (2.1)  $E$  is a weakly approximately primary submodule of  $S$ . ■

As direct application of proposition (2.2) we get the following corollaries.

**Corollary (2.3)**

Let  $S$  be a multiplication  $T$ -module, and  $E$  be a proper submodule of  $S$ . Then  $E$  is a weakly approximately primary if and only if wherever  $(0) \neq mm' \subseteq E$ , for  $m, m' \in S$ , implies that  $m' \subseteq E + \text{Soc}(S)$  or  $m^n \subseteq E + \text{Soc}(S)$  for some positive integer  $n$ .

**Corollary (2.4)**

Let  $S$  be a multiplication  $T$ -module, and  $E$  be a proper submodule of  $S$ . Then  $E$  is a weakly approximately primary if and only if wherever  $(0) \neq mH \subseteq E$ , for  $m \in S, H$  is a submodule of  $S$ , implies that  $H \subseteq E + \text{Soc}(S)$  or  $m^n \subseteq E + \text{Soc}(S)$  for some positive integer  $n$ .

**Corollary (2.5)**

Let  $S$  be a multiplication  $T$ -module, and  $E$  be a proper submodule of  $S$ . Then  $E$  is a weakly approximately primary if and only if wherever  $(0) \neq Km \subseteq E$ , for  $K$  is a submodule of  $S, m \in S$ , implies that  $m \subseteq E + \text{Soc}(S)$  or  $K^n \subseteq E + \text{Soc}(S)$  for some positive integer  $n$ .

It is well known that cyclic  $T$ -module is multiplication [3], we get the following corollaries.

**Corollary (2.6)**

Let  $S$  be a cyclic  $T$ -module, and  $E$  is a proper submodule of  $S$ . Then  $E$  is a weakly approximately primary if and only if wherever  $(0) \neq HD \subseteq E$ , where  $H, D$  are submodules of  $S$ , implies that  $D \subseteq E + \text{Soc}(S)$  or  $H^n \subseteq E + \text{Soc}(S)$  for some positive integer  $n$ .

**Corollary (2.7)**

Let  $S$  be a cyclic  $T$ -module, and  $E$  be a proper submodule of  $S$ . Then  $E$  is a weakly approximately primary if and only if wherever  $(0) \neq mm' \subseteq E$ , for  $m,$

$m' \in S$ , implies that  $m' \subseteq E + \text{Soc}(S)$  or  $m^n \subseteq E + \text{Soc}(S)$  for some positive integer  $n$ .

**Corollary (2.8)**

Let  $S$  be a cyclic  $T$ -module, and  $E$  be a proper submodule of  $S$ . Then  $E$  is a weakly approximately primary if and only if wherever  $(0) \neq mH \subseteq E$ , for  $m \in S, H$  is a submodule of  $S$ , implies that  $H \subseteq E + \text{Soc}(S)$  or  $m^n \subseteq E + \text{Soc}(S)$  for some positive integer  $n$ .

**Corollary (2.9)**

Let  $S$  be a cyclic  $T$ -module, and  $E$  be a proper submodule of  $S$ . Then  $E$  is a weakly approximately primary if and only if wherever  $(0) \neq Km \subseteq E$ , for  $K$  is a submodule of  $S, m \in S$ , implies that  $m \subseteq E + \text{Soc}(S)$  or  $K^n \subseteq E + \text{Soc}(S)$  for some positive integer  $n$ .

The following are another characterizations of a weakly approximately primary submodules in class of multiplication modules.

**Corollary (2.10)**

Let  $S$  be a multiplication  $T$ -module, and  $E$  be a proper submodule of  $S$ . Then  $E$  is a weakly approximately primary if and only if  $m' \subseteq E + \text{Soc}(S)$  or  $m \subseteq S\text{-rad}(E + \text{Soc}(S))$ .

**Corollary (2.11)**

Let  $S$  be a multiplication  $T$ -module, and  $E$  be a proper submodule of  $S$ . Then  $E$  is a weakly approximately primary if and only if wherever  $(0) \neq mH \subseteq E$ , for  $m \in S, H$  is a submodule of  $S$ , implies that  $H \subseteq E + \text{Soc}(S)$  or  $m \subseteq S\text{-rad}(E + \text{Soc}(S))$ .

**3. Characterized weakly approximately primary submodules by their residues.**

In this part of this paper we introduced many characterizations of weakly approximately primary submodules by their residues.

First we need to discuss the following fact in this remark.

**Remark (3.1)**

The residues of weakly approximately primary submodule of  $T$ -module  $S$  need not to be a weakly approximately primary ideals of  $T$ , so, the following example explained that.

