



Partial b-Rectangular Metric Space with Some Results

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DOI: <http://dx.doi.org/10.25130/tjps.27.2022.030>

ARTICLE INFO.

Article history:

-Received: 2 / 9 / 2021

-Accepted: 26 / 10 / 2021

-Available online: / / 2022

Keywords: b-rectangular metric space, contraction mapping, fixed point, partial b-metric space, partial b-rectangular metric space.

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ABSTRACT

A new generalization of metric space called partial b-rectangular metric space is introduced. Also, the relation between this generalization and the other generalizations for example a b-rectangular metric space is given. Moreover, we have proved Banach theorem and Kannan theorem of fixed Point in partial b-rectangular metric space. Furthermore, some definitions and results dealing with partial b-rectangular metric space are discussed.

1. Introduction

The definition of metric space was presented by Freshet in 1906 [1]. Many authors after Freshet gave several generalization of metric space. In [2], Shukla S. gave the concept of partial b-metric space. The notion of b-rectangular metric space was given in [3]. Banach in 1922 introduced the famous and important type of fixed point which is Banach contraction principle [4]. In 1969, Kannan gave another type of fixed point which is Kannan fixed point theorem [5]. Many fixed point theorems were stated and proved in generalization metric space [6-14].

Definition 1[2]

Let S be a non-empty set. A function $L: S \times S \rightarrow [0, \infty)$ is a partial b-metric on S if $\forall a, b$ and c in S :

(L1): $L(a, a) = L(a, b) = L(b, b)$ iff $a = b$.

(L2): $L(a, a) \leq L(a, b)$.

(L3): $L(a, b) = L(b, a)$.

(L4): $L(a, b) \leq k[L(a, c) + L(c, b)] - L(c, c)$, for some $k \in \mathbb{R}, k \geq 1$.

Then the pair (S, L) is said to be partial b-metric space, where $k \in \mathbb{R}, k \geq 1$ is the coefficient number of (S, L, k) .

Definition 2[3]

Let S be a non-empty set. A function $N: S \times S \rightarrow [0, \infty)$ is a b-rectangular metric on S if $\forall a, b, c$ and d in S :

(N1): $N(a, b) = 0$ iff $a = b$.

(N2): $N(a, b) = N(b, a)$.

(N3): $N(a, b) \leq k[N(a, c) + N(c, d) + N(d, b)]$, for some $k \in \mathbb{R}, k \geq 1$.

Then the pair (S, N) is said to be b-rectangular metric space, where $k \in \mathbb{R}, k \geq 1$ is the coefficient number of (S, N, k) .

2. Partial b-Rectangular Metric Space

In this section we generalize the definitions of partial b-metric space and b-rectangular metric space, we named partial b-rectangular metric space.

Definition 3.

Let S be a non-empty set. A function $p_{br}: S \times S \rightarrow [0, \infty)$ is a partial b-rectangular metric on S if satisfies the following conditions: for a, b in S and for $c \neq d$ in $S - \{a, b\}$:

(PBR1): $p_{br}(a, a) = p_{br}(a, b) = p_{br}(b, b)$ if and only if $a = b$.

(PBR2): $p_{br}(a, a) \leq p_{br}(a, b)$.

(PBR3): $p_{br}(a, b) = p_{br}(b, a)$.

(PBR4): $p_{br}(a, b) \leq k[p_{br}(a, c) + p_{br}(c, d) + p_{br}(d, b)] - p_{br}(c, c) - p_{br}(d, d)$ for some $k \in \mathbb{R}, k \geq 1$.

Then the pair (S, p_{br}) is said to be partial b-rectangular metric space, where $k \in \mathbb{R}, k \geq 1$ is the coefficient number of (S, p_{br}, k) .

We explain that by some examples and remarks.

Example 4.

Let $S = \{0, \frac{1}{4}, \frac{1}{2}, 1\}$. Define $p_{br}: S \times S \rightarrow (0, \infty)$ as $p_{br}(a, b) = \alpha + |a - b|$, α is a real number greater

than zero. Then (S, p_{br}) is a partial b-rectangular metric space.

Proof: We see that the properties (PBR1), (PBR2) and (PBR3) are verified for all $a, b \in S$.

To proof (PBR4) consider:

$$\begin{aligned} p_{br}(a, b) &= \alpha + |a - b| \\ &= \alpha + |a - c + c - d + d - b| - 2\alpha + 2\alpha \\ &\leq \alpha + |a - c| + \alpha + |c - d| + \alpha + |d - b| - \alpha - \alpha \\ &\leq k[(\alpha + |a - c|) + (\alpha + |c - d|) + (\alpha + |d - b|)] - \alpha - \alpha \text{ (for some } k \in \mathbb{R}, k \geq 1) \\ &= k[p_{br}(a, c) + p_{br}(c, d) + p_{br}(d, b)] - p_{br}(c, c) - p_{br}(d, d) \\ &\forall a, b \in S \text{ and for } c \neq d \in S - \{a, b\} \end{aligned}$$

Hence (S, p_{br}) is a partial b-rectangular metric space.

Example 5.

Let $A = \{0, 1, 2, 3, 4, 5\}$, $A^2 = S$. Define $p_{br}: S \times S \rightarrow [0, \infty)$ as follows:

$$p_{br}(a, b) = 2 + (|a_1 - a_2| + |b_1 - b_2|), \quad \text{for each } a = (a_1, b_1), b = (a_2, b_2) \text{ in } S. \text{ Then } (S, p_{br}) \text{ is a partial b-rectangular metric space.}$$

Proof: We notice that the properties (PBR1), (PBR2) and (PBR3) are verified for all $a, b \in S$.

To proof (PBR4) consider:

$$\begin{aligned} p_{br}(a, b) &= 2 + (|a_1 - a_2| + |b_1 - b_2|) \\ &\leq 2 + (|a_1 - a_3| + |b_1 - b_3|) + 2 + (|a_3 - a_4| + |b_3 - b_4|) + 2 + (|a_4 - a_2| + |b_4 - b_2|) - 2 - 2 \\ &\leq k[2 + (|a_1 - a_3| + |b_1 - b_3|) + 2 + (|a_3 - a_4| + |b_3 - b_4|) + 2 + (|a_4 - a_2| + |b_4 - b_2|)] - 2 - 2 \\ &= k[p_{br}(a, c) + p_{br}(c, d) + p_{br}(d, b)] - p_{br}(c, c) - p_{br}(d, d) \\ &\forall a, b \in S \text{ and for } c \neq d \in S - \{a, b\}, \text{ for some } k \in \mathbb{R}, k \geq 1 \end{aligned}$$

Hence (S, p_{br}) is a partial b-rectangular metric space.

Remarks 6.

1. If $a, b \in S$ such that $p_{br}(a, b) = 0$ we have $a = b$. The opposite is not necessary true.
2. b-rectangular metric space is a partial b-rectangular metric space with the self-distance between two points equals zero but the convers not necessary true.

Note 7. In (Examples 4, 5) we have $p_{br}(a, a) \neq 0, \forall a = b \in S$. So the opposite of remark 1 not true and (S, p_{br}) is not a b-rectangular metric space.

3. Fixed Point in Partial b-Rectangular Metric Space

In this section we prove Banach contraction principