Using Zero-Truncated Poisson-Exponential Distribution in Case Increasing Failure Rate Function

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Abstract

In this paper, a new distribution based on zero-truncated Poisson and Exponential distributions is introduced, called zero-truncated Poisson-exponential distributions, which is suitable for models with increasing failure rate function. This distribution arises on latent complementary risk problem base, in which only the maximum lifetime value is visible among all risks, but no information is available about which factor is responsible for component failure. In addition to introducing the model, we determine important features such as distribution function, reliability, survival and failure rate functions. In addition, after obtaining the rth ordinary moments, we achieve a formula for the mean and variance of the proposed distribution. Then we make inference about the model parameters and use the EM algorithm to calculate the maximum likelihood estimator. Finally, we describe the results of the proposed method using real data.

Keywords: Exponential distributions, Poisson distributions, maximum likelihood estimator, EM algorithm, failure rate function.

1. **Introduction:**

The lifetime of a system or organism is an important issue in reliability and its study is of major importance in other sciences, especially in biological sciences and engineering[4]. An important part of such studies is devoted to mathematical description of lifetime by failure time distribution[12]. In the field of statistics, survival analysis is a topic that deals with the time it takes for biological organisms and mechanical systems to die or break down[7]. This issue is called reliability analysis in engineering theory and settlement analysis in economics and sociology[10]. In general, survival analysis involves modeling for the time of death and failure, which are considered events[1]. In survival analysis, an event takes place when a living thing dies or when the system breaks down[20,8].

The study of the lifetime of systems, organisms, instruments, devices, structures, and materials is of utmost significance in the fields of engineering and the biological sciences. Reliability is directly related to the lifetime of a system, making it a crucial aspect to consider[15,3]. The mathematical description of the lifetime by failure time distribution is a key aspect of these investigations[21,17]. Depending on the type of failure, a distribution may be determined by physical considerations[19]. However, in many cases, the decision is made based on how the observed failure times match the distribution[10]. When conducting reliability studies, it is helpful to use the shape and monotonicity of the failure rate function as a tool to reduce down the possibilities available to choose among because they are an accurate reflection of some of the system's characterizes[5]. Modeling lifetime data frequently involves the utilization of failure rate functions. If these functions are monotone, they can express the desired properties of the lifetime distribution[6].

Exponential distributions are frequently concerned with how long it will take for an event to happen. For instance, an exponential distribution can be applied to the period of time (beginning right now) until an earthquake occurs. Other examples include the amount of time, measured in months, that a car battery will last and the length of time, measured in minutes, that a long-distance business conversation will endure[1,5,19]. You can also show that the value of the cash in your pocket or purse roughly follows an exponential distribution[2,4].

The zero-truncated Poisson distribution, often known as the ZTP distribution, is a specific discrete probability distribution that is used in probability theory[13]. Its support is the set of positive integers, in addition to these names, this distribution is sometimes referred to as the conditional Poisson distribution or the positive Poisson distribution[7]. Given that the value of the random variable is not zero, this is the conditional probability distribution of a Poisson-distributed random variable[2].

2. Zero-truncated Poisson-exponential distribution:

We introduce the Poisson-exponential distribution. We will explain the density function of this distribution. Then we try to obtain the properties of this distribution, such as the cumulative distribution function, moments, survival function, and failure rate function[3].

2.1 Density Function:

Let $Z_1, Z_2, \cdots Z_W$ be a random sample from exponential distribution with the following density function

$$f_Z(z;\tau) = \tau e^{-\tau z} \quad z,\tau \in \mathbb{R}^+ \qquad (1)$$

and W follows a zero-truncated Poisson distribution with the following density function

$$p(W=w) = \frac{e^{-\theta}\theta^w}{\Gamma(w+1)(1-e^{-\theta})} \qquad w \in \mathbb{N} \quad and \quad \theta \in \mathbb{R}^+ \qquad (2)$$

Where Z's and W are independent. Let $V = max(Z_1, Z_2, \cdots Z_W)$, then the cumulative density function of V given $W = w$ is

$$\begin{aligned} F(V|W=w,) &= p(V \le v\,|W=w) \\ &= p(max(Z_1, Z_2, \cdots Z_W) \le v\,|W=w) \\ &= p(max(Z_1, Z_2, \cdots Z_w) \le v) \\ &= p(Z_1 \le v, Z_2 \le v, \cdots, Z_w \le v\,) \\ &= p(Z_1 \le v)p(Z_2 \le v)\cdots p(Z_w \le v) \\ &= \prod_{i=1}^{w} p(Z_i \le v) \\ &= [\,F_Z(w;\tau)]^w \\ &= [1 - e^{-\tau v}]^w \qquad (3) \end{aligned}$$

Therefore, the density function of V given $W = w$ will be:

$$f(v|w;\tau) = \frac{d}{dv}F(V|W=w,\theta) = w\tau e^{-\tau v}[1 - e^{-\tau v}]^{w-1}$$

and the joint density function of V and W is given by:

$$\begin{aligned} f(v,w;\tau,\theta) &= f(v|w;\tau)p(W=w) \\ &= w\tau e^{-\tau v}[1 - e^{-\tau v}]^{w-1}\frac{e^{-\theta}\theta^w}{\Gamma(w+1)(1-e^{-\theta})} \qquad (4) \end{aligned}$$

Now, we can derive the marginal density function of V as

$$\begin{aligned}
f(v;\tau,\theta) &= \sum_{w=1}^{\infty} f(v,w;\tau,\theta) \\
&= \sum_{w=1}^{\infty} w\tau e^{-\tau v}[1- e^{-\tau v}]^{w-1} \frac{e^{-\theta}\theta^{w}}{\Gamma(w+1)(1-e^{-\theta})} \\
&= \frac{\tau e^{-\tau v}e^{-\theta}}{(1-e^{-\theta})} \sum_{w=1}^{\infty} w[1- e^{-\tau v}]^{w-1} \frac{\theta^{w}}{\Gamma(w+1)} \\
&= \frac{\tau\theta e^{-\tau v}e^{-\theta}}{(1-e^{-\theta})} \sum_{w=1}^{\infty} [1- e^{-\tau v}]^{w-1} \frac{\theta^{w-1}}{\Gamma(w)} \\
&= \frac{\tau\, e^{-\theta}}{(1-e^{-\theta})}\, \theta e^{-\tau v} \sum_{w=1}^{\infty} \frac{(\theta[1- e^{-\tau v}])^{w-1}}{\Gamma(w)} \\
&= \frac{\tau\, e^{-\theta}}{(1-e^{-\theta})}\, \theta e^{-\tau v}\left(e^{\theta[1- e^{-\tau v}]}\right) \\
&= \frac{\tau\theta}{(1-e^{-\theta})}\, e^{-\tau v-\theta e^{-\tau v}} \qquad\qquad v,\theta,\tau \\
&\in \mathbb{R}^{+}
\end{aligned}$$

This density function is known as Poisson-Exponential (PE) distribution.