**Example (3.2)**

The submodule  $\langle \overline{10} \rangle$  of the  $Z$ -module  $Z_{60}$  is weakly approximately primary, since  $\text{Soc}(Z_{60}) = \langle \overline{2} \rangle$  [1, Example (2.3)]. That is wherever  $(0) \neq r\bar{h} \in \langle \overline{10} \rangle$  for  $r \in Z, \bar{h} \in Z_{60}$  implies that  $\bar{h} \in \langle \overline{10} \rangle + \text{Soc}(Z_{60}) = \langle \overline{10} \rangle + \langle \overline{2} \rangle = \langle \overline{2} \rangle$  or  $r \in \sqrt{[\langle \overline{10} \rangle + \text{Soc}(Z_{60}) :_Z Z_{60}]} = \sqrt{[\langle \overline{2} \rangle :_Z Z_{60}]} = 2Z$ . Thus, if  $0 \neq 2.5 \in \langle \overline{10} \rangle$  for  $2 \in Z, 5 \in Z_{60}$  implies that  $2 \in \sqrt{[\langle \overline{10} \rangle + \text{Soc}(Z_{60}) :_Z Z_{60}]} = Z$ . But the residue  $[\langle \overline{10} \rangle :_Z Z_{60}] = 10Z$  is not weakly approximately primary ideal of  $Z$  since  $0 \neq 2.5 \in 10Z$  for  $2, 5 \in Z$  but  $5 \notin 10Z + \text{Soc}(Z) = 10Z + (0) = 10Z$  and  $2 \notin \sqrt{[\langle \overline{10} \rangle + \text{Soc}(Z) :_Z Z]} = 10Z$ . ■

Before we introduced the first characterization we need to recall the following lemma

**Lemm(3.3) [6, prop.3.25]**

"If  $S$  is a regular  $T$ -module, then  $\text{Soc}(S) = \text{Soc}(T)S$ ".

**Proposition (3.4)**

Let  $S$  be a multiplication regular  $T$ -module, and  $E$  is a proper submodule of  $S$ . Then  $E$  is a weakly approximately primary if and only if  $[E:{}_T S]$  is a weakly approximately primary ideal of  $T$ .

**Proof**

( $\Rightarrow$ ) Let  $(0) \neq IJ \subseteq [E:{}_T S]$  where  $I, J$  are ideals of  $T$ , implies that  $(0) \neq IJS \subseteq E$ , it follows that  $(0) \neq KL \subseteq E$ , where  $K = IS$ ,  $L = JS$  (since  $S$  is a multiplication). Hence by proposition (2.2)  $L \subseteq E + \text{Soc}(S)$  or  $K^n \subseteq E + \text{Soc}(S)$  for some positive integer  $n$ , that is  $JS \subseteq E + \text{Soc}(S)$  or  $I^n S \subseteq E + \text{Soc}(S)$ . But  $S$  is a regular, then by lemma (3.3)  $\text{Soc}(S) = \text{Soc}(T)S$ . Thus we have  $JS \subseteq [E:{}_T S]S + \text{Soc}(T)S$  or  $I^n S \subseteq [E:{}_T S]S + \text{Soc}(T)S$ , it follows that  $J \subseteq [E:{}_T S] + \text{Soc}(T)$  or  $I^n \subseteq [[E:{}_T S] + \text{Soc}(T):_T T]$ , implies that  $J \subseteq [E:{}_T S] + \text{Soc}(T)$  or  $I \subseteq \sqrt{[[E:{}_T S] + \text{Soc}(T):_T T]}$ . Hence by lemma(2.1)

$[E:{}_T S]$  is a weakly approximately primary ideal of  $T$ .

( $\Leftarrow$ ) Let  $(0) \neq IL \subseteq E$ , for  $I$  is an ideal of  $T$  and  $L$  is a submodule of  $S$ , since  $S$  is a multiplication then  $L = JS$  for some ideal  $J$  of  $T$ . Thus,  $(0) \neq IJS \subseteq E$ , that is  $(0) \neq IJ \subseteq [E:{}_T S]$ . But  $[E:{}_T S]$  is a weakly approximately primary ideal of  $T$ , then by lemma (2.1) we have  $J \subseteq [E:{}_T S] + \text{Soc}(T)$  or  $I^n \subseteq [E:{}_T S] + \text{Soc}(T)$  for some positive integer  $n$ , it follows that  $JS \subseteq [E:{}_T S]S + \text{Soc}(T)S$  or  $I^n S \subseteq [E:{}_T S]S + \text{Soc}(T)S$ . But  $S$  is a regular, then by lemma (3.3)  $\text{Soc}(T)S = \text{Soc}(S)$ , so,  $L \subseteq E + \text{Soc}(S)$  or  $I^n \subseteq [E + \text{Soc}(S):_T S]$ . Hence by lemma(2.1)  $E$  is a weakly approximately primary submodule of  $S$ . ■

