MGP and EGP Rings

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الملخص

في هذا البحث تم دراسة الحلقات التي يكون فيها كل مثالي أيمن أعظم (أساسي) هو من النمط -GP الأيسر. هذه الحلقات تسمى حلقات من النمط –MGP (-EGP) يمنى. أعطينا الخواص الأساسية لهذه الحلقات وعلاقاتها مع حلقات اخرى مثل الحلقات المنتظمة بقوة من النمط - p، والحلقات المتحايدة اليسرى الكاملـــة، وكذلك مع الحلقات المنتظمة الضعيفة من النمط – S.

ABSTRACT

The purpose of this paper is to study the rings in which every maximal (essential) right ideal is a left GP- ideal. Such rings will be called right MGP- rings (EGP- rings). We give the basic properties of such rings and their connection with strongly p - regular rings, fully left idempotent rings, and S - weakly regular rings.

1. Introduction:

In this paper, all rings are assumed to be associative ring with identity. An ideal I of a ring R is said to be right (left) pure ideal if for every $a \in I$ there exists $b \in I$ such that $a = ab$ $(a = ba)$. This concept was introduced by Fieldhouse [3],[4].

Recall that:

1- R is called reduced if R has no non-zero nilpotent element.
2- According to Cohn [2], a ring R is called reversible if $ab = 0$ implies $ba = 0$, for $a, b \in R$. It is easy to see that R is reversible if and only if right (left) annihilator of a in R is two sided ideal [2].
3- A ring R is said to be a right (left) Kasch ring if every maximal right (left) ideal is a right (left) annihilator [8].
4- According to [5], a ring R is called strongly p - regular if for every $x \in R$, there exists a positive integer n such that $x^n R = x^{2n} R$.
5- A ring R is called right (left) quasi-duo if every maximal right (left) ideal of R is two sided ideal [1].
6- A ring R is said to be ERT if every essential right ideal is two-sided [1].

2- Maximal Generalized Pure Rings (MGP- rings)

In this section, some basic properties of MGP- rings are given. Also the relations between such rings and strongly p - regular, weakly p - regular rings are given.

Following [10], an ideal I of a ring R is said to be right (left) GP- ideal if for every $a \in I$, there exists $b \in I$ and a positive integer n such that $a^n = a^n b$ $(a^n = ba^n)$.

Definition 2.1.[10]: A ring R is called a right (left) MGP- rings if and only if every maximal right (left) ideal is left (right) GP- ideal.

Theorem 2.2: Let R be a ring with $l(a^n) \subseteq r(a)$, for any $a \in R$ and a positive integer n. If R is a right MGP- ring, then the Jacobson radical of R is zero $(J(R) = (0))$.

Proof: Let $0 \neq a \in J(R)$. If $aR + r(a) \neq R$, then there exists a maximal right ideal M containing $aR + r(a)$. Since R is right MGP- ring, then M is left GP- ideal, so there exists $b \in M$ and a positive integer n such that $a^n = ba^n$, $(1-b)a^n = 0$ hence $(1-b) \in l(a^n) \subseteq r(a) \subseteq M$. So $1 \in M$, a contradiction. Therefore $aR + r(a) = R$, and so $ar + d = 1$ for some $r \in R$ and $d \in r(a)$, this implies that $a = a^2 r$. Since $a \in J(R)$, then there exists an invertible element v in R such that $(1-ar)v = 1$, so $(a - a^2 r)v = a$ yields $a = 0$. This proves that $J(R) = 0$. #

Theorem 2.3: Let M be a maximal right ideal of R. Then R is MGP- ring if and only if for every $a \in M$, $M + l(a^n) = R$ for some positive integer n.

Proof: Let M be a maximal right ideal of R and $a \in M$, since R is MGP- ring, then M is a left GP- ideal so there exists $b \in M$ such that $a^n = ba^n$ for some positive integer n. This implies that $(1-b) \in l(a^n)$. Therefore $R = M + l(a^n)$.
Conversely, assume that $M + l(a^n) = R$ for every $a \in M$ and a positive integer n, then $t + s = 1$ for some $t \in M$ and $s \in l(a^n)$. So $ta^n + sa^n = a^n$ and this implies $a^n = ta^n$. Whence R is MGP- ring. #

Lemma 2.4.[6]: Every strongly p - regular is p - regular ring.