The density function of Poisson-Exponential (PE) distribution has two parameters τ and θ. The parameter τ controls the scale of distribution and the parameter θ controls the shape of distribution. Figure 2.1 presents the density function of Poisson-Exponential distribution with different values for parameters. It is observed that when $0 < \theta < 1$, the density function is a strictly non-increasing function and it takes maximum value at $V = 0$ which means that the mode of this distribution happens at $V = 0$ and equals to

$$mode(V) = \frac{\tau\theta}{1-e^{-\theta}}$$

But when $\theta \geq 1$, the distribution is unimodal and we can obtained as:

$$\frac{d}{dv} f(v;\tau,\theta) = \frac{\tau\theta}{(1-e^{-\theta})}\left[-\tau e^{-\tau v-\theta e^{-\tau v}} + \theta\tau e^{-2\tau v-\theta e^{-\tau v}}\right] = 0$$

Then we get:

$$v = \frac{\log\theta}{\tau} \qquad (5)$$

And the modal value will be:

$$modal(V) = \frac{\tau e^{-1}}{1 - e^{-\theta}}$$

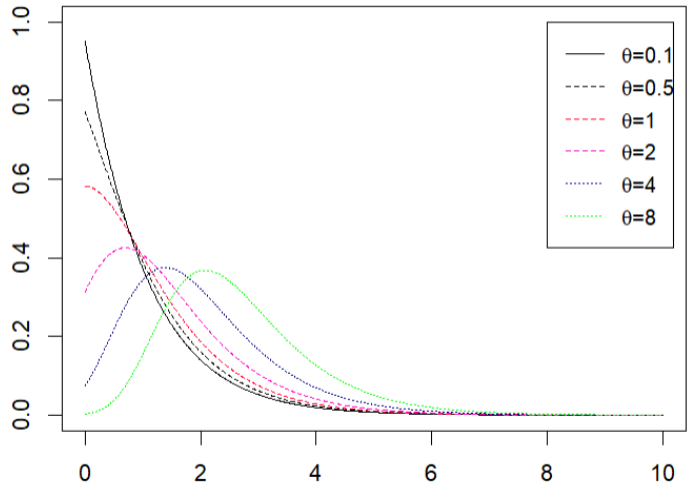

Figure 1: Density function of PE distribution for different values of θ when $\tau = 1$.

On the other hand, if we look at the probability density function, we can conclude that when θ approaches zero, by using the hospital rule , we have.

$$\lim_{\theta \to 0} f(v; \tau, \theta) = \tau e^{-\tau v} \frac{\lim_{\theta \to 0} \theta\, e^{-\theta e^{-\tau v}}}{\lim_{\theta \to 0} (1 - e^{-\theta})}$$

$$= \tau e^{-\tau v} \frac{\lim_{\theta \to 0} (e^{-\theta e^{-\tau v}} - \theta^2 e^{-\theta e^{-\tau v}})}{\lim_{\theta \to 0} e^{-\theta}}$$

$$= \tau e^{-\tau v}$$

So, the PE probability density function approaches the exponential probability density function with parameter. τ

In the following sections, we obtain some properties of the Poisson-Exponential distribution.

2.2 Cumulative Distribution Function:

The cumulative distribution function of a PE distribution is given by:

$$F(v; \tau, \theta) = \int_0^v f(t; \tau, \theta) dt$$

$$= \int_0^v \frac{\tau\theta}{(1-e^{-\theta})}\, e^{-\tau t-\theta e^{-\tau t}} dt$$

$$= \frac{1}{(1-e^{-\theta})}\int_0^v \tau\theta\, e^{-\tau t-\theta e^{-\tau t}} dt$$

$$= \frac{1}{(1-e^{-\theta})} e^{-\theta e^{-\tau t}}|_0^v$$

$$= \frac{1}{(1-e^{-\theta})} e^{\theta e^{-\tau v}} - \frac{e^{-\theta}}{(1-e^{-\theta})}$$

$$= \frac{e^{-\theta e^{-\tau v}}-e^{-\theta}}{1-e^{-\theta}} \qquad v,\theta,\tau \in \mathbb{R}^+ \qquad (6)$$

Let m be the median of PE distribution, then:

$$F(m;\tau,\theta) = \frac{e^{-\theta e^{-\tau m}}-e^{-\theta}}{1-e^{-\theta}}$$

On the other hand, we know $F(m) = \frac{1}{2}$. Then,

$$\frac{e^{-\theta e^{-\tau m}}-e^{-\theta}}{1-e^{-\theta}} = \frac{1}{2}$$

$$e^{-\theta e^{-\tau m}}-e^{-\theta} = \frac{1-e^{-\theta}}{2}$$

$$e^{-\theta e^{-\tau m}} = e^{-\theta} + \frac{1-e^{-\theta}}{2}$$

$$e^{-\theta e^{-\tau m}} = \frac{1+e^{-\theta}}{2}$$

$$-\theta e^{-\tau m} = \log\left(\frac{1+e^{-\theta}}{2}\right)$$

$$e^{-\tau m} = -\theta^{-1}\log\left(\frac{1+e^{-\theta}}{2}\right)$$

$$-\tau m = \log\left(-\theta^{-1}\, log\left(\frac{1+e^{-\theta}}{2}\right)\right)$$

$$-m = \frac{1}{\tau}\log\left(-\theta^{-1}\, log\left(\frac{1+e^{-\theta}}{2}\right)\right)$$

Therefore, the median will be:

$$m = -\frac{1}{\tau}\log\left(-\theta^{-1}\, log\left(\frac{1+e^{-\theta}}{2}\right)\right) \qquad (7)$$

2.3 Survival and Failure Rate Function:

The survival function and failure rate function of PE distribution with parameters τ and θ, will be obtained as:

$$S(V;\tau,\theta) = 1 - F(v;\tau,\theta)$$
$$= 1 - \frac{e^{-\theta e^{-\tau v}} - e^{-\theta}}{1 - e^{-\theta}}$$
$$= \frac{1 - e^{-\theta} - e^{-\theta e^{-\tau v}} + e^{-\theta}}{1 - e^{-\theta}}$$
$$= \frac{1 - e^{-\theta e^{-\tau v}}}{1 - e^{-\theta}}$$

Figure 2 displays some survival function shapes of PE distribution for some values of the parameter θ when the parameter τ considered as fixed value $(\tau = 1)$.

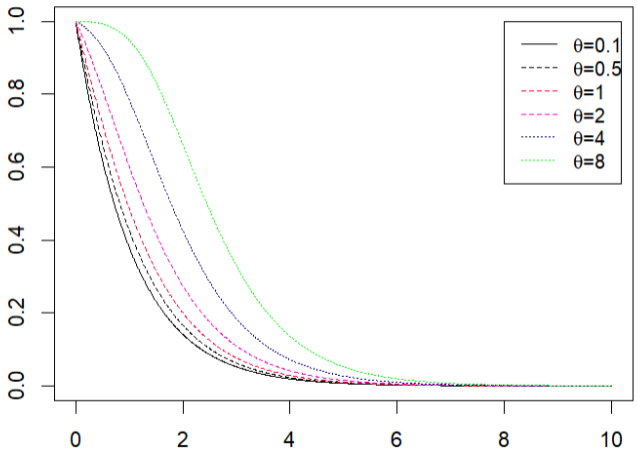

Figure 2: Survival function of PE distribution for different values of θ when τ=1.