The following corollary is a direct consequence of proposition (3.4)

**Corollary (3.5)**

Let  $S$  be a cyclic regular  $T$ -module, and  $E$  is a proper submodule of  $S$ . Then  $E$  is a weakly approximately primary if and only if  $[E:{}_T S]$  is a weakly approximately primary ideal of  $T$ .

Before we introduced the next characterization we need to recall the following lemmas

**Lemma (3.6) [1, coro.(2.7)]**

"A proper submodule  $E$  of an  $T$ -module  $S$  is a weakly approximately primary if and only if wherever  $(0) \neq aF \subseteq E$  for  $a \in S$ ,  $F$  is a submodule of  $S$ , implies that  $F \subseteq E + \text{Soc}(S)$  or  $a \in \sqrt{[E + \text{Soc}(S):_T S]}$ ".

**Lemma (3.7) [6, coro.(2.1.4)(1)]**

"If  $S$  is a faithful multiplication  $T$ -module then  $\text{Soc}(S) = \text{Soc}(T)S$ ".

**Proposition (3.8)**

A proper submodule  $E$  of a faithful multiplication  $T$ -module  $S$  is a weakly approximately primary if and only if  $[E:{}_T S]$  is a weakly approximately primary ideal of  $T$ .

**Proof**

( $\Rightarrow$ ) Let  $(0) \neq aI \subseteq [E:{}_T S]$ , for  $a \in T$ ,  $I$  is an ideal of  $T$ , implies that  $(0) \neq aIS \subseteq E$ . Since  $E$  is a weakly approximately primary, then by lemma (3.6) we have  $IS \subseteq E + \text{Soc}(S)$  or  $a \in \sqrt{[E + \text{Soc}(S):_T S]}$ , it follows

that  $IS \subseteq E + \text{Soc}(S)$  or  $a^n S \subseteq E + \text{Soc}(S)$  for some positive integer  $n$ . Since  $S$  is faithful multiplication, then by lemma (3.7)  $\text{Soc}(S) = \text{Soc}(T)S$ . Thus, we have  $IS \subseteq [E:{}_T S]S + \text{Soc}(T)S$  or  $a^n S \subseteq [E:{}_T S]S + \text{Soc}(T)S$ , it follows that  $I \subseteq [E:{}_T S] + \text{Soc}(T)$  or  $a^n \subseteq [E:{}_T S] + \text{Soc}(T) = [[E:{}_T S] + \text{Soc}(T):_T T]$ , that is  $I \subseteq [E:{}_T S] + \text{Soc}(T)$  or  $a \in \sqrt{[[E:{}_T S] + \text{Soc}(T):_T T]}$ . Hence by lemma (3.6)

$[E:{}_T S]$  is a weakly approximately primary ideal of  $T$ .

( $\Leftarrow$ ) Let  $(0) \neq mH \subseteq E$ , for  $m \in S$ ,  $H$  is a submodule of  $S$ . Since  $S$  is a multiplication then  $m = Tm = JS$ ,  $H = IS$  for some ideals  $I, J$  of  $T$ , that is  $(0) \neq JIS \subseteq E$ , so  $(0) \neq JI \subseteq [E:{}_T S]$ . Since  $[E:{}_T S]$  is a weakly approximately primary ideal of  $T$ , then by lemma(2.1) we have  $I \subseteq [E:{}_T S] + \text{Soc}(T)$  or  $J^n \subseteq [[E:{}_T S] + \text{Soc}(T):_T T] = [E:{}_T S] + \text{Soc}(T)$ , it follows that  $IS \subseteq [E:{}_T S]S + \text{Soc}(T)S$  or  $J^n S \subseteq [E:{}_T S]S + \text{Soc}(T)S$ . But  $S$  is faithful multiplication,  $\text{Soc}(T)S = \text{Soc}(S)$ , that is  $H \subseteq E + \text{Soc}(S)$  or  $I^n S \subseteq E + \text{Soc}(S)$ , it follows that  $H \subseteq E + \text{Soc}(S)$  or  $m^n \subseteq E + \text{Soc}(S)$ . Hence by corollary (2.4)  $E$  is a weakly approximately primary submodule of  $S$ . ■

The following corollary is direct application of proposition (3.8)

**Corollary (3.9)**

A proper submodule  $E$  of cyclic faithful  $T$ -module  $S$  is a weakly approximately primary if and only if  $[E:{}_T S]$  is a weakly approximately primary ideal of  $T$ . To introduced next characterization we need to recall the following lemma

**Lemma (3.10) [2, coro.1.26]**

"If  $S$  is a non-singular  $T$ -module, then  $\text{Soc}(T)S = \text{Soc}(S)$ ".