Recall that, R is said to be W.R.D (weakly right duo), if for each $a \in R$ there exists a positive integer n such that $a^n R = Ra^n R$.

Theorem 2.5.[7]: Let R be W.R.D ring. Then the following are equivalent:
1- R is a p - regular.
2- Every ideal of R is a left GP- ideal.

Theorem 2.6: If R is a right Kasch ring and reversible, then the following statements are equivalent:
1- R is a strongly p - regular ring.
2- R is a right MGP- ring.

Proof: $(1) \Rightarrow (2)$:

Assume that R is strongly p - regular ring then by Lemma 2.4 and Theorem 2.5, R is a right MGP- ring.

$(2) \Rightarrow (1)$:

Assume that R is a right MGP- ring. Let M be any maximal right ideal of R. Since R is a right Kasch ring, then $M = r(a^n)$ for some $a \in R$ and a positive integer n. For any $x \in M$, we have $a^n x = 0$, and so $a^n Rx = 0$. This implies that $Rx \subseteq r(a^n) = M$, which proves that M is a two sided ideal of R. We claim that $b^n R + r(b^n) = R$. If not, there is a maximal right ideal N of R such that $b^n R + r(b^n) \subseteq N$. Since R is MGP- ring, then N is a left GP- ideal and $b^n \in N$. Then there exists $y \in N$ such that $b^n = yb^n$. Hence $(1-y)b^n = 0$ and so $(1-y) \in l(b^n) = r(b^n) \subseteq N$ (because R is reversible). Thus $1 \in N$, a contradiction.

Therefore $b^n R + r(b^n) = R$. In particular $b^n u + v = 1$ for some $u \in R$ and $v \in r(b^n)$, so $b^n = b^{2n} u$. Thus R is strongly p - regular. #

Theorem 2.7: Let R be a right Kasch ring and reversible. If R is a right MGP- ring. Then for each completely prime ideal P of R, $P = \bigcup_{x \notin P} r(x)$.

Proof: By Theorem 2.6, R is a strongly p - regular ring. Let $Q = \bigcup_{x \notin P} r(x)$ and show that $Q = P$. If $x \in Q$, then $x \in r(y)$ for some $y \notin P$, thus $yx = 0 \in P$ and hence $x \in P$. Therefore $Q \subseteq P$.

On the other hand, by the Lemma 2.4, P is p - regular ideal, thus for each $x \in P$, there exists $u \in P$ and a positive integer n such that $x^n = x^n u x^n$, which implies that $x^n(1 - ux^n) = 0 \in P$. Then $x^n \in l(1 - ux^n) = r(1 - ux^n)$ (because R is reversible ring). Since $(1 - ux^n) \notin P$ for otherwise $1 \in P$, which is impossible. Thus $x^n \in Q$, so that $P \subseteq Q$, whence $Q = P$. #

Following [9], a ring R is called right (left) weakly p - regular if, for every $x \in R$ there exists a positive integer n such that $x^n \in x^n R x^n R$ $(x^n \in R x^n R x^n)$. R is weakly p - regular if it is both right and left weakly p - regular.

The next result gives us a sufficient condition for MGP- ring to be weakly p - regular ring.

Theorem 2.8: Let R be a semi prime ring with each non-zero right ideal contains a non-zero two sided ideal. If R is an MGP- ring, then it is weakly p - regular ring.

Proof: Assume that $0 \neq a \in R$ such that $a^2 = 0$, then by assumption there is a non-zero two sided ideal of R with $I \subseteq aR$. We claim that $l(a) \cap I \neq (0)$, for if $Ia = 0$ then $I \subseteq l(a)$ and we are done. If $Ia \neq 0$, then $Ia \subseteq I \cap l(a) \neq (0)$ Now, $(I \cap l(a))^2 \subseteq l(a) I \subseteq l(a) aR = (0)$. Since R is semi prime ring, then $I \cap l(a) = (0)$ which is a contradiction, consequently R is reduced. We show that $Rx^n R + r(x^n) = R$ for any $x \in R$ and a positive integer n. Suppose that there exists $y \in R$ such that $Ry^n R + r(y^n) \neq R$. Then there exists a maximal right ideal M of R containing $Ry^n R + r(y^n)$. Since R is a right MGP- ring, then M is a left GP- ideal. So there exists $a \in M$ and a positive integer m such that $(y^n)^m = a(y^n)^m$ implies $(1 - a) \in l((y^n)^m) = r((y^n)^m) = r(y^n) \subseteq M$. Hence $(1 - a) \in M$ and so $1 \in M$ implies that $M = R$ which is a contradiction. Therefore $Rx^n R + r(x^n) = R$ for any $x \in R$. In particular $cx^n d + u = 1$ for some $u \in r(x^n)$ and $c, d \in R$, hence $x^n = x^n c x^n d$. So R is a right weakly p - regular ring. Since R is reduced it is also can be easily verified that R is weakly p - regular ring. #