The failure function will be obtained as follows:

$$h(V;\tau,\theta) = \frac{f(V;\tau,\theta)}{S(V;\tau,\theta)} = \frac{\tau\theta e^{-\tau v - \theta e^{-\tau v}}}{1 - e^{-\theta e^{-\tau v}}}$$

Figure 3 displays the failure rate function shapes of the PE distribution for some values of the parameter θ when the parameter τ considered as fixed value $(\tau = 1)$. Figure shows that each shape starts from an initial value, which is minimum value of the failure rate function, and increases with time and finally stabilizes at a value. So, it seems that both the initial and final value are finite.

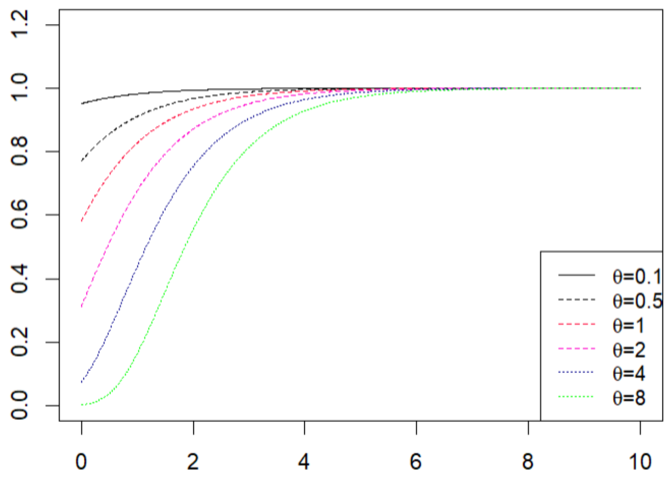

Figure 3: Failure rate function of PE distribution for different values of θ when τ=1.

Now, we find the initial and final values of the failure rate function shapes of the PE distribution. The initial value can be obtained when $V = 0$, so:

$$h(0;\tau,\theta) = \frac{\tau\theta e^{-\tau 0-\theta e^{-\tau 0}}}{1-e^{-\theta e^{-\tau 0}}} = \frac{\tau\theta e^{-\theta}}{1-e^{-\theta}} = \frac{\tau\theta}{e^{\theta}-1}$$

And to get the final value, we need to set $V = \infty$, then:

$$h(\infty;\tau,\theta) = \lim_{v\to\infty} \frac{\tau\theta e^{-\tau v-\theta e^{-\tau v}}}{1-e^{-\theta e^{-\tau v}}} = \lim_{v\to\infty} \frac{\tau\theta e^{-\tau v}}{e^{\theta e^{-\tau v}}-1} = \frac{0}{0}$$

By using hospital rule, we have:

$$h(\infty;\tau,\theta) = \lim_{v\to\infty} \frac{\tau^2\theta e^{-\tau v}}{\tau\theta e^{-\tau v+\theta e^{-\tau v}}} = \lim_{v\to\infty} \tau e^{-\theta e^{-\tau v}} = \tau$$

2.4 Generating random number:

Since we have a closed form for the cumulative distribution function of the PE distribution, we can use the probability integral transformation to generate a random number. Consider the uniform distribution on interval (0,1).To generate a random number, we need the following equation:

$$F(V;\tau,\theta) = U$$

Then, we solve above equation subjected to v:

$$\frac{e^{-\theta e^{-\tau v}} - e^{-\theta}}{1-e^{-\theta}} = u$$

$$e^{-\theta e^{-\tau v}} - e^{-\theta} = u(1-e^{-\theta})$$

$$e^{-\theta e^{-\tau v}} = u(1-e^{-\theta}) + e^{-\theta}$$
$$-\theta e^{-\tau v} = \log\left(u\left(1-e^{-\theta}\right) + e^{-\theta}\right)$$
$$e^{-\tau v} = -\frac{1}{\theta}\log\left(u\left(1-e^{-\theta}\right) + e^{-\theta}\right)$$
$$-\tau v = \log\left(-\frac{1}{\theta}\log\left(u\left(1-e^{-\theta}\right) + e^{-\theta}\right)\right)$$

And finally

$$v = -\frac{1}{\tau}\log\left(-\frac{1}{\theta}\log\left(u\left(1-e^{-\theta}\right) + e^{-\theta}\right)\right) \tag{8}$$

2.5 Moment generating function:

The moment generating function of the Poisson-Exponential distribution can be derived as:

$$M_V(t) = E(e^{tv}) = \int_0^\infty e^{tv} f_V(v)dv$$
$$= \int_0^\infty e^{tv} \frac{\tau\theta}{(1-e^{-\theta})} e^{-\tau v - \theta e^{-\tau v}} dv$$
$$= \frac{\tau\theta}{(1-e^{-\theta})}\int_0^\infty e^{tv}\, e^{-\tau v} e^{-\theta e^{-\tau v}} dv$$
$$= \frac{\tau\theta}{(1-e^{-\theta})}\int_0^\infty e^{(t-\tau)v} \sum_{i=0}^{\infty}\frac{(-\theta e^{-\tau v})^i}{i!}\, dv$$
$$= \frac{\tau\theta}{(1-e^{-\theta})}\sum_{i=0}^{\infty}\frac{(-\theta)^i}{i!}\int_0^\infty e^{(t-\tau v)-\tau v i} dv$$
$$= \frac{\tau\theta}{(1-e^{-\theta})}\sum_{i=0}^{\infty}\frac{(-\theta)^i}{i!}\int_0^\infty e^{v(t-\tau-\tau i)} dv$$
$$= \frac{\tau\theta}{(1-e^{-\theta})}\sum_{i=0}^{\infty}\frac{(-\theta)^i}{i!}\frac{1}{t-\tau-\tau i} e^{v(t-\tau-\tau i)}\Big|_0^\infty$$
$$= \frac{\tau\theta}{(1-e^{-\theta})}\sum_{i=0}^{\infty}\frac{(-\theta)^i}{i!}\frac{-1}{t-\tau-\tau i}$$
$$= \frac{\theta}{(1-e^{-\theta})}\sum_{i=0}^{\infty}\frac{(-\theta)^i}{i!}\frac{1}{1+i-\frac{t}{\tau}} \tag{9}$$

The generalized hypergeometric function is defined as:

$$F_{p,q}(\boldsymbol{u},\boldsymbol{n},k) = \sum_{j=0}^{\infty}\frac{k^j}{\mathrm{j!}}\frac{\prod_{i=1}^{p}\Gamma(u_i+j)\Gamma^{-1}(u_i)}{\prod_{i=1}^{p}\Gamma(n_i+j)\Gamma^{-1}(n_i)} \tag{10}$$

