On Sπ – Weakly Regular Rings, II

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الملخص

الهدف الرئيسي من هذا البحث هو دراسة الحلقات المنتظمة الضعيفة من النمط– Sp . بالإضافة إلى بعض الخواص الأساسية لهذه الحلقات وعلاقاتها مع الحلقات الأخرى مثل الحلقات من النمط -CS ، الحلقات من النمط –MGP والحلقات من النمط -SSGP .

ABSTRACT

The main purpose of this paper is to study right(left) Sp –Weakly regular rings. also we give some properties of Sp –Weakly regular rings, and the connection between such rings and CS-rings, MGP-rings and SSGP-rings.

1- Introduction

Throughout this paper. R is an associative ring with identity. A ring R is said to be right(left) **S-weakly regular ring** if for each $a \in R$, $a \in aRa^2R(a \in Ra^2Ra)$.

This concept was introduced by W.B. Vasantha kandasamy [7]. As a generalization of this concept the authors in [3] defined Sp –**Weakly regular ring** that is a ring such that for each $a \in R$, there exists a positive integer n, $a^n \in a^n R a^{2n} R(a^n \in R a^{2n} R a^n)$. In the present work we develop further properties of Sp –Weakly regular rings, and we give the connection of Sp –Weakly regular rings with other rings.

Recall that:-

1- An ideal I of a ring R is called **right(left) GP- ideal** if for every $a \in R$, there exists $b \in R$ and a positive integer n such that $a^n = a^n b(a^n = ba^n)$. [6]
2- A ring R is said to be a **right MGP-ring** if and only if every maximal right ideal is left GP-ideal.[6]
3- R is called **reduced** if it has no non nilpotent elements.

4- According to cohn [1], a ring R is called **reversible** if $ab=0$ implies $ba=0$ for $a,b\in R$. It is easy to see that R is reversible if and only if right (left) annihilator of a in R is two sided ideal.
5- A ring R is said to be **right SSGP-ring** if every simple singular right R-module is GP- injective [4].
6- $J(R)$ denote the Jacobson radical.

2- Sp – Weakly regular rings

In this section we give some of basic properties, as well as a connection between CS-rings, MGP-rings, SSGP-rings and Sp – Weakly regular rings.

begins with the following definition .

Definition 2.1:[3]

R is called **right (left)** Sp **– Weakly regular ring** if for each $a\in R$, there exists a positive integer $n=n(a)$ depending on a such that $a^n\in a^nRa^{2n}R(a^n\in Ra^{2n}Ra^n)$. R is called Sp **– Weakly regular ring** if it is both right and left Sp – Weakly regular ring.

Theorem2.2:

Let R be Sp – Weakly regular ring and $a^nR=Ra^n$ with $r(a^n)\subseteq r(a)$ for every $a\in R$ and a positive integer n, then $J(R)=(0)$.

Proof:

Let $0\neq a\in J(R)$, if $aR+r(a)\neq R$, then there exists a maximal right ideal M containing $aR+r(a)$, since R is Sp – Weakly regular ring, then there exists $b,c\in R$ and a positive integer n such that $a^n=a^nba^{2n}c$, so $a^n(1-ba^{2n}c)=0$ implies that $(1-ba^{2n}c)\in r(a^n)\subseteq r(a)\subseteq M$ so $1\in M$ a contradiction. Therefore $aR+r(a)=R$, in particular $ab+d=1$ for some $b\in R, d\in r(a)$, so $a^2b=a$ implies $a(1-ab)=0$ since $a\in J(R)$, so there exists an invertible element v in R such that $(1-ab)v=1$ multiply in the left by a, $(a-a^2b)v=a$ implies $a=0$. Therefore $J(R)=(0)$.

Theorem 2.3:

Let R be a ring with out identity and with out divisors of zero. Then R is Sp – Weakly regular ring if and only if for every $a\in R$ there exists $b,c\in R$ and a positive integer n with $a^n=a^nba^{2n}c$.

Proof:

Given a ring R with out identity and with out divisors of zero. Let R be $Sp-$Weakly regular ring. Then for every $a \in R$ we have $a^n \in a^n R a^{2n} R$, thus $a^n = a^n b a^{2n} c$ for some $b, c \in R$.

Conversely, if $a^n = a^n b a^{2n} c$ for every $a \in R$, and a positive integer n we have obviously R to the $Sp-$ Weakly regular ring and given R has no identity and zero divisors.

The following result is a connection between MGP- ring and $Sp-$Weakly regular ring by adding reversible condition.

Theorem 2.4:

Let R be a right MGP-ring and reversible ring. Then R is a right $Sp-$Weakly regular ring.

Proof:

Let R be a right MGP- ring, to prove R is a right $Sp-$Weakly regular ring, let $Ra^{2n}R + r(a^n) = R$, if not then there exists a maximal right ideal M containing $Ra^{2n}R + r(a^n)$ such that $Ra^{2n}R + r(a^n) \subseteq M$, since R is a right MGP- ring, then every maximal right ideal of R is left GP-ideal. Then for all $a \in M$, there exists $b \in M$ and a positive integer n such that $a^n = ba^n$ then $a^n - ba^n = 0$, implies $(1-b)a^n = 0$ and $(1-b) \in l(a^n) = r(a^n) \subseteq M$ (R is reversible ring), hence $1 \in M$, a contradiction, hence $Ra^{2n}R + r(a^n) = R$. Then $ba^{2n}c + d = 1$, for some $b, c \in R$ and $d \in r(a^n)$ implies that $a^n b a^{2n} c + a^n d = a^n$. Therefore $a^n \in a^n R a^{2n} R$, so R is a right $Sp-$Weakly regular ring.

Example:

Let Z_{12} be the ring of integers module 12, then the maximal ideals, $I = \{o,3,6,9\}$, $J = \{o,2,4,6,8,10\}$ are GP-ideal, hence Z_{12} is a right MGP-ring and a right $Sp-$Weakly regular ring.

Definition 2.5:[2]

A ring R is said to be **right (left) CS- ring** if every non zero right (left) ideal is essential in a direct summand.

The next result gives a sufficient condition for CS- ring to be $Sp-$Weakly regular ring.

Theorem 2.6:

Let R be a reversible right CS- ring and $r(a^{2n}) \subseteq r(a^n)$ for every $a \in R$ and a positive integer n. Then R is a Sp −Weakly regular ring.

Proof:

Let $0 \neq a \in R$ such that $Ra^{2n}R + r(a^n) = R$, if not then there exists a maximal right ideal M containing $Ra^{2n}R + r(a^n)$ since M is a direct summand, there exists K right ideal such that $M \oplus K = R$, this meaning $Ra^{2n}R + r(a^n) \cap K = 0$, which implies that $Ra^{2n}RK \subseteq MK \subseteq M \cap K = 0$, then $K \in r(a^{2n}) \subseteq r(a^n)$, hence $K \subseteq r(a^n) \subseteq M$ but $M \cap K = 0$, contradiction, if $Ka \neq 0$, $Ka \subseteq M \cap K = 0$ implies that $Ka = 0$, contradiction, hence $Ra^{2n}R + r(a^n) = R$, in particular $ra^{2n}s + t = 1$, where $r, s \in R$ and $t \in r(a^n)$, so $ra^{2n}sa^n + ta^n = a^n$ henc1e $ta^n = 0$ (since $t \in r(a^n) = l(a^n)$ R is reversible ring), implies $ra^{2n}sa^n = a^n$, then $a^n \in Ra^{2n}Ra^n$ as well as $a^n ra^{2n}s + a^n t = a^n$, so $a^n \in a^n Ra^{2n}R$. Therefore R is a Sp −Weakly regular ring.

The following result is due to S.B. Nam [4].

Lemma 2.7:[4]

Let R be a SSGP- ring and $l(a) \subseteq r(a)$ for every $a \in R$. Then R is a reduced ring.

Proposition 2.8:

Let R be a SSGP- ring and $l(a) \subseteq r(a)$ for every $a \in R$. Then R is Sp −Weakly regular ring.

Proof:

Assume that R is SSGP- ring and $l(a) \subseteq r(a)$ for every $a \in R$, then by Lemma 2.8, R is reduced ring. We will show that $Ra^{2n}R + r(a^n) = R$, for any $a \in R$ and a positive integer n. Suppose that $Ra^{2n}R + r(a^n) \neq R$. Then there exists a maximal right ideal M of R containing $Ra^{2n}R + r(a^n)$. Thus R/M is GP- injective, so any R- homomorphism of $a^{2n}R \to R/M$ extends to one of R into R/M. Let $f : a^{2n}R \to R/M$ be defined by $f(a^{2n}t) = t + M$, for all $t \in R$. Since R is reduced ring, then f is a well-defined R- homomorphism. Now, R/M is GP-injective. So there exists $c \in R$ such that $1 + M = f(a^{2n}) = ca^{2n} + M$. Hence $1 - ca^{2n} \in M$ and so $1 \in M$, which is a contradiction. Therefore $Ra^{2n}R + r(a^n) = R$. In particular, $1 = ba^{2n}d + x$ for some $b, d \in R$ and $x \in r(a^n)$. Therefore $a^n = a^n ba^{2n} d$ .So $a^n R = a^n Ra^{2n} R$ and hence R is Sp –Weakly regular ring.

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