**Proposition (3.11)**

A proper submodule  $E$  of a non-singular multiplication  $T$ -module  $S$  is a weakly approximately primary if and only if  $[E:{}_T S]$  is a weakly approximately primary ideal of  $T$ .

**Proof**

( $\Rightarrow$ ) Let  $(0) \neq rt \in [E:{}_T S]$ , for  $r, t \in T$ , implies that  $(0) \neq r(tS) \subseteq E$ . Since  $E$  is a weakly approximately primary submodule, then by lemma (3.6) we have  $tS \subseteq E + \text{Soc}(S)$  or  $r^n S \subseteq E + \text{Soc}(S)$ . But  $S$  is a non-singular, then by lemma (3.10)  $\text{Soc}(S) = \text{Soc}(T)S$ . Thus  $tS \subseteq [E:{}_T S]S + \text{Soc}(T)S$  or  $r^n S \subseteq [E:{}_T S]S + \text{Soc}(T)S$ , it follows that  $t \in [E:{}_T S] + \text{Soc}(T)$  or  $r^n \in [E:{}_T S] + \text{Soc}(T) = [[E:{}_T S] + \text{Soc}(T):_T T]$ . Hence  $[E:{}_T S]$  is a weakly approximately primary ideal of  $T$ .

( $\Leftarrow$ ) Let  $(0) \neq Km \subseteq E$ , for  $K$  is a submodule of  $S$ ,  $m \in S$ . Since  $S$  is a multiplication then  $K = JS$ ,  $m = Tm = IS$ , for some ideals  $I, J$  of  $T$ , it follows that  $(0) \neq JIS \subseteq E$ , implies that  $(0) \neq JI \subseteq [E:{}_T S]$ . But  $[E:{}_T S]$  is a weakly approximately primary ideal of  $T$ , then by lemma (2.1) we have  $I \subseteq [E:{}_T S] + \text{Soc}(T)$  or  $J^n \subseteq [[E:{}_T S] + \text{Soc}(T):_T T] = [E:{}_T S] + \text{Soc}(T)$ , for some positive integer  $n$ . That is  $IS \subseteq [E:{}_T S]S + \text{Soc}(T)S$  or  $J^n S \subseteq [E:{}_T S]S + \text{Soc}(T)S$ . But  $S$  is a non-

singular, then  $\text{Soc}(T)S = \text{Soc}(S)$ . Thus, we have  $m \subseteq E + \text{Soc}(S)$  or  $K^n \subseteq E + \text{Soc}(S)$ . Hence by corollary (2.5)  $E$  is a weakly approximately primary submodule of  $S$ . ■

The following corollary is a direct consequence of proposition (3.11).

**Corollary (3.12)**

A proper submodule  $E$  of a cyclic non-singular  $T$ -module  $S$  is a weakly approximately primary if and only if  $[E:{}_T S]$  is a weakly approximately primary ideal of  $T$ .

We recall the following lemmas before we introduced the next characterization

**Lemma (3.13) [6, prop (3.24)]**

"If  $S$  is a projective  $T$ -module, then  $\text{Soc}(S) = \text{Soc}(T)S$ ".

**Lemma (3.14) [1, coro (2.8)]**

"A proper submodule  $E$  of a  $T$ -module  $S$  is a weakly approximately primary if and only if where ever  $(0) \neq Ix \subseteq E$  for  $I$  is an ideal of  $T$ ,  $x \in S$ , implies that  $x \in E + \text{Soc}(S)$  or  $I \subseteq \sqrt{[E + \text{Soc}(S) :_T S]}$ "

**Proposition (3.15)**

A proper submodule  $E$  of a projective multiplication  $T$ -module  $S$  is a weakly approximately primary if and only if  $[E:{}_T S]$  is a weakly approximately primary ideal of  $T$ .