Where $\boldsymbol{u} = [u_1, u_2, \cdots, u_p]$ and $\boldsymbol{n} = [n_1, n_2, \cdots, n_q]$. By using (10) in (9), the moment generating function can be rewritten as:

$$M_V(t) = \frac{\theta}{\left(1-\frac{t}{\tau}\right)(1-e^{-\theta})}\ F_{1,1}(1-\frac{t}{\tau}, 2-\frac{t}{\tau}, -\theta)$$

2.6 Moments:

The rth moment of a Poisson-Exponential distribution can be derived from equation (9) or can be calculated directly by its definition. The rth moment of a Poisson-Exponential distribution will be:

$$E(V^r;\tau,\theta) = \int v^r f(v;\tau,\theta)dv$$

$$= \int_0^\infty v^r \frac{\tau\theta}{(1-e^{-\theta})}\ e^{-\tau v-\theta e^{-\tau v}}dv$$

$$= \int_0^\infty v^r \frac{\tau\theta}{(1-e^{-\theta})}\ e^{-\tau v-\theta e^{-\tau v}}dv$$

$$= \frac{\tau\theta}{(1-e^{-\theta})}\int_0^\infty v^r e^{-\tau v}\ e^{-\theta e^{-\tau v}}dv$$

$$= \frac{\tau\theta}{(1-e^{-\theta})}\int_0^\infty v^r e^{-\tau v}\ \sum_{j=0}^{\infty}\frac{(-\theta e^{-\tau v})^j}{j!}dv$$

$$= \frac{\tau\theta}{(1-e^{-\theta})}\sum_{j=0}^{\infty}\frac{(-\theta)^j}{j!}\int_0^\infty v^r e^{-\tau v(1+j)}dv$$

$$= \frac{\tau\theta}{(1-e^{-\theta})}\sum_{j=0}^{\infty}\frac{(-\theta)^j}{j!}\int_0^\infty v^r e^{-\tau v(1+j)}dv$$

The equation under the integration is the kernel of the gamma distribution with the parameters $r+1$ and $\tau(1+j)$, therefore:

$$E(V^r;\tau,\theta) = \frac{\tau\theta}{(1-e^{-\theta})}\sum_{j=0}^{\infty}\frac{(-\theta)^j}{j!}\ \frac{\Gamma(r+1)}{[\tau(1+j)]^{r+1}}$$

$$= \frac{\theta\Gamma(r+1)}{\tau^r(1-e^{-\theta})}\sum_{j=0}^{\infty}\frac{(-\theta)^j}{j!}\ \frac{1}{[(1+j)]^{r+1}} \qquad (11)$$

We use (2-3) in (2-4), we obtain the following formula for the rth:moment

$$E(V^r;\tau,\theta) = \frac{\theta\Gamma(r+1)}{\tau^r(1-e^{-\theta})}\sum_{j=0}^{\infty}\frac{(-\theta)^j}{j!}\,\frac{1}{[(1+j)]^{r+1}} \qquad (12)$$

Thus, the mean and variance of Poisson-Exponential distribution will be:

$$E(V;\tau,\theta) = \frac{\theta}{\tau(1-e^{-\theta})}\,F_{2,2}([1,1],[2,2],-\theta)$$

And:

$$\begin{aligned} var(V;\tau,\theta) &= E(V^2;\tau,\theta) - E^2(V;\tau,\theta) \\ &= \frac{\theta\Gamma(3)}{\tau^2(1-e^{-\theta})}\,F_{3,3}([1,1,1],[2,2,2],-\theta) \\ &\quad - \left[\frac{\theta}{\tau(1-e^{-\theta})}\,F_{2,2}([1,1],[2,2],\theta)\right]^2 \\ &= \frac{\theta}{\tau^2(1-e^{-\theta})}\left\{2F_{3,3}([1,1,1],[2,2,2],\theta) - \frac{\theta}{(e^{\theta}-1)}\,F_{2,2}^2([1,1],[2,2],\theta)\right\} \end{aligned}$$

3. Estimation methods for parameter:

3.1 Maximum Likelihood Estimator:

Let $\boldsymbol{v_{obs}} = (v_1, v_2, \cdots, v_n)$ be an observed sample of size n from a Poisson-Exponential distribution. The likelihood function based on the observations is given by[2]:

$$\mathcal{L}(\boldsymbol{v_{obs}},\tau,\theta) = \prod_{i=1}^{n} f(v_i;\ \tau,\theta) = \prod_{i=1}^{n}\frac{\tau\theta}{(1-e^{-\theta})}\,e^{-\tau v_i-\theta e^{-\tau v_i}}$$

$$= \frac{(\tau\theta)^n}{(1-e^{-\theta})^n}\,e^{-\tau\sum_{i=1}^{n}v_i-\theta\sum_{i=1}^{n}e^{-\tau v_i}} \qquad (13)$$

Now, we obtain the log- likelihood function

$$\begin{aligned} \ell(\boldsymbol{v_{obs}},\tau,\theta) &= \log\big(\mathcal{L}(v,\tau,\theta)\big) \\ &= n\log(\tau) + n\log(\theta) - n\log\big(1-e^{-\theta}\big) - \tau\sum_{i=1}^{n}v_i \\ &\quad - \theta\sum_{i=1}^{n}e^{-\tau v_i} \qquad (14) \end{aligned}$$

It is possible to directly obtain the maximum likelihood estimator of τ and θ by maximizing the log-likelihood function (3-1). Therefore, we get the derivative of (3-1) subject to τ and θ:

$$\frac{\partial}{\partial\tau}\ell(\boldsymbol{v_{obs}},\tau,\theta)=\frac{n}{\tau}-\sum_{i=1}^{n}v_i+\theta\sum_{i=1}^{n}v_ie^{-\tau v_i}=0 \qquad (15)$$

$$\frac{\partial}{\partial\theta}\ell(\boldsymbol{v_{obs}},\tau,\theta)=\frac{n}{\theta}-\frac{ne^{-\theta}}{1-e^{-\theta}}-\sum_{i=1}^{n}e^{-\tau v_i}=0 \qquad (16)$$

To get the estimator of θ, we can solve equation (15) which means:

$$\theta\sum_{i=1}^{n}v_ie^{-\tau v_i}=\sum_{i=1}^{n}v_i-\frac{n}{\tau}$$

$$\theta=\frac{\sum_{i=1}^{n}v_i-\frac{n}{\tau}}{\sum_{i=1}^{n}v_ie^{-\tau v_i}}$$

Consequently, conditional upon the value of $\hat{\tau}$ where $\hat{\tau}$ is the maximum likelihood estimator of τ, the maximum likelihood estimator of θ is given by:

$$\hat{\theta}=\frac{\sum_{i=1}^{n}v_i-\frac{n}{\hat{\tau}}}{\sum_{i=1}^{n}v_ie^{-\hat{\tau}v_i}} \qquad (17)$$

In the following theorems 13 and 14, we present the conditions that we need in order to achieve the existence and uniqueness of the maximum likelihood estimator provided in (16), whether the other parameter is unknown or is given.

3.2 EM Algorithm:

Since we cannot get the maximum likelihood estimation of PE distribution parameters, we use the Expectation–maximization algorithm (EM algorithm) to derive the estimations[21].

Consider $v=(v_1,v_2,\cdots,v_n)^T$ as observations and $m=(m_1,m_2,\cdots,m_n)^T$ as missing data. To start the algorithm, we need the hypothetical distribution of complete data. So;

$$f(v_i,m_i|\tau,\theta)=m_i\tau e^{-\tau v_i}[1-e^{-\tau v_i}]^{m_i-1}\frac{e^{-\theta}\theta^{m_i}}{\Gamma(m_i+1)(1-e^{-\theta})} \qquad m_i:1,2,\cdots$$

the likelihood function of $(\boldsymbol{v},\boldsymbol{m})$ is given by:

$$L(\theta,\tau|\boldsymbol{v},\boldsymbol{m}) = \prod_{i=1}^{n} f(v_i,m_i|\tau,\theta)$$

$$= \prod_{i=1}^{n} m_i\tau e^{-\tau v_i}[1-\ e^{-\tau v_i}]^{m_i-1}\frac{e^{-\theta}\theta^{m_i}}{\Gamma(m_i+1)(1-e^{-\theta})}$$

$$= \left(\prod_{i=1}^{n} m_i\right)\left(\frac{\tau}{e^{\theta}-1}\right)^{n}\left(\prod_{i=1}^{n}\frac{[1-\ e^{-\tau v_i}]^{m_i-1}}{\Gamma(m_i+1)}\right)e^{-\tau\sum_{i=1}^{n} v_i}\ \theta^{\sum_{i=1}^{n} m_i}$$

And the log-likelihood function is given by:

$$\ell(\theta,\tau|\boldsymbol{v},\boldsymbol{m}) = \sum_{i=1}^{n}\log(m_i) + n\log(\tau) - nlog\left(e^{\theta}-1\right) - \tau\sum_{i=1}^{n} v_i + \sum_{i=1}^{n}(m_i-1)\log(1-\ e^{-\tau v_i}) + \sum_{i=1}^{n} m_i\log(\theta) \qquad (18)$$

For the EM algorithm, we need to find $\widehat{m}_i = E(M_i|\hat{\tau},\hat{\theta},v_i)$. Therefore, the conditional distribution of M_i given v_i and (τ,θ) must be derived:

$$f(m_i|\tau,\theta,v_i) = \frac{f(m_i,v_i;\ \tau,\theta)}{f(\ v_i;\ \tau,\theta)}$$

$$f(m_i|\tau,\theta,v_i) = \frac{m_i\tau e^{-\tau v_i}[1-\ e^{-\tau v_i}]^{m_i-1}\frac{e^{-\theta}\theta^{m_i}}{\Gamma(m_i+1)(1-e^{-\theta})}}{\frac{\tau\theta}{(1-e^{-\theta})}\ e^{-\tau v_i-\theta e^{-\tau v_i}}}$$

$$= \frac{m_i\tau e^{-\tau v_i}[1-\ e^{-\tau v_i}]^{m_i-1}\frac{e^{-\theta}\theta^{m_i}}{\Gamma(m_i+1)(1-e^{-\theta})}}{\frac{\tau\theta}{(1-e^{-\theta})}\ e^{-\tau v_i-\theta e^{-\tau v_i}}}$$

$$= \frac{[\theta(1-\ e^{-\tau v_i})]^{m_i-1}}{\Gamma(m_i)}e^{-\theta+\theta e^{-\tau v_i}} \qquad m_i: 1,2,\cdots$$

Thus:

$$E(M_i|\tau,\theta,v_i) = \sum_{m_i=1}^{\infty} m_i\, f(m_i|\tau,\theta,v_i)$$

$$= \sum_{m_i=1}^{\infty} m_i \frac{[\theta(1- e^{-\tau v_i})]^{m_i-1}}{\Gamma(m_i)} e^{-\theta+\theta e^{-\tau v_i}}$$

$$= e^{-\theta+\theta e^{-\tau v_i}} \sum_{m_i=1}^{\infty} m_i \frac{[\theta(1- e^{-\tau v_i})]^{m_i-1}}{\Gamma(m_i)}$$

By substitution $t = m_i - 1$, we have:

$$E(M_i|\tau,\theta,v_i) = e^{-\theta+\theta e^{-\tau v_i}} \sum_{t=0}^{\infty} (t+1) \frac{[\theta(1- e^{-\tau v_i})]^{t}}{\Gamma(t+1)}$$

$$= e^{-\theta+\theta e^{-\tau v_i}} \left[\sum_{t=0}^{\infty} (t+1) \frac{[\theta(1- e^{-\tau v_i})]^{t}}{\Gamma(t+1)} \times \frac{e^{-\theta(1- e^{-\tau v_i})}}{e^{-\theta(1- e^{-\tau v_i})}}\right]$$

$$= \frac{e^{-\theta+\theta e^{-\tau v_i}}}{e^{-\theta(1- e^{-\tau v_i})}} \left[\sum_{t=0}^{\infty} t \frac{[\theta(1- e^{-\tau v_i})]^{t}}{\Gamma(t+1)} e^{-\theta(1- e^{-\tau v_i})} + \sum_{t=0}^{\infty} \frac{[\theta(1- e^{-\tau v_i})]^{t}}{\Gamma(t+1)} e^{-\theta(1- e^{-\tau v_i})}\right]$$

Inside the both summation operation, we have Poisson distribution with parameter $\theta(1- e^{-\tau v_i})$, therefore:

$$E(M_i|\tau,\theta,v_i) = \frac{e^{-\theta+\theta e^{-\tau v_i}}}{e^{-\theta(1- e^{-\tau v_i})}} [\theta(1- e^{-\tau v_i}) + 1]$$

And finally, we obtain:

$$E(M_i|\tau,\theta,v_i) = \theta(1- e^{-\tau v_i}) + 1.$$

Now, if we consider:

$$\widehat{m}_i = E(M_i|\tau,\theta,v_i) = \theta(1- e^{-\tau v_i}) + 1 \qquad (19)$$

We can directly implement the M-step of EM algorithm which consists in maximizing the log-likelihood of complete data subject to (τ,θ) when the missing data m_i is replaced by $\widehat{m}_i = E(M_i|\tau,\theta,v_i)$. Therefore, if we define:

$$K(\tau,\theta) = \sum_{i=1}^{n} \log(\widehat{m}_i) + n\log(\tau) - nlog\left(e^{\theta} - 1\right) - \tau \sum_{i=1}^{n} v_i + \sum_{i=1}^{n} (\widehat{m}_i - 1)\log(1- e^{-\tau v_i}) + \sum_{i=1}^{n} \widehat{m}_i \log(\theta)$$

Now, need to find the value τ and θ which maximize the function $K(\tau,\theta)$. We derive the first derivation of $K(\tau,\theta)$ subject τ and θ and set them by zero.

$$\frac{\partial}{\partial \tau} K(\tau, \theta) = \frac{n}{\tau} + \sum_{i=1}^{n} \frac{y_i(\hat{m}_i - 1)e^{-\tau v_i}}{1 - e^{-\tau v_i}} - \sum_{i=1}^{n} y_i = 0$$