Theorem 2.9.[6]: Let R be duo ring. Then R is p - regular if and only if R is strongly p - regular.

Lemma 2.10.[13]:If R is left or right quasi- duo and $J(R) = (0)$ .Then R is reduced ring.

Proposition 2.11.[6]: Let R be W.R.D. Then the following statement are equivalent:
1- R is a weakly p - regular ring.

2- R is a strongly p - regular ring.

The following theorem extends Theorem 2.6.

Proposition 2.12: Let R be a right duo ring. Then the following statements are equivalent:
1- R is a strongly p - regular ring.
2- R is a p - regular ring.
3- R is a right weakly p - regular ring.
4- R is a right MGP- ring and $J(R)$ is nil.

Proof: $(1) \Rightarrow (2)$: Follows from Theorem 2.9.
$(2) \Rightarrow (3)$: Obvious.
$(3) \Rightarrow (4)$: Follows from Proposition 2.11 and Theorem 2.5. Thus every maximal right ideal is GP- ideal. Now, we show that $J(R)$ is nil.

Let $r \in J(R)$. Then $r^n R = r^n R r^n R$ for some positive integer n, and so $r^n(1-s)=0$ for some $s \in Rr^n R$. But since $r \in J(R)$, $(1-s)$ is invertible and thus we have $r^n = 0$, showing that $J(R)$ is nil.
$(4) \Rightarrow (1)$: Assume (4), then from Theorems 2.5 and 2.9, R is strongly p - regular. #

Lemma 2.13.[10]: Let R be a reduced ring. Then every GP- ideal is pure ideal.

By the Proposition 2.12, Lemmas 2.10 and 2.13, we have the following theorem.

Theorem 2.14: Let R be a right quasi-duo and $J(R)=0$. Then the following statements are equivalent:
1- R is a strongly p - regular ring.
2- R is a p - regular ring.
3- R is a right (left) weakly p - regular ring.
4- R is a weakly p - regular ring.
5- R is a right MGP- ring.
6- Every maximal right ideal is a left pure.

3- EGP- rings

In this section, we introduce a new essential GP- ideals which are called EGP- rings. We give some of their basic properties, as well as a connection between EGP- rings and $S-$ weakly regular rings, strongly regular rings.

Definition 3.1: A ring R is said to be right EGP- rings, if every essential right ideal of R is a left GP- ideals.

Recall that [11], a ring R is a right (left) $S-$ weakly regular if for each $a \in R$, $a \in aRa^2R$ $(a \in Ra^2Ra)$. A ring R is called $S-$ weakly regular ring if it is both right and left $S-$ weakly regular ring.

We start this section by recalling the following propositions:

Proposition 3.2.[12]: A ring R is $S-$ weakly regular ring if and only if R is a reduced weakly regular ring.

Proposition 3.3.[10]: Let R be a duo ring. Then R is regular if and only if every ideal I of R is left pure.

Now, the following result is given:

Theorem 3.4: Let R be a ring with $aR = Ra$. Then R is a reduced, right EGP- ring if and only if R is an $S-$ weakly regular ring.

