Complex Dynamics of the family $\lambda \frac{\sinh^2(z)}{z^3}$

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الملخص

الهدف من البحث هو دراسة الدينمية المعقدة لعائلة الدوال الميرومورفكية المتسامية $F=\{\lambda>0,\ \lambda\frac{\sinh^2(z)}{z^3}\}$, سنبرهن أن فترات معينة من الأعداد الحقيقية تنتمي إلى مجموعة جوليا في حين تنتمي فترات أخرى الى مجموعة فاتو كما سنبرهن على أن مجموعة جوليا تحوي على $R^+ \cup iR^+$ كما سنبرهن أن مجموعة فاتو تحتوي على مكونات معينة ولمختلف قيم المعلمة $l>0$. بالإضافة الى ذلك,سنعطي وصفا لمجاميع جوليا على أنها أنغلاق مجموعة النقاط في الفضاء العقدي والتي تكون فيها غاية الدالة هو المالانهاية لمختلف قيم المعلمة $l>0$ وسنستخدم هذا الوصف لإعطاء صور لهذه المجاميع بواسطة الحاسوب.

ABSTRACT

The aim of this work is to study the complex dynamics of the family $\boldsymbol{F}=\{\lambda\frac{\sinh^2(z)}{z^3}\ ,\ l>0\}$, of transcendental meromorphic functions. We prove that certain intervals are contained in J(*f*) or in F(*f*) and we prove that Julia set contains $R^+ \cup iR^+$ and show that Fatou set of the functions in *F* contains certain components for different values of $l>0$. We characterize Julia set of $f_l \in F$, for different values of $l>0$, as the closure of the escaping points, and use this characterization to give computer images for these sets.

1. Introduction:

In the early of twentieth century, the iteration theory of complex functions originated in works of Julia and Fatou. There had been a long period of inactivity after that. During the end of 20-th century, a renewed interest in the study of iteration theory started due to beautiful computer graphics and a wide ranging applications in engineering and science associated with it [1,2,3,4,8, 10]. Iteration occurs in many parts of

mathematics. For example many algorithms of numerical mathematics based on it. Given a set D and a function $f : D \to D$ the iterates of f are defined by $f^1 = f$ and $f^n = fof^{n-1}$ for n>1.The main problem in iteration theory is to study the behavior of the sequence $\langle f^n \rangle$ as n goes to infinity.

In iteration theory, for complex dynamical systems, most of the works have been centered around the dynamics of entire and rational functions. However, in comparison to the investigations of entire and rational functions, not much works have been done in this direction for transcendental meromorphic functions.

The initial works on study of transcendental meromorphic functions started at 80-th by Devaney, Keen, Bergweiler, and others [2,6,7]. The central objects studied in complex dynamical systems of a function is its Julia and Fatou sets. The Fatou set of a function f, denoted by F(f), is defined to be the set of all complex numbers where the family of iterates $\{f^n : n>0\}$ of f forms a normal in the sense of Montel . Recall that the family $F=\{f^n : n>0\}$ of complex maps defined on open set U of C called normal if every infinite sequence in F contains a subsequence which either: converges uniformly on a compact subset of U, or converges uniformly to ∞ on U[4].

2. Definitions:

Julia set (or chaotic set), denoted by J(f), is the complement of F(f). The escaping points set of meromorphic function f(z), denoted by I(f), is defined by $I(f)= \{ z \in C : f^{(n)}(z) \to \infty$ as $n \to \infty$ and $f^{(n)}(z) \neq \infty \}$.

A point $w \in C$ is said to be critical point of f if $f'(z) = 0$. The value $f(w)$ corresponding to a critical point w is called a critical value of f. A point $v \in C$ is called asymptotic value of f if there is a continuous function $a(t)$ satisfying that $\lim_{t \to \infty} f(a(t)) = v$. A singular value of f is defined to be either critical value or asymptotic value of f. Finally the function f is called critically finite if it has a finite numbers of critical and asymptotic values.

3. Helping Results:

Let B be a class of meromorphic functions having bounded singular values. In this work, we need the following characterization given by Zheng [10], for transcendental meromorphic functions:

Theorem (3.1): Let $f_1 \in$ B, then $J(f) = \overline{I(f)}$.