Then,

$$\frac{n}{\tau} = \sum_{i=1}^{n} y_i - \sum_{i=1}^{n} \frac{y_i(\hat{m}_i - 1)e^{-\tau v_i}}{1 - e^{-\tau v_i}}$$

Consequently:

$$\tau = \frac{n}{\sum_{i=1}^{n} y_i - \sum_{i=1}^{n} \frac{y_i(\hat{m}_i - 1)e^{-\tau v_i}}{1 - e^{-\tau v_i}}} \tag{20}$$

And about θ, we have:

$$\frac{\partial}{\partial \theta} K(\tau, \theta) = \frac{\sum_{i=1}^{n} \hat{m}_i}{\theta} - \frac{ne^{\theta}}{e^{\theta} - 1} = 0$$

Thus:

$$\sum_{i=1}^{n} \hat{m}_i \left(e^{\theta} - 1\right) = \theta n e^{\theta}$$

Finally:

$$\theta = \frac{\sum_{i=1}^{n} \hat{m}_i \left(e^{\theta} - 1\right)}{ne^{\theta}} \tag{21}$$

Then, the E-step and the M-step of the algorithm are given by:

E-step: *Given a current estimation $\hat{\tau}^{(t)}$ and $\hat{\theta}^{(t)}$, compute $\hat{m}_i^{(t)}$.*

M-step*: Update $\hat{\tau}^{(t)}$ and $\hat{\theta}^{(t)}$by the following formulas*

$$\hat{\tau}^{(t+1)} = \frac{n}{\sum_{i=1}^{n} y_i - \sum_{i=1}^{n} \frac{y_i\left(\hat{m}_i^{(t)} - 1\right)e^{-\hat{\tau}^{(t+1)} v_i}}{1 - e^{-\hat{\tau}^{(t+1)} v_i}}}$$

And

$$\hat{\theta}^{(t+1)} = \frac{\sum_{i=1}^{n} \hat{m}_i^{(t)}}{n}\left(1 - e^{-\hat{\theta}^{(t)}}\right)$$

The EM algorithm for both parameters τ and θ is not explicit and a Newton-Raphson algorithm for the M-step of the EM cycle is required.

4. Simulation

In this part, first we conduct a simulation study to discuss about the estimation problem in the Poisson-Exponential distribution and the we explain the efficiency of the proposed distribution in the real-life data set.

4.1 Simulation study:

We use (8) to generate data from Poisson-exponential distribution. The sample size is considered as a parameter to see the effect of the sample size on the estimation process. The sample size values are chosen as $n = 30, 50, 100, 200$. Moreover, we set different values for the distribution parameters. We set $\tau = 2$ and various values for the θ such as $\theta = 0.5, 1, 2, 4, 6$. Since, we have to use EM algorithm which is explained in subsection 3.2 to estimate parameters, we need to check the Theorem 3.1 and Theorem 3.2 to ensure that there are unique estimations for the parameters. Here for the EM algorithm, we didn't consider a value for the maximum number iterations and the algorithm will be only convergence if the norm of estimation be less than 10^{-6}. In addition to, we can calculate the variances and covariance of the parameters by using the observed and expected fisher information and also it can be calculated directly by repeating the generating data. Therefore, we repeat generating data 1000 times.

Table 1 presents the results of estimating variance and covariance of parameters using different methods. Three first columns are the estimated values of the variance $\left(Var(\hat{\tau})\ and\ Var(\hat{\theta})\right)$ and covariance $\left(cov(\hat{\tau}, \hat{\theta})\right)$ of parameters. The rest columns are the mean of the obtained corresponding values from the observed and expected fisher information matrix.

Table 1: The mean of the variances and covariance of the MLE of the parameters $\boldsymbol{\vartheta} = (\boldsymbol{\tau}, \boldsymbol{\theta})$.

n	ϑ	Simulated			Expected Fisher information			Observed Fisher information		
		$Var(\hat{\tau})$	$Var(\hat{\theta})$	$cov(\hat{\tau},\hat{\theta})$	$Var(\hat{\tau})$	$Var(\hat{\theta})$	$cov(\hat{\tau},\hat{\theta})$	$Var(\hat{\tau})$	$Var(\hat{\theta})$	$cov(\hat{\tau},\hat{\theta})$
30	(0.5, 2)	0.514	1.474	0.731	0.548	1.494	0.745	0.523	1.372	0.691
	(1, 2)	0.422	1.347	0.614	0.445	1.445	0.644	0.444	1.405	0.6290
	(2, 2)	0.318	1.215	0.495	0.334	1.490	0.535	0.335	1.472	0.532
	(4, 2)	0.239	1.859	0.459	0.250	2.479	0.525	0.250	2.545	0.530
	(6, 2)	0.215	3.334	0.514	0.224	5.636	0.681	0.225	5.640	0.685
50	(0.5, 2)	0.358	0.934	0.488	0.369	0.942	0.492	0.357	0.898	0.469
	(1, 2)	0.302	0.858	0.417	0.315	0.904	0.435	0.309	0.887	0.425
	(2, 2)	0.240	0.838	0.346	0.247	0.901	0.161	0.245	0.907	0.459
	(4, 2)	0.193	1.164	0.325	0.196	1.340	0.344	0.195	1.345	0.344
	(6, 2)	0.179	2.040	0.374	0.181	1.587	0.412	0.182	2.834	0.428
100	(0.5, 2)	0.240	0.528	0.305	0.243	0.534	0.308	0.240	0.523	0.302
	(1, 2)	0.213	0.490	0.270	0.216	0.504	0.276	0.214	0.500	0.273
	(2, 2)	0.182	0.482	0.235	0.184	0.494	0.239	0.184	0.497	0.238
	(4, 2)	0.158	0.644	0.224	0.159	0.684	0.229	0.159	0.677	0.228
	(6, 2)	0.151	1.086	0.248	0.151	1.204	0.257	0.152	1.182	0.257
200	(0.5, 2)	0.182	0.327	0.214	0.183	0.328	0.214	0.182	0.324	0.214
	(1, 2)	0.168	0.307	0.197	0.169	0.311	0.198	0.169	0.320	0.198
	(2, 2)	0.152	0.302	0.179	0.153	0.305	0.180	0.153	0.305	0.179
	(4, 2)	0.141	0.383	0.173	0.141	0.392	0.174	0.142	0.392	0.174
	(6, 2)	0.137	0.605	0.186	0.137	0.639	0.188	0.137	0.6300.469	0.188

From Table 1, one can realize that for large value of n, the values obtained from the expected and observed information matrices are rather close to the ones that were simulated. In addition to this, we had expected that the approximation becomes closer and closer to being accurate as the value of n increases which was occurred in our simulation study. For large value of sample size, the variances and covariance of the MLEs which calculated from the observed information matrix are very close to those calculated from expected information matrix.

Table 2: The averages of the 1000 MLEs, $av(\hat{\vartheta})$, the average number of iterations to convergence, $av(k)$, their standard errors, $SE(\hat{\vartheta})$ and

$SE(k)$, and the coverage probabilities of the 95% confidence intervals for parameters of the PED, C.P. The initial values are given by $\vartheta^{(0)} = (\theta^{(0)}, \tau^{(0)})$.

n	ϑ	$\vartheta^{(0)}$	$av(\hat{\vartheta})$	$SE(\hat{\vartheta})$	C.P	$av(k)$	$SE(k)$
30	(0.5, 2)	(0.5, 2)	(0.984, 2.352)	(0.872, 0.579)	(0.973, 0.970)	213.8	4.80
	(1, 2)	(1, 2)	(1.411, 2.276)	(1.034,0.5590	(0.967, 0.944)	224.35	3.23
	(2, 2)	(2, 2)	(2.327,2.197)	(1.186, 0.466)	(0.985, 0.968)	226.57	3.12
	(4, 2)	(4, 2)	(4.401, 2.093)	(1.658, 0.984)	(0.981,0.986)	234.17	3.25
	(6, 2)	(6, 2)	(6.748, 2.085)	(2.479, 0.367)	(0.977, 0.955)	213.34	3.47
	(0.5, 2)	(1,1.5)	(0.954, 2.292)	(0.900, 0.562)	(0.965, 0.963)	267.12	5.12
	(1, 2)	(2, 3.5)	(1.275, 2.180)	(1.050, 0.553)	(0.967, 0. 964)	246.82	4.80
	(2, 2)	(3.6, 1.2)	(2.259, 2.135)	(1.148, 0.481)	(0.985, 0.940)	2.47.35	5.36
	(4, 2)	(5.9, 3)	(4.424, 2.042)	(1.654, 0.388)	(0.973, 0.941)	236.17	4.29
	(6, 2)	(7.5, 1.1)	(6.872, 2.097)	(2.401, 0.343)	(0.982, 0.953)	227.17	8.44
50	(0.5, 2)	(0.5, 2)	(0.792, 2.175)	(0.745, 0.462)	(0.962, 0.968)	142.16	3.76
	(1, 2)	(1, 2)	(1.206, 2.133)	(0.815, 0.452)	(0.968, 0.952)	141.13	4.92
	(2, 2)	(2, 2)	(2.123, 2.058)	(0.872, 0.355)	(0.974, 0.953)	152.21	4.45
	(4, 2)	(4, 2)	(4.292, 2.076)	(1.114, 0.288)	(0.975, 0.947)	123.26	3.59
	(6, 2)	(6, 2)	(6.561, 2.075)	(1.752, 0.272)	(0.964, 0.931)	124.81	4.73
	(0.5, 2)	(1,1.5)	(0.752, 2.154)	(0.721, 0.449)	(0.962, 0.965)	163.63	6.38
	(1, 2)	(2, 3.5)	(1.177, 2.115)	(0.925, 0.442)	(0.962, 0.954)	162.02	4.81
	(2, 2)	(3.6, 1.2)	(2.094, 2.053)	(0.886, 0.368)	(0.975, 0.947)	174.68	4.35
	(4, 2)	(5.9, 3)	(4.313, 2.067)	(1.195, 0.291)	(0.969, 0.942)	144.69	5.87
	(6, 2)	(7.5, 1.1)	(6.472, 2.058)	(1.699, 0.272)	(0.962, 0.948)	167.14	6.26
100	(0.5, 2)	(0.5, 2)	(0.703, 2.171)	(0.528, 0.325)	(0.955, 0.956)	142.96	265
	(1, 2)	(1, 2)	(1.149, 2.131)	(0.603, 0.302)	(0.952, 0.953)	142.83	3.69
	(2, 2)	(2, 2)	(2.162, 2.123)	(0.602, 0.242)	(0.955, 0.957)	140.85	3.45
	(4, 2)	(4, 2)	(4.215, 2.125)	(0.762, 0.202)	(0.959, 0.952)	142.94	4.28
	(6, 2)	(6, 2)	(6.324, 2.125)	(1.065, 0.183)	(0.955, 0.958)	153.40	1.55
	(0.5, 2)	(1,1.5)	(0.704, 2.171)	(0.538, 0.325)	(0.954, 0.952)	151.72	4.44
	(1, 2)	(2, 3.5)	(1.203, 2.162)	(0.609, 0.303)	(0.950, 0.968)	151.83	2.86
	(2, 2)	(3.6, 1.2)	(2.173, 2.134)	(0.641, 0.259)	(0.962, 0.956)	144.66	2.29
	(4, 2)	(5.9, 3)	(4.167, 2.119)	(0.747, 0.196)	(0.960, 0.958)	154.14	3.16
	(6, 2)	(7.5, 1.1)	(6.179, 2.124)	(1.136, 0.223)	(0.958, 0.948)	164.14	3.06
200	(0.5, 2)	(0.5, 2)	(0.654, 2.131)	(0.416, 0.246)	(0.957, 0.953)	146.71	3.55
	(1, 2)	(1, 2)	(1.118, 2.128)	(0.447, 0.228)	(0.954, 0.955)	139.53	3.42
	(2, 2)	(2, 2)	(2.132, 2.113)	(0.438, 0.183)	(0.947, 0.954)	142.48	3.16
	(4, 2)	(4, 2)	(4.163, 2.109)	(0.532, 0.143)	(0.956, 0.952)	134.58	2.49
	(6, 2)	(6, 2)	(6.237, 2.132)	(0.745, 0.129)	(0.954, 0.953)	149.03	3.50
	(0.5, 2)	(1,1.5)	(0.667, 2.138)	(0.427, 0.265)	(0.962, 0.952)	153.82	5.63
	(1, 2)	(2, 3.5)	(1.132, 2.132)	(0.456, 0.229)	(0.951, 0.951)	149.91	3.21
	(2, 2)	(3.6, 1.2)	(2.149, 2.121)	(0.429, 0.194)	(0.958, 0.947)	134.70	4.48
	(4, 2)	(5.9, 3)	(4.175, 2.121)	(0.545, 0.146)	(0.956, 0.949)	152.75	3.98
	(6, 2)	(7.5, 1.1)	(6.142, 2.102)	(0.798, 0.162)	(0.952, 0.954)	163.32	4.26

4.2 Real Application:

In this part we use the data set analyzed by Lawless (2003). The data set is the lifetimes of the number of million revolutions before failure for each one of the 23 ball bearings on an endurance test of deep groove ball bearings.

To begin, in order to determine the form of a lifetime data failure rate function, we are going to look at a graphical technique that is based on the TTT plot. If the TTT plot is concave (convex), it has been demonstrated that the failure rate function is increasing function (or decreasing). Despite the fact that the TTT plot is simply a sufficient requirement, and not a required one, for identifying the failure rate function shape, it is nonetheless utilized in this application as a subjective indicator of the form of the function.

Table 3: The data set is the lifetimes of the number of million revolutions before failure for each one of the 23 ball bearings on an endurance test of deep groove ball bearings.

17.88	28.92	33	41.12	45.6	48.8	51.84	51.96	54.12	55.56	76.8
68.44	68.88	84.12	93.12	98.64	105.12	105.84	127.92	128.04	173.4	

Figure 4 display the empirical TTT plot of ball bearings data. Since it has a concave shape, which indicates that the hazard rate function is an increasing function and may be handled appropriately by a Poisson-Exponential distribution.

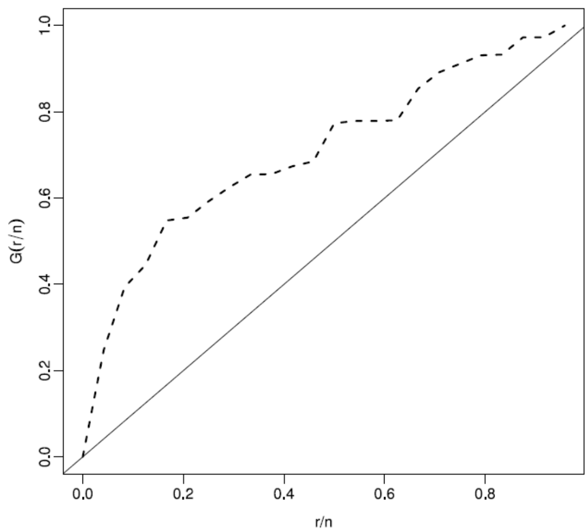

Figure 4: Empirical TTT plot for the data set.

Consequently, we first fit the Poisson-Exponential distribution to this data set. In order to calculate the MLEs of the parameters, we utilized use of the EM-algorithm and employed $\theta = 1$ and $\tau = 1/\bar{y}$ as our initial values. $\tau = 1/\bar{y}$ is a moment estimate that may be derived from an independent exponential observation that has been distributed equally. This option is not failsafe, so it goes without saying that you should run the method many times, each time starting with a different value.

Table 4 displays the MLEs and associated 95 percent confidence intervals for the Poisson-Exponential distribution parameters. These values were derived using the Fisher information matrix at the MLEs, which is given by:

$$I(\hat{\vartheta}) = \begin{pmatrix} -0.412 & 137.47 \\ 137.47 & -73647.59 \end{pmatrix}$$

In the Poisson-Exponential distribution, when $\theta = 0$, it equals the exponential distribution. Therefore, for comparison, we can consider the exponential distribution. We can use the following test to decide between the Poisson-Exponential distribution and the exponential distribution:

$$H_0: \theta = 0 \qquad vs \quad H_1: \theta > 0$$

Which equals testing the exponential distribution against the Poisson-exponential distribution. To evaluate, we will use the likelihood test ratio statistics, which are given by:

$$W_n = 2\,(\ell_{PE} - \ell_E)$$

Where ℓ_{PE} and ℓ_E are the log-likelihood functions when the model of data is the Poisson-Exponential distribution (PED) and the exponential distribution (ED), respectively. It is shown by Maller and Zhou (1995) that for a large enough sample size, W_n follows a chi-squared distribution with one degree of freedom.

The log-likelihood functions of both models are presented in Table 4-4. One can calculate $W_n = 16.574$ with a p-value< 0.0000, which shows that the Poisson-Exponential distribution is better model for the data set.

We can also compare the two models by using the Akaike information criterion (AIC) and the Bayesian information criterion (BIC). For both criteria, the model with the smallest value is evaluated as the best model for the data. The AIC and BIC of the Poisson-Exponential distribution and the exponential distribution are reported in Table 4-4. The results for AIC and BIC show that the Poisson-Exponential distribution is the better model for the data set.

Table 4: MLEs and corresponding 95% confidence intervals along with the log-likelihood function, AIC and BIC.

distribution	Parameter		$\ell(.)$	AIC	BIC
	θ	τ			
PED	7.439 (2.368, 12.311)	0.036 (0.024, 0.048)	-113.152	23.304	235.515
ED	-	0.014 (0.008, 0.020)	-121.439	244.879	247.439

Figure 5 shows the empirical Kaplan-Merier hazard rate estimation for both the Poisson-exponential distribution and the exponential distribution. This figure also confirms the previous results.

Another alternative distribution we can consider is the generalization of the exponential distribution (GED) with the following probability density function:

$$f(y) = \frac{\alpha\theta\tau}{(1-e^{\theta})^{\alpha}} \exp(-\theta - \tau y + \theta e^{-\tau y}) \left(1 - \exp(-\theta + \theta e^{-\tau y})\right)^{\alpha-1} \qquad y > 0$$

Where $\alpha, \theta, \tau > 0$. The MLEs of the parameters by considering the data set are:

$$\hat{\alpha} = 5.753 \qquad \hat{\theta} = 0.242 \qquad \hat{\tau} = 0.0319$$

with $\ell_{GED} = -112.974$, $AIC = 231.949$ and $BIC = 239.765$. This results also confirm that again the Poisson-exponential distribution is better model for the data set.

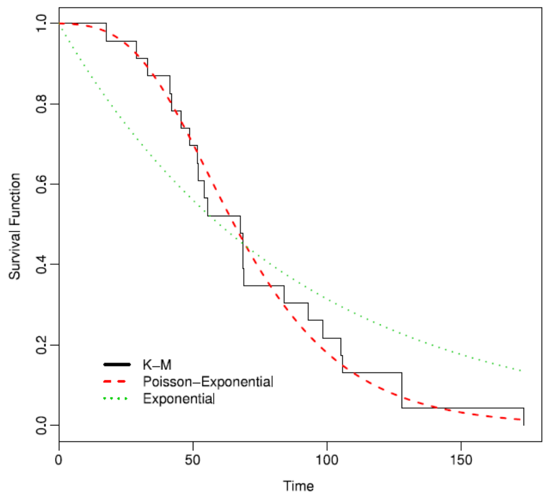

Figure 5: Kaplan-Merier curve with estimated of the Poisson-Exponential and Exponential distributions.

5. Conclusions

To investigate the estimation of proposed distribution parameters, we conducted simulation study for the values of parameters that considered previously but used different initial values such that some are close to the real value of parameters but some have large difference with them. To investigate the convergence of the proposed EM algorithm we noted the number of the iterations. Table 4-2 illustrates the mean of the obtained MLE of the parameters in each step of generating data, which is denoted by $av(\hat{\vartheta})$ in the table along with the standard error of the simulated estimators, $SE(\hat{\vartheta})$. For the convergence of the algorithm, we determinate the mean of the iterations of the EM algorithms a long with the standard error of those values which are denoted by $av(k)$ and $SE(k)$ in the table respectively. In addition, the coverage probability of the 95 percent confidence intervals for the parameters of the proposed distribution is displayed in Table 4-2, which is denoted by C.P. The findings are as follows:

1. Convergence has been accomplished in every scenario, even when the initial values are inadequate, and thus highlights the numerical stability of the EM method.
2. The disparities between the average estimates and the actual values are virtually never more than the empirical standard error of one estimate. Based on these findings, it appears that the EM estimations have maintained their accuracy.
3. An increase in the size of the sample leads to a decrease in the standard errors of the MLEs.
4. The empirical coverage probabilities are quite close to the amount of coverage that is considered to be the nominal value. Particularly when the number of people in the sample rises.

6. References:

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