**Proof**

( $\Rightarrow$ ) Let  $(0) \neq Ir \subseteq [E:{}_T S]$  for  $I$  is an ideal of  $T$  and  $r \in T$ , so  $(0) \neq I(rS) \subseteq E$ . But  $E$  is a weakly approximately primary, then by lemma (2.1) we have  $rS \subseteq E + \text{Soc}(S)$  or  $I^n \subseteq [E + \text{Soc}(S) :_T S]$  for some positive integer  $n$ , that is  $rS \subseteq E + \text{Soc}(S)$  or  $I^n S \subseteq E + \text{Soc}(S)$ . But  $S$  is a projective then by lemma (3.13)  $\text{Soc}(S) = \text{Soc}(T)S$ . Thus  $rS \subseteq [E:{}_T S]S + \text{Soc}(T)S$  or  $I^n S \subseteq [E:{}_T S]S + \text{Soc}(T)S$ , implies that  $r \in [E:{}_T S] + \text{Soc}(T)$  or  $I^n \subseteq [E:{}_T S] + \text{Soc}(T) = [[E:{}_T S] + \text{Soc}(T) :_T T]$

Hence by lemma (3.14)  $[E:{}_T S]$  is a weakly approximately primary ideal of  $T$ .

( $\Leftarrow$ ) Let  $(0) \neq mm' \subseteq E$  for  $m, m' \in S$ . Since  $S$  is a multiplication, then  $m = Tm = IS$ ,  $m' = Tm' = JS$  for some ideals  $I, J$  of  $T$ . That is  $(0) \neq IJS \subseteq E$ , so  $(0) \neq IJ \subseteq [E:{}_T S]$ . But  $[E:{}_T S]$  is a weakly approximately primary ideal of  $T$ , then by lemma (2.1) we have  $J \subseteq [E:{}_T S] + \text{Soc}(T)$  or  $I^n \subseteq [E:{}_T S] + \text{Soc}(T)$ , implies that  $JS \subseteq [E:{}_T S]S + \text{Soc}(T)S$  or  $I^n S \subseteq [E:{}_T S]S + \text{Soc}(T)S$  for some positive integer  $n$ . But  $S$  is a projective, then by lemma (3.13)  $\text{Soc}(T)S = \text{Soc}(S)$ , that is  $m' \subseteq E + \text{Soc}(S)$  or  $m^n \subseteq E + \text{Soc}(S)$ . So by corollary (2.3)  $E$  is a weakly approximately primary submodule of  $S$ . ■

The proof of the following corollary is direct from proposition (3.15).

**Corollary (3.16)**

A proper submodule  $E$  of a cyclic projective  $T$ -module  $S$  is a weakly approximately primary if and only if  $[E:{}_T S]$  is a weakly approximately primary ideal of  $T$ .

**Characterized weakly approximately ideals with some kind of weakly approximately submodules.**

In this section we characterized a weakly approximately ideal  $A$  with submodule  $AS$ . Before we introduced the first characterization we recall the following lemma.

**Lemma (4.1) [8, coro. of Theo (9)]**

"Let  $S$  be a finitely generated multiplication  $T$ -module, and  $I, J$  are ideals of  $T$ . Then  $IS \subseteq JS$  if and only if  $I \subseteq J + \text{ann}_T(S)$ ".

**Proposition (4.2)**

Let  $S$  be a finitely generated multiplication regular  $T$ -module and  $A$  be an ideal of  $T$  with  $\text{ann}_T(S) \subseteq A$ . Then  $A$  is a weakly approximately primary if and only if  $AS$  is a weakly approximately primary submodule of  $S$ .

**Proof**

( $\Rightarrow$ ) Let  $(0) \neq IK \subseteq AS$ , for  $I$  is an ideal of  $T$ ,  $K$  is a submodule of  $S$ . Since  $S$  is a multiplication, then  $K = JS$  for some ideal  $J$  of  $T$ , so  $(0) \neq IJS \subseteq AS$ . But  $S$  is a finitely generated multiplication then by lemma (4.1) we have  $(0) \neq IJ \subseteq A + \text{ann}_T(S)$ . Since  $\text{ann}_T(S) \subseteq A$ , then  $A + \text{ann}_T(S) = A$ , so  $(0) \neq IJ \subseteq A$ . It is given that  $A$  is a weakly approximately primary ideal, implies that by lemma (2.1)  $J \subseteq A + \text{Soc}(T)$  or  $I^n \subseteq [A + \text{Soc}(T) :_T T] = A + \text{Soc}(T)$ , for some positive integer  $n$ . Hence  $JS \subseteq AS + \text{Soc}(T)S$  or  $I^n S \subseteq AS + \text{Soc}(T)S$ . But  $S$  is a regular, then by lemma (3.3) we have  $\text{Soc}(T)S = \text{Soc}(S)$ . Thus  $K \subseteq AS + \text{Soc}(S)$  or  $I^n S \subseteq AS + \text{Soc}(S)$ , it follows that  $K \subseteq AS + \text{Soc}(S)$  or  $I^n \subseteq [AS + \text{Soc}(S) :_T S]$ . Hence by lemma (2.1)  $AS$  is a weakly approximately primary submodule of  $S$ .