In [5], we studied the real dynamics of the family $\boldsymbol{F}= \lambda\frac{\sinh^2(z)}{z^3}$. In present work we study the complex dynamics of this family. We show that the functions in this family are non-critically finite and all the critical values are bounded. We characterize the Julia set of $f_l \in \mathrm{F}$, as the escaping points of $f_l, \overline{I(f_l)}$, for various values of $l > 0$. For different values of the parameter $l > 0$, we prove that Julia and Fatou sets contains certain intervals. Finally we use the characterization of Julia set to give computer images of Julia set for different value of $l > 0$

In [5],it was shown that the critical parameter value of the family *F* is $l_1 = 1.2203$, and we proved the following theorem :

Theorem (3.2): Let $f_l \in \boldsymbol{F}$. Lt T_0 be set of backward orbit of the pole z=0, then: **1**. For $0 < l < l_1$, $f_l^n(x) \to a_l \; for x \in (a, r_l)/T_0$,

$f_l^n(x) \to -a_l \; for x \in (-r_l, -a)/T_0$. $For x \in \{(0,a) \cup (r_l, \infty)\}\; f_l^n(x) \to \infty$

and, $for x \in \{(-a,0) \cup (-\infty, -r_l)\}\; f_l^n(x) \to -\infty$, where a is a positive solution of the equation $f_l(x) = r_l$, a_l is an attracting fixed point of $f_l(x)$ and r_l is a repelling fixed point of $f_l(x)$.

2. For $l = l_1$, $f_l^n(x) \to x_1 \; for x \in (m, x_1)/T_0 \; and \; f_l(x) \to -x \; for \; x \in (-x, -m)/T_0$

Moreover $f_l^n(x) \to \infty$ for

$x \in \{(0,m),(x_1,\infty)\}/T_0 \; and \; f_l^n(x) \to -\infty \; for \; x \in \{((-\infty, x_1) \cup (-m,0))/T_0\}$.,

where x_1 and $-x_1$ are the rationally indifferent fixed points of $f_l(x)$. and m is a positive solution of the equation $f_l(x) = x_1$

3. For $l > l_1$, $f_{l)}^n(x) \to \infty \; for x \in (0,\infty)/T_0 \; and \; f_l^n(x) \to -\infty \; for \; x \in (-\infty, 0)/T_0$.

The following theorem gives a classification of the periodic components of the Fatou set of a function *f*. Recall that a maximal connected domain *U* contained in the Fatou set of a function *f* is called a component of F(*f*) .

Theorem (3.3)[9].

Let f be a complex function. Let *U* be a periodic component, of period *p*, in the Fatou set of *f*. Then one of the following possibilities is true:

1. U contains an attracting periodic point z_0 of period p. Then for $z \in U$ $z \in U \quad f^{(np)}(z) \to z_0 asn \to \infty$ and U is called the **basin of attraction of** z_0.

2. ∂U contains a periodic point z_0, $f^{(np)}(z) \to z_0$ as $n \to \infty$ for $z \in U$ and $f^{(p)}(z_0) = 1$. In this case U is called **Parabolic domain.**

3. There exists an analytic homeomorphisim $j : U \to D$, where D is the unit disk, such that $j \circ f^{(p)} \circ j^{-1}(z) = e^{2piia} z$ for some $a \in R/Q$. In this case U is called **Siegel disk**.

4. There exists an analytic homeomorphisim $j : U \to A$, where $A = \{z \in C : 1 < |z| < r\}$ and $r>1$, such that $j \circ f^{(p)} \circ j^{-1}(z) = e^{2piia} z$ for some $a \in R/Q$. In this case U is called **Herman ring.**

5. There exists $z_0 \in U$ such that $f^{(np)}(z) \to z_0$ as $n \to \infty$ for $z \in U$ but $f^{(np)}(z_0)$ is not defined. In this case, U is called **Baker domain.**

If U is not periodic, then U is called **wandering domain**, i.e. if U is wandering domain, then $U^m \neq U^n$ for all $m \neq n$.

Theorem (3.4)[10]: Let f be a transcendental meromorphic function and let $D = \{U_0, U_1, ..., U_{n-1}\}$ be a n-periodic cycle of components of F(f) . Then:
1. If D is a cycle of attracting basins or parabolic domains, then some U_k, k=1,2,…,n-1, must intersects the set of singular values.

