

Detour Hosoya Polynomials of Some Compound Graphs

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المخلص

في هذا البحث قمنا بتعريف متعددة حدود هوسويا نسبة الى مسافة اطول بعد $H^*(G; x)$. تم ايجاد متعددة حدود هوسويا نسبة الى مسافة أطول بعد لبعض البيانات الخاصة مثل بيان الدرب و بيان الدارة. و كذلك تم الحصول على كل من $H^*(G_1 \cdot G_2; x)$ و $H^*(G_1 : G_2; x)$ ، $H^*(G_1 \cdot G_2; x)$.

ABSTRACT

In this paper we will introduce a new graph distance based polynomial; Detour Hosoya polynomials of graphs $H^*(G; x)$. The Detour Hosoya polynomials $H^*(G; x)$ for some special graphs such as paths and cycles are obtained. Moreover the Detour Hosoya polynomials $H^*(G_1 \cdot G_2; x)$, $H^*(G_1 : G_2; x)$ and $H^*(G_1 \cdot G_2; x)$ are obtained.

Keywords: Detour distance, compound graphs, Hosoya polynomials.

1. Introduction

The concept of Hosoya polynomial was first put forward in 1988 by Hosoya [1]. Several authors, such as [1], [2], [3], [4], [5], [6], [7], [8], [13] and [15] had obtained Hosoya polynomials for special graphs, graphs having some kind of regularity and for compound graphs obtained by using some well-known binary operations in graph theory.

In this paper, we consider finite connected graphs without loops or multiple edges. For undefined concepts and notations see [9] and [12].

Ordinarily, when we wish to proceed from a point A to a point B we take a route which involves the least distance. We have all been faced with detour sign which require us to take a route from A to B that involves a greater distance. In any such detour route from A to B we assume that there is no possible shortcut along the route, for otherwise this should have been part of the route initially. When one is driving along such a detour, it sometimes seems that we are using the longest route possible from A to B .

(again subject to the “no shortcut” condition). In this paper we investigate longest detour routes in graphs.

The distance $d(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest u - v path in G . For a nonempty set S of vertices of G , the subgraph $\langle S \rangle$ of G induced by S as its vertex set while an edge of G belongs to $\langle S \rangle$ if it joins two vertices of S . If P is a u - v path of length $d(u, v)$, then the subgraph $\langle V(P) \rangle$ induced by the vertices of P is P itself. This observation suggests the following concept. The detour distance $d^*(u, v)$ between u and v in G is the length of a longest induced u - v path, that is a longest u - v path P for which $\langle V(P) \rangle = P$. An induced u - v path of length $d^*(u, v)$ is called a *detour path* [10].

Observe that $d^*(u, v) \geq d(u, v)$ for all vertices u and v of G and that $d^*(u, v) = d(u, v) = 1$ if and only if u and v are adjacent. Also, note that $d^*(u, v) = d^*(v, u)$ for all vertices u and v of G . Therefore the detour distance is symmetric. However, the triangle inequality does not hold in general. Consider the wheel W_p of order $p \geq 6$ with center at the vertex w ; then: $d^*(u, v) = p - 3 \geq 2 = d^*(u, w) + d^*(w, v)$, for every two vertices u and v of W_p , $u, v \neq w$, that are both adjacent to a common vertex $x \neq w$. Therefore, in general, the detour distance is not a metric on the vertex set of G [10].

The *detour eccentricity* $e^*(v)$ of a vertex v is defined by $e^*(v) = \max\{d^*(v, w) : w \in V(G)\}$. The *detour eccentricity set* $e^*(G)$ of a connected graph G is the set consisting of all detour eccentricities of G , that is $e^*(G) = \{e^*(v) : v \in V(G)\}$. The *detour radius* $rad^*(G)$ of G is the minimum detour eccentricity, while the *detour diameter* $diam^*(G)$ of G is the maximum detour eccentricity.

For completeness we define $d^*(u, v) = 0$ if and only if $u = v$.

A connected graph G is called a *detour graph* if $d^*(u, v) = d(u, v)$ for all vertices u and v of G . No cycle of length 5 or more is a detour graph.

On the other hand, all trees and all complete graphs are detour graphs. If u and v are distinct vertices of a graph G such that $d^*(u, v) = 1$ or 2 , then $d^*(u, v) = d(u, v)$ [10], the converse is not true in general, that is if $d(u, v) = 2$, then $d^*(u, v) \geq 2$, as for the wheel W_p , $p \geq 6$.

The concept of Hosoya polynomial $H(G; x)$ of a graph G was put forward by Hosoya [13], and defined as

$H(G; x) = \sum_{k=0}^{d(G)} C(G, k) x^k$; where $C(G, k)$ is the number of pairs of vertices in G that are distance k apart, and $d(G)$ is the diameter of the graph G .

In this paper, the concept of *Hosoya polynomials of detour distance* of a connected graph G (or simply *detour Hosoya polynomial of a graph G*) has been defined by

$$H^*(G; x) = \sum_{k=0}^{d^*(G)} C^*(G, k) x^k = \sum_{\{u, v\} \in V(G)} x^{d^*(u, v)} \quad \dots(1)$$

in which $C^*(G, k)$ is the number of pairs of vertices in G with detour distance k , and $d^*(G)$ is the detour diameter of G .

It is clear that if G is a detour graph, then $H^*(G; x) = H(G; x)$.

The sum $W^*(G)$ of detour distances between all pairs of vertices of the graph G is known as the *Wiener index of detour distance* of the graph G (or simply *detour Wiener index* of the graph G), that is

$$W^*(G) = \sum_{u, v} d^*(u, v),$$

where the sum is taken over all unordered pairs $\{u, v\}$ of distinct vertices in G .

It is clear that

$$W^*(G) = \frac{d}{dx} H^*(G; x) \Big|_{x=1}.$$

We illustrate these ideas in the following example.

Example 1.1. Let G be a graph of order $p = 9$, depicted in figure 1.1(a).

It is clear that

$$e^*(v_1) = 5, \quad e^*(v_2) = 4, \quad e^*(v_3) = 4, \quad e^*(v_4) = 3, \quad e^*(v_5) = 4, \\ e^*(v_6) = 3, \quad e^*(v_7) = 4, \quad e^*(v_8) = 5 \text{ and } e^*(v_9) = 5.$$

Hence

$$e^*(G) = \{5, 4, 4, 3, 4, 3, 4, 5, 5\}, \text{diam}^*(G) = 5 \text{ and } \text{rad}^*(G) = 3.$$

A detour $v_1 - v_9$ path is given in Figure 1.1(b). Therefore $d^*(v_1, v_9) = 5$, and this gives us the maximum detour distance among all detour distances of pairs of vertices of $V(G)$.

The path P_1 is not a detour $v_1 - v_9$ path, because $\langle V(P_1) \rangle^1 P_1$ (see figures 1.1(c) and 1.1(d)).

By direct calculations, we get that

$$C^*(G, 0) = p = 9, \quad C^*(G, 1) = 10, \quad C^*(G, 2) = 9, \\ C^*(G, 3) = 9, \quad C^*(G, 4) = 6 \text{ and } C^*(G, 5) = 2.$$

Hence, the detour Hosoya polynomial of G is

$$H^*(G; x) = 9 + 10x + 9x^2 + 9x^3 + 6x^4 + 2x^5,$$

and

$$W^*(G) = \frac{d}{dx} H^*(G; x) \Big|_{x=1} = 89.$$

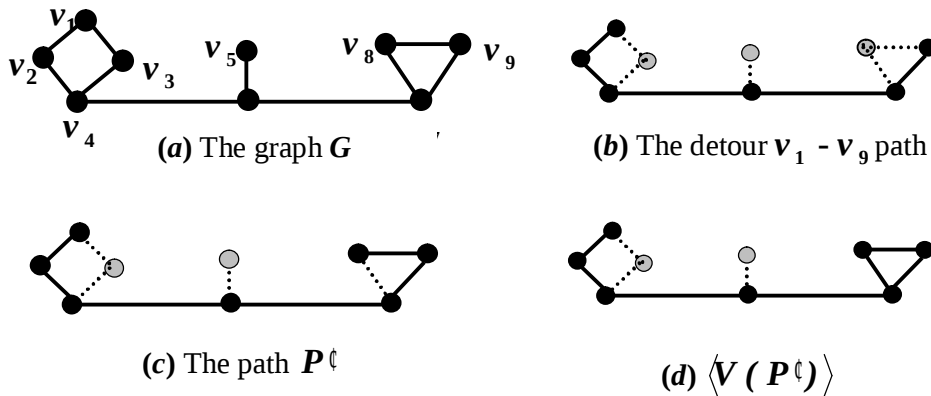


Figure 1.1.

In 1993, Gutman [8], established few additional properties of the respective graph polynomials. He obtained Hosoya polynomials of some special graphs and obtained formula for the Hosoya polynomials of some compound graphs, namely $G_1 \cdot G_2$ and $G_1 : G_2$ which are defined in the following: Let G_1 and G_2 be vertex-disjoint connected graphs, and let $u \in V(G_1)$ and $v \in V(G_2)$. Then, the graph $G_1 \cdot G_2$ is obtained from

G_1 and G_2 by identifying the two vertices u and v . This means that G_1 and G_2 have exactly one vertex in common in the compound graph $G_1 \cdot G_2$. The graph $G_1 : G_2$ is obtained from G_1 and G_2 by introducing a new edge joining the two vertices u and v . In this paper, formulas for $H^*(G_1 \cdot G_2; x)$ and $H^*(G_1 : G_2; x)$ in terms of the detour Hosoya polynomials of G_1 and G_2 will be obtained.

2. Detour Hosoya Polynomials of Some Special Graphs

Let P_n , K_n and S_n denotes the path, complete and star graphs of n vertices respectively. It is known that [10] all trees and complete graphs are detour graphs. This leads us to the following result.

Proposition 2.1

$$\begin{aligned} (a) \quad H^*(P_n; x) &= \sum_{k=0}^{n-1} (n-k)x^k. \\ (b) \quad H^*(K_n; x) &= n + \frac{1}{2}n(n-1)x. \\ (c) \quad H^*(S_n; x) &= n + (n-1)x + \frac{n-1}{2}x^2. \quad \blacksquare \end{aligned}$$

Proposition 2.2 Let C_p be a cycle of order $p \geq 5$, then

$$H^*(C_p; x) = \begin{cases} p(1 + x + \sum_{k=\frac{p+1}{2}}^{p-2} x^k) & \text{if } p \text{ is odd} \\ p(1 + x + \frac{1}{2}x^{\frac{p}{2}} + \sum_{k=\frac{p}{2}+1}^{p-2} x^k) & \text{if } p \text{ is even} \end{cases}$$

Proof. Let u, v be any two distinct vertices of C_p . We will consider the following cases:

- (1) If $uv \in E(C_p)$ then $d^*(u, v) = 1$ and $C^*(G, 1) = p$.
- (2) If $uv \notin E(C_p)$, then $d^*(u, v) = p - d(u, v)$,
where $d(u, v)$ denotes the ordinary distance.

We know that[11], for an odd p , the ordinary Hosoya polynomial of C_p is

$$\text{given by } H(C_p; x) = p + px + \sum_{k=2}^{\frac{p-1}{2}} x^k.$$

Hence

$$H^*(C_p; x) = p + px + \sum_{k=p-2}^{\frac{p-1}{2}} x^k$$

or

$$H^*(C_p; x) = p + px + \sum_{k=\frac{p+1}{2}}^{p-2} x^k.$$

Similarly, we prove the formula for the case when p is even.

This completes the proof. ■

Proposition 2.3 Let W_p be a wheel graph of $p + 1$ vertices, then

$$H^*(W_p; x) = p + 2(p-1)x + (p-1) \sum_{k=\frac{p}{2}}^{\frac{p-3}{2}} x^k, \quad \text{if } p \text{ is even}$$

$$H^*(W_p; x) = p + 2(p-1)x + (p-1) \sum_{k=\frac{p+1}{2}}^{\frac{p-1}{2}} x^k, \quad \text{if } p \text{ is odd}$$

Proof. For $uv \in E(W_p)$, $d_{W_p}^*(u, v) = d_{C_{p-1}}^*(u, v)$.

Hence, for $k \geq 2$

$$C^*(W_p, k) = C^*(C_{p-1}, k).$$

Thus,

$$H^*(W_p; x) = 1 + (p-1)x + H^*(C_{p-1}, x).$$

Now, using Proposition 2 we obtain the required result. ■

Proposition 2.4 Let $K_{t,s}$ be a complete bipartite graph with partite subsets of sizes t and s , then

$$H^*(K_{t,s}; x) = (t+s) + (ts)x + \sum_{i=2}^t \sum_{j=2}^s x^{i+j}.$$

Proof. Obvious ■

The following result gives us the Wiener index of the detour distance of the special graphs P_n , K_n , S_n , C_p , W_p and $K_{t,s}$.