( $\Leftarrow$ ) Let  $(0) \neq rt \subseteq A$ , for  $r, t \in T$ , implies that  $(0) \neq rtS \subseteq AS$ . But  $AS$  is a weakly approximately primary submodule of  $S$ , then by lemma (3.6) we have  $tS \subseteq AS + \text{Soc}(S)$  or  $r^n S \subseteq AS + \text{Soc}(S)$  for some positive integer  $n$ . Since  $S$  is a regular then by lemma (3.3)  $\text{Soc}(S) = \text{Soc}(T)S$ , so  $t \in A + \text{Soc}(T)$  or  $r^n \in A + \text{Soc}(T) = [A + \text{Soc}(T) :_T T]$ . Hence  $A$  is a weakly approximately primary ideal of  $T$ . ■

It is well-known that cyclic  $T$ -module is a finitely generated we get the following corollary

**Corollary (4.3)**

Let  $S$  be a cyclic regular  $T$ -module and  $A$  be an ideal of  $T$  with  $\text{ann}_T(S) \subseteq A$ . Then  $A$  is a weakly approximately primary ideal if and only if  $AS$  is a weakly approximately primary submodule of  $S$ .

**Proposition (4.4)**

Let  $S$  be a finitely generated multiplication projective  $T$ -module and  $A$  be an ideal of  $T$  with  $\text{ann}_T(S) \subseteq A$ . Then  $A$  is a weakly approximately primary ideal if and only if  $AS$  is a weakly approximately primary submodule of  $S$ .

**Proof**

( $\Rightarrow$ ) Let  $(0) \neq KL \subseteq AS$ , where  $K, L$  are submodules of  $S$ . Since  $S$  is a multiplication, then  $L = IS$ ,  $K = JS$  for some ideals  $I, J$  of  $T$ , it follows that  $(0) \neq JIS \subseteq AS$ . But  $S$  is a finitely generated, then by lemma (4.1) we have  $(0) \neq JI \subseteq A + \text{ann}_T(S)$ . Since  $\text{ann}_T(S) \subseteq A$ , implies that  $A + \text{ann}_T(S) = A$ , so  $(0) \neq JI \subseteq A$ . But  $A$  is a weakly approximately primary ideal of  $T$ , then by

lemma (2.1) we have  $I \subseteq A + \text{Soc}(T)$  or  $J^n \subseteq [A + \text{Soc}(T) : T] = A + \text{Soc}(T)$ , for some positive integer  $n$ , it follows that  $IS \subseteq AS + \text{Soc}(T)S$  or  $J^n S \subseteq AS + \text{Soc}(T)S$ . Since  $S$  is a projective, then by lemma (3.13) we have  $\text{Soc}(T)S = \text{Soc}(S)$ . Thus  $L \subseteq AS + \text{Soc}(S)$  or  $K^n \subseteq AS + \text{Soc}(S)$ . Hence by proposition (2.2)  $AS$  is a weakly approximately primary submodule of  $S$ .

( $\Leftarrow$ ) Let  $(0) \neq Ir \subseteq A$ , for  $I$  is an ideal of  $T$ ,  $r \in T$ , implies that  $(0) \neq IrS \subseteq AS$ . Since  $AS$  is a weakly approximately primary submodule of  $S$ , then by lemma (3.14) we have  $rS \subseteq AS + \text{Soc}(S)$  or  $I^n S \subseteq AS + \text{Soc}(S)$ , for some positive integer  $n$ . Since  $S$  is a projective, then by lemma (3.13) we have  $\text{Soc}(T)S = \text{Soc}(S)$ . Hence  $rS \subseteq AS + \text{Soc}(T)S$  or  $I^n S \subseteq AS + \text{Soc}(T)S$ , it follows that  $r \in A + \text{Soc}(T)$  or  $I^n \in A + \text{Soc}(T) = [A + \text{Soc}(T) : T]$ . So by lemma (3.14)  $A$  is a weakly approximately primary ideal of  $T$ . ■

#### **Proposition (4.5)**

Let  $S$  be a finitely generated multiplication non-singular  $T$ -module and  $A$  be an ideal of  $T$  with  $\text{ann}_T(S) \subseteq A$ . Then  $A$  is a weakly approximately primary ideal if and only if  $AS$  is a weakly approximately primary submodule of  $S$ .

#### **Proof**

( $\Rightarrow$ ) Follows by proposition (2.2) lemma (4.1) and lemma (3.10)

( $\Leftarrow$ ) Follows by lemma (3.6), lemma (4.1) and lemma (3.10). ■

#### **Proposition (4.6)**

Let  $S$  be a faithful finitely generated multiplication  $T$ -module and  $A$  be an ideal of  $T$ . Then  $A$  is a weakly approximately primary ideal if and only if  $AS$  is a weakly approximately primary submodule of  $S$ .

#### **Proof**

( $\Rightarrow$ ) Let  $(0) \neq mH \subseteq AS$ , for  $m \in S$ ,  $H$  is a submodules of  $S$ . Since  $S$  is a multiplication, then  $H = JS$ ,  $m = Tm = IS$ , for some ideals  $I, J$  of  $T$ , so  $(0) \neq IJS \subseteq AS$ . It follows by lemma (4.1),  $(0) \neq IJ \subseteq A + \text{ann}_T(S)$ . Since  $S$  is a faithful then  $\text{ann}_T(S) = (0)$ , so,  $(0) \neq IJ \subseteq A$ . But  $A$  is a weakly approximately primary ideal of  $T$ , then by lemma (2.1) we have  $J \subseteq A + \text{Soc}(T)$  or  $I^n \subseteq [A + \text{Soc}(T) : T] = A + \text{Soc}(T)$ , it follows that  $JS \subseteq AS + \text{Soc}(T)S$  or  $I^n S \subseteq AS + \text{Soc}(T)S$ . But  $S$  is a faithful multiplication, then by lemma (3.7) we have  $\text{Soc}(T)S = \text{Soc}(S)$ . Hence  $H \subseteq AS + \text{Soc}(S)$  or  $m^n \subseteq AS + \text{Soc}(S)$ . Thus by corollary (2.4)  $AS$  is a weakly approximately primary submodule of  $S$ .

( $\Leftarrow$ ) Let  $(0) \neq ab \in A$ , for  $a, b \in S$ , it follows that  $(0) \neq abS \subseteq AS$ . But  $AS$  is a weakly approximately primary submodule, then by lemma (3.6) we have  $bS \subseteq AS + \text{Soc}(S)$  or  $a^n S \subseteq AS + \text{Soc}(S)$ . Since  $S$  is faithful multiplication then  $\text{Soc}(S) = \text{Soc}(T)S$ , so we have  $bS \subseteq AS + \text{Soc}(T)S$  or  $a^n S \subseteq AS + \text{Soc}(T)S$ , it follows that  $b \in A + \text{Soc}(T)$  or  $a^n \in A + \text{Soc}(T) = [A + \text{Soc}(T) : T]$ . Hence  $A$  is a weakly approximately primary ideal of  $T$ . ■

The following corollary is direct application of proposition (4.4), proposition (4.5) and proposition (4.6).

#### **Corollary (4.7)**

Let  $S$  be a cyclic (projective, non-singular, faithful)  $T$ -module and  $A$  be an ideal of  $T$  with  $\text{ann}_T(S) \subseteq A$ . Then  $A$  is a weakly approximately primary ideal if and only if  $AS$  is a weakly approximately primary submodule of  $S$ .

“Recall that a  $T$ -module  $S$  is weak cancellation if  $AS = BS$ , implies that  $A + \text{ann}_T(S) = B + \text{ann}_T(S)$  for  $A, B$  are ideals of  $T$ ” [8].

The following fact appear in [8, prop (3.9)] we needed before we introduce the next propositions.

#### **Lemma (4.8)**

“Let  $S$  be a multiplication  $T$ -module. Then  $S$  is a finitely generated if and only if  $S$  is a cancellation”.

#### **Proposition (4.9)**

Let  $S$  be a finitely generated multiplication regular  $T$ -module, and  $E$  be a proper submodule of  $S$ . Then the following statements are equivalents:

1.  $E$  is a weakly approximately primary submodule of  $S$ .
2.  $[E :_T S]$  is a weakly approximately primary ideal of  $T$ .
3.  $E = AS$  for some weakly approximately primary ideal  $A$  of  $T$  with  $\text{ann}_T(S) \subseteq A$ .