2. if D is a Siegel disks or Herman rings, then U_k is a proper subset of the closure of the forward orbits of the singular values, for each k=1,2,…,n-1.

4. Main Results:

In the following, we study the complex dynamics of the family F starting with the following proposition:

Proposition (4.1): Let $f_l \in \boldsymbol{F}$, then the function $f_l(z)$ is non-critically finite and all the singular values of it are bounded.

If we derive the function $f_l(z) = \lambda \dfrac{\sinh^2(z)}{z^3}$ and solve the equation $f_l'(z) = 0$, then one can easily show that $f_l(z)$ has infinite number of critical values, two of them are real and the other are purely imaginary. Moreover, if we assume that $\{iy_n : n \in N\}$ is the sequence of imaginary critical value then $\lambda \left| \dfrac{\sinh^2(iy)}{iy^3} \right| \leq \dfrac{\lambda}{\left| y_n^3 \right|} < \lambda M$, where $M = \max\{1/y_n^3 : n \in N\}$.

Therefore the function $f_\lambda(z)$ is non-critically finite and all the singular values of it are bounded.

Theorem (4.2): Let $f= f_l \in \boldsymbol{F}$, then $J(f)=\overline{I(f)}$.

Proof: By proposition (4.1), $f_l \in B$, where B is the class of meromorphic functions having bounded singular values. Then, by theorem (3.1), $J(f)=\overline{I(f)}$.

Theorem (4.3): let $f_l \in \boldsymbol{F}$. Then

1. For $0 < l < l_1$, *the* int*ervals* $\{(-r_l,-a)\cup(a,r_l)\}/T_0$ contained in $F(f_l)$, and the intervals $(0,a)\cup(r_l,\infty)$ are contained in $J(f_l)$.

2. For $l = l_1$, $\{(-m,-x_1)\cup(x_1,m)\}/T_0$ contained in the Fatou set of f_l, $F(f_l)$, and $\{(0,m)\cup(x_0,\infty)\}$ are contained in the Julia set of f_l ,$J(f_l)$.

3. For $l > l_1$, $J(f_l)$ contains $R^+ \cup iR^+$.

Proof:

1.For $0 < l < l_1$,by theorem (3.2), for $x\in(a,r_l)/T_0$, $f_l^n(x)\to a_l$ and $x\in(-r_l,-a)/T_0$, $f_l^n(x)\to -a_l$. Thus for $x\in(a,r_l)/T_0$, x is belong to the basin of attraction of the attracting fixed point a_l, $B(a_l)$.But $B(a_l)\subseteq F(f_l)$. Thus $(r_l,a)/T_0 \subseteq F(f_l)$. Similarly $(-a,-r_l)/T_0 \subseteq B(-a_l)$, where $B(-a_l)$ is the basin of attraction of $-a_l$. Hence $(-a,-r_l)/T_0 \subseteq F(f_l)$. Therefore for $0 < l < l_1$,$\{(-r_l,-a)\cup(a,r_l)\}/T_0 \subseteq F(f_l)$. Also, by theorem (3.2) for $x\in\{(0,l)\cup(r_l,\infty)\}/T_0$, $f_l^n(x)\to\infty$. Hence, by theorem (3.3), $\{(0,l)\cup(r_l,\infty)\}/T_0 \subseteq J(f_l)$. But the pole z=0 and its pre images are contained in $J(f_l)$. Thus $\{(0,l)\cup(r_l,\infty)\}$is contained in $J(f_l)$

2. For this case the proof is similar to that in part (1)

3. For $l > l_1$, by theorem (3.2), $f_l^n(x)\to\infty$ *for* $x\in(0,\infty)/T_0$. Thus $(0,\infty)/T_0$ is contained in $J(f_l)$. Since the pole z=0 and its pre images are contained in $J(f_l)$, then R^+ is contained in $J(f_l)$. But f_l maps iR^+ into R^+. Thus $J(f_l)$ contains the positive parts of the real and imaginary axes $R^+ \cup iR^+$. .

Theorem (4.4): For $f_l \in F$, we have the following:

1. For $0 < l < l_1$, the Fatou set, F(f_l), contains only two basins of attraction B(a_l) and B(-a_l). Also, F(f_l) has no any parabolic domains.

2. For $l = l_1$, F(f_l) has no parabolic domains other than that associated with the indifferent fixed points x_1 and $-x_1$.