Proposition 2.5

$$(1) W^*(P_n) = \frac{1}{6}n(n^2 - 1).$$

$$(2) W^*(K_n) = \frac{1}{2}n(n - 1).$$

$$(3) W^*(S_n) = (n - 1)^2.$$

$$(4) \text{ For } p \geq 5, W^*(C_p) = \begin{cases} \frac{1}{8}p(3p^2 - 12p + 17), & \text{if } p \text{ is odd} \\ \frac{1}{8}p(3p^2 - 12p + 16), & \text{if } p \text{ is even} \end{cases}.$$

$$(5) \text{ For } p \geq 6, W^*(W_p) = \begin{cases} \frac{1}{8}(p - 1)(3p^2 - 18p + 39), & \text{if } p \text{ is odd} \\ \frac{1}{8}(p - 1)(3p^2 - 18p + 40), & \text{if } p \text{ is even} \end{cases}.$$

$$(6) W^*(K_{t,s}) = ts + t(t - 1) + s(s - 1).$$

3. Detour Hosoya Polynomials of Some Compound Graphs

Let u be a vertex of a connected graph G of order p . The number of pairs of vertices of G containing the vertex u such that $d_G^*(u, v) = k$, " $v \in V(G)$ ", will be denoted by $C^*(u, G; k)$.

We define the polynomial

$$H^*(u, G; x) = \sum_{k=0}^{e^*(u)} C^*(u, G; k) x^k \quad \dots(2)$$

It is clear that

$$H^*(G; x) = \frac{1}{2} \sum_{u \in V(G)} H^*(u, G; x) + \frac{1}{2} p \quad \dots(3)$$

Let G_1 and G_2 be two disjoint connected graphs of orders p_1 and p_2 respectively. Moreover, let w be the vertex obtained by identifying the vertex u of G_1 with the vertex v of G_2 in order to construct the compound graph $G_1 \cdot G_2$. The compound graph $G_1 : G_2$ is obtained by introducing a new edge joining the vertex u of G_1 with the vertex v of G_2 .

Now, we are ready to present formulas for $H^*(G_1 \cdot G_2; x)$ and $H^*(G_1 : G_2; x)$ in terms of $H^*(G_1; x)$ and $H^*(G_2; x)$.

Theorem 3.1 If G_1 and G_2 are disjoint connected graphs, then

$$H^*(G_1 \cdot G_2; x) = H^*(G_1; x) + H^*(G_2; x) + H^*(u, G_1; x) \cdot H^*(v, G_2; x) - H^*(u, G_1; x) - H^*(v, G_2; x).$$

Proof: Let s, t be any two vertices of $G_1 \cdot G_2$ such that $d_{G_1 \cdot G_2}^*(s, t) = k$. We will consider the following cases:

(1) If $s, t \hat{=} V(G_1)$, then $C^*(G_1 \cdot G_2; k) = C^*(G_1, k)$, which produces the polynomial $H^*(G_1; x)$.

(2) If $s, t \hat{=} V(G_2)$, then $C^*(G_1 \cdot G_2; k) = C^*(G_2, k)$, which produces the polynomial $H^*(G_2; x)$.

(3) $s \hat{=} V(G_1)$ and $t \hat{=} V(G_2)$: In this case, any longest induced (s, t) -path P will contain the vertex w . If P^ℓ is a longest (s, w) -path and $P^\mathfrak{C}$ is a longest (t, w) -path with $\langle V(P^\ell) \rangle = P^\ell$ and $\langle V(P^\mathfrak{C}) \rangle = P^\mathfrak{C}$, then

$$V(P) = V(P^\ell) \cup V(P^\mathfrak{C}), \text{ and } \langle V(P) \rangle = \langle V(P^\ell) \cup V(P^\mathfrak{C}) \rangle,$$

because no vertex of P^ℓ , other than w is adjacent with a vertex of $P^\mathfrak{C}$, other than w .

Therefore $P^\ell \cdot P^\mathfrak{C} = \langle V(P) \rangle = P$.

Hence, $d_{G_1 \cdot G_2}^*(s, t) = d_{G_1}^*(s, w) + d_{G_2}^*(t, w)$.

This produces the polynomial $H^*(u, G_1; x) \cdot H^*(v, G_2; x)$. Notice that the polynomial $H^*(u, G_1; x)$ is counted twice in the Cases (1) and (3), and also $H^*(v, G_2; x)$ is counted twice in the Cases (2) and (3).