#### **Proof**

(1) $\Rightarrow$ (2) Follows from proposition (3.4)

(2) $\Rightarrow$ (3) Since  $[E :_T S]$  is a weakly approximately primary ideal and  $\text{ann}_T(S) = [0 :_T S] \subseteq [E :_T S]$  then by proposition (4.2)  $[E :_T S]S$  is a weakly approximately primary submodule of  $S$ , since  $S$  is a multiplication, then  $E = [E :_T S]S = AS$ , where  $A = [E :_T S]$  is a weakly approximately primary ideal of  $T$ .

(3) $\Rightarrow$ (2) Since  $E = AS$  with  $A$  is a weakly approximately primary ideal of  $T$  such that  $\text{ann}_T(S) \subseteq A$ . From other hand we have  $S$  is a multiplication, then  $E = [E :_T S]S$ . But  $S$  is a finitely generated, it follows by lemma (4.8) that  $S$  is a weak cancellation. That is  $A + \text{ann}_T(S) = [E :_T S] + \text{ann}_T(S)$ . But  $\text{ann}_T(S) \subseteq A$  and  $\text{ann}_T(S) \subseteq [E :_T S]$ , it follows that  $A + \text{ann}_T(S) = A$  and  $[E :_T S] + \text{ann}_T(S) = [E :_T S]$ . But  $A$  is a weakly approximately primary ideal, it follows that  $[E :_T S]$  is a weakly approximately primary ideal of  $T$ . ■

Form proposition (3.11), proposition (4.5) and lemma (4.8) we get the following result.

#### **Proposition (4.10)**

Let  $S$  be a finitely generated multiplication non-singular  $T$ -module, and  $E$  be a proper submodule of  $S$ . Then the following statements are equivalents:

1.  $E$  is a weakly approximately primary submodule of  $S$ .
2.  $[E :_T S]$  is a weakly approximately primary ideal of  $T$ .
3.  $E = AS$  for some weakly approximately primary ideal  $A$  of  $T$  with  $\text{ann}_T(S) \subseteq A$ .

Form proposition (3.15), proposition (4.4) and lemma (4.8) we get the following result.

**Proposition (4.11)**

Let  $S$  be a finitely generated multiplication projective  $T$ -module, and  $E$  be a proper submodule of  $S$ . Then the following statements are equivalents:

1.  $E$  is a weakly approximately primary submodule of  $S$ .
2.  $[E:_T S]$  is a weakly approximately primary ideal of  $T$ .
3.  $E = AS$  for some weakly approximately primary ideal  $A$  of  $T$  with  $\text{ann}_T(S) \subseteq A$ .

Form proposition (3.8), proposition (4.6) and lemma (4.8) we get the following result.

**Proposition (4.12)**

Let  $S$  be a faithful finitely generated multiplication  $T$ -module, and  $E$  be a proper submodule of  $S$ . Then the following statements are equivalents:

1.  $E$  is a weakly approximately primary submodule of  $S$ .
2.  $[E:_T S]$  is a weakly approximately primary ideal of  $T$ .
3.  $E = AS$  for some weakly approximately primary ideal  $A$  of  $T$ .

We end this section by the following corollary.

**Corollary (4.13)**

Let  $S$  be a cyclic (projective, non-singular and faithful)  $T$ -module, and  $E$  be a proper submodule of  $S$ . Then the following statements are equivalents:

1.  $E$  is a weakly approximately primary submodule of  $S$ .
2.  $[E:_T S]$  is a weakly approximately primary ideal of  $T$ .
3.  $E = AS$  for some weakly approximately primary ideal  $A$  of  $T$  with  $\text{ann}_T(S) \subseteq A$ .

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## تشخيصات المقاسات الجزئية المتقاربة الابتدائية الضعيفة في بعض انواع المقاسات

خالد يونس جهاد ، بثينة نجاد شهاب

قسم الرياضيات ، كلية التربية ابن الهيثم ، جامعة بغداد ، بغداد ، العراق

**الملخص**

هدفنا في هذا البحث هو تقديم العديد من تشخيصات المقاسات الجزئية المتقاربة الابتدائية الضعيفة في صنف المقاسات الضربية. بالإضافة الى ذلك تشخص المقاسات الجزئية المتقاربة الابتدائية الضعيفة بواسطة بواقيها في صنف المقاسات الضربية بمساعده بعض انواع المقاسات الاخرى مثل المقاسات الغير احادية والاسقاطية و المنتظمة والمخلصة.

كذلك تشخص المثاليات المتقاربة الابتدائية الضعيفة مع بعض انواع المقاسات الجزئية المتقاربة الابتدائية الضعيفة في نفس صنف المقاسات اعلاه بمساعده المقاسات المنتهيه التولدز