Proof: Assume that R is a reduced right EGP- ring. Let $a \in R$ and $I = Ra^2R + r(a)$. We claim that I is an essential right ideal of R. Suppose this is not true, then there exists a non-zero ideal J of R such that $I \cap J = (0)$. Then $(Ra^nR)J \subseteq IJ \subseteq I \cap J = (0)$. Since $a^2R \subseteq Ra^2R$, then $a^2R \cap J = (0)$. But $(a^2R)J \subseteq a^2R \cap J = (0)$ implies $J = (0)$, a contradiction; hence I is an essential right ideal. Since R is right EGP- ring, then I is a left GP- ideal, for every $a \in I$ there exists $b \in I$ and a positive integer n such that $a^n = ba^n$. Since $b \in I$ and $I = Ra^2R + r(a)$, then $b = ca^2d + h$ for some $c, d \in R$ and $h \in r(a)$ hence $a^n = ba^n = ca^2da^n + ha^n$. Since R is reduced then $r(a) = l(a) = l(a^n)$. So $a^n = ca^2da^n + 0$ implies that $(1 - ca^2d)a^n = 0$ and $(1 - ca^2d) \in l(a^n) = r(a^n) = r(a)$ hence $a = aca^2d$. Therefore R is an $S-$ weakly regular ring.

Conversely, assume that R is an $S-$ weakly regular ring. Then by Proposition 3.2, R is reduced weakly regular ring. Let $a \in R$ and $aR = aRaR = aRRa = aRa$ since $(aR = Ra)$. So R is regular and by Proposition 3.3, R is EGP- ring. #

Lemma 3.5.[9]: Let R be a weakly regular ring. Then $J(R) = (0)$.

However, we have the following proposition:

Proposition 3.6: Let R be a right duo ring. Then the following statements are equivalent:

1- R is a strongly regular ring.
2- R is a regular ring.
3- R is a right weakly regular ring.
4- R is an $S-$ weakly regular ring.
5- R is a right EGP- ring and $l(a^n) \subseteq r(a)$ for every $a \in R$ and a positive integer n.

Proof: $(1) \Rightarrow (2) \Rightarrow (3)$: They are obvious.

$(3) \Rightarrow (4)$: Assume (3), then by Lemmas 3.5, 2.10 and Proposition 3.2, R is an $S-$ weakly regular ring.

$(4) \Rightarrow (5)$: It follows from Theorem 3.4.

$(5) \Rightarrow (1)$: Assume that R is right EGP- ring. For any $a \in T$, let $T = aR + r(a)$ be a right ideal, by a similar method of the proof used in Theorem 3.4, T is an essential ideal. Since R is a right EGP- ring, then T is a left GP- ideal. For every $a \in T$, there exists $b \in T$ and a positive integer n such that $a^n = ba^n$, which implies $(1 - b) \in l(a^n) \subseteq r(a) \subseteq T$, so $1 \in T$ and $T = R$. Therefore $aR + r(a) = R$.

In particular $ar + d = 1$ for some $r \in R$ and $d \in r(a)$. Then $a = a^2r$. Thus R is strongly regular ring. #

Definition 3.7.[8]: A ring R is called fully right (left) idempotent if every right (left) ideal of R is idempotent.

Theorem 3.8: If R is ERT, then the following condition are equivalent:

1- R is a fully left idempotent ring.

2- R is a right EGP- ring.

Proof: $(1) \Rightarrow (2)$: Assume (1), and let E be an essential right ideal of R then it is an ideal of R (since R is ERT). Since R is a fully left idempotent ring, then for any $x \in E$, $Rx = (Rx)^2$ which implies that $x = ax$ for some $a \in RxR \subseteq E$. Therefore $x \in Ex$ for each $x \in E$. So E is left pure implies E is a left GP- ideal. Thus R is EGP- ring.

$(2) \Rightarrow (1)$: Assume (2), for any $a \in R$, set $L = Ra^n R + l(Ra^n R)$, and let K be a complement right ideal of R such that $L \oplus K$ is an essential right ideal of R. Now, $KRa^n R \subseteq K \cap Ra^n R \subseteq K \cap L = (0)$ implies that $K \subseteq l(Ra^n R)$. Whence $K \subseteq K \cap L = (0)$. This shows that L is an essential right ideal of R which is an ideal of R, by hypothesis R is a right EGP- ring. Therefore L is left GP- ideal and $a \in L$ implies that $a^n = da^n$ for some $d \in L$ and a positive integer n. If $d = u + v$, $u \in Ra^n R$ and $v \in l(Ra^n R)$, then $a^n = ua^n + va^n = ua^n \in Ra^n Ra^n$ which implies that $a^n \in (Ra^n)^2$; whence $Ra^n = (Ra^n)^2$. Therefore R is a fully left idempotent ring. #

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