Now, adding the polynomials obtained from the cases (1), (2) and (3), we get the required result. ■

Theorem 3.2 If G_1 and G_2 are disjoint connected graphs, then

$$H^*(G_1 : G_2, x) = H^*(G_1, x) + H^*(G_2, x) + x \cdot H^*(u, G_1; x) \cdot H^*(v, G_2; x).$$

Proof. Let s, t be any two distinct vertices of the compound graph $G_1 : G_2$. We consider the following cases:

(1) If $s, t \hat{=} V(G_1)$, then we get the polynomial $H^*(G_1; x)$.

(2) If $s, t \hat{=} V(G_2)$, then we get the polynomial $H^*(G_2; x)$.

(3) $s \hat{=} V(G_1)$ and $t \hat{=} V(G_2)$: In this case, any longest (s, t) -path will contain the edge uv , and as in the proof of Theorem 6(Case 3), this produces the polynomial

$$x \cdot H^*(u, G_1; x) \cdot H^*(v, G_2; x)$$

Now, adding the polynomials obtained from the cases (1), (2) and (3), we get the required result. ■

Definition 3.3 Let G_1 and G_2 be disjoint connected graphs of orders p_1 and p_2 , respectively. Let $G_2^{(i)}$ be the i^{th} copy of G_2 . The **Corona** $G_1 \circ G_2$, is the graph[13] constructed from $G_1 \cup p_1 G_2$ with additional edges $\bigcup_{i=1}^{p_1} \{v_i u : u \hat{=} V(G_2^{(i)})\}$, as depicted in Fig. 3.1, in which $V(G_1) = \{v_1, v_2, \dots, v_{p_1}\}$.

It is clear that

$$p(G_1 \circ G_2) = p_1(1 + p_2) = p,$$

and

$$q(G_1 \circ G_2) = q(G_1) + p_1(p_2 + q(G_2)) = q.$$

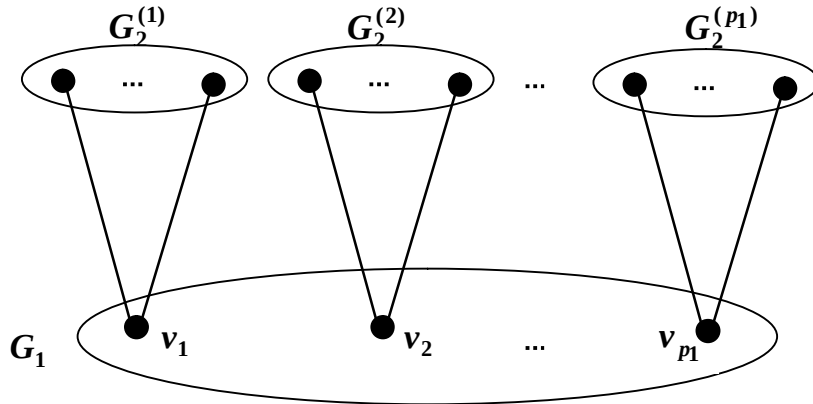


Fig. 3.1 The Corona $G_1 \circ G_2$

The next theorem computes the detour Hosoya polynomial of the corona $G_1 \circ G_2$.

Theorem 3.4 Let G_1 and G_2 be two disjoint connected graphs, then

$$H^*(G_1 \cup G_2; x) = (1 + p_2 x)^2 H^*(G_1; x) + p_1 H^*(G_2; x) - p_1 p_2 x (1 + p_2 x).$$

Proof. Let s, t be any two distinct vertices of $G_1 \cup G_2$. We will consider the following cases:

Case 1. If $s, t \in V(G_1)$, then we get the polynomial $H^*(G_1; x)$.

Case 2. If $s, t \in V(G_2^{(i)})$, for $i = 1, 2, \dots, p_1$, then we get the polynomial $p_1 H^*(G_2; x)$.

Case 3. $s \in V_2^{(i)}$ and $t = v_j$ (or $s = v_i$ and $t \in V_2^{(j)}$) for $i, j = 1, 2, \dots, p_1$, then

(i) If $i = j$, then we get the polynomial $p_1 p_2 x$.

(ii) If $i \neq j$, then we get the polynomial $2p_2 x [H^*(G_1; x) - p_1]$.

Case 4. If $s \in V_2^{(i)}$ and $t \in V_2^{(j)}$ for $i, j = 1, 2, \dots, p_1$, $i \neq j$, then we get the polynomial $p_2^2 x^2 [H^*(G_1; x) - p_1]$.

Now, adding the polynomials obtained from the above cases and simplifying, we get the required result. ■

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