3. For $l > l_1$, F(f_l) has no basins of attraction, parabolic domains, Siegel disks or Herman rings.

Proof:

1. For $0 < l < l_1$. Let $z \in B(a_l)$, by theorem (3.2), $f_l^n(z) \to a_l$. Hence the sequence $\{f_l^n(z)\}$ forms a normal family. Thus $z \in$ F(f_l). Therefore $B(a_l) \subseteq$ F(f_l). Also, for any $y \in B(-a_l)$, then $f_l^n(y) \to -a_l$. Hence $\{f_l^n(y)\}$ forms a normal family. Thus $y \in$ F(f_l).Therefore $B(-a_l) \subseteq$ F(f_l). Again by theorem (3.2) part (a), the forward orbits of all singular values are either tend to a_l or -a_l or infinity. Hence by theorem (3.3), F(f_l) has no other basins. Also, by theorem (3.3) does not contains any parabolic domains.

2. For $l = l_1$, by theorem (3.2), $f_l^n(x) \to x_1 \; for x \in (m, x_1)/T_0$ and $f_l^n(x) \to \infty for x \in (x_1, \infty)/T_0$.

Define $U_1 = \{z \in C : f_l^n(z) \to x_1 \text{ as } n \to \infty\}$. Then the indifferent fixed points x_1 belongs to the boundary of U_1. Hence U_1 is a parabolic domains in F(f_l).

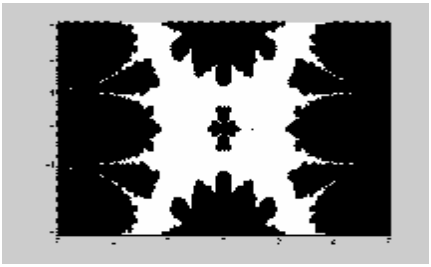
Similarly, $U_2 = \{z \in C : f_l^n(z) \to -x_1 \text{ as } n \to \infty\}$ is a parabolic domain of the indifferent fixed point $-x_1$. Again by theorem (3.2), the forward orbits of all singular values of f_l tends to x_1, $-x_1$, or infinity. Thus, using theorem (3.3), F(f_l) .has no basins of attraction and has no parabolic domains other than U_1 and U_2.

3. For $l > l_1$, by theorem (3.2), $f_l^n(x) \to \infty for x \in (0, \infty)/T_0$. Therefore the forward orbits of the critical values of f_l are tend to infinity. Further, since the asymptotic value 0 is also a pole. Thus the orbit of 0 is tend to infinity.

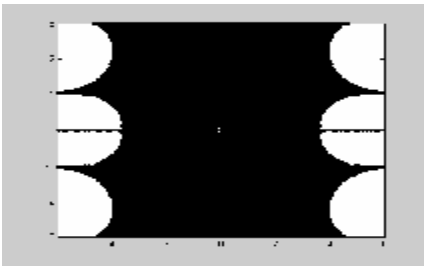
Hence, F(f_l) has no any basins, parabolic domains, Siegel disks or Herman rings.

Finally we give computer images for Julia sets of the functions in the family F for $0 < l < l_1$, $l = l_1$ and $l > l_1$ where $l_1 = 1.2203$ is the critical parameter value of the family F ,see[5].

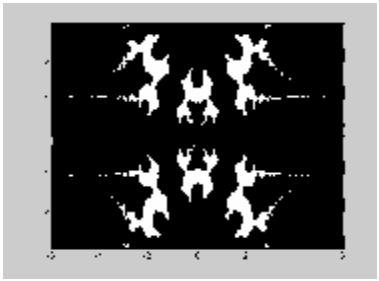
We used simple algorithm with Matlab 6.5 Program. A window has been selected in the plane and divided into k x k grids with width d, we choose a large numbers N and M. For each grid point, compute the orbit up to maximum iteration N. If, at i< N, the modulus of the orbit is grater than M, then the grid point colored black and the iteration stopped. If not then original grid point left white. The output generated by this algorithm black and white pictures. The black points represent the approximated Julia set.



Julia set of the function $l\frac{\sinh^2(z)}{z^3}$ *for* $l < l_1$



Julia set of the function $l\frac{\sinh^2(z)}{z^3}$ *for* $l = l_1$



Julia set of the function $l\frac{\sinh^2(z)}{z^3}$ *for* $l > l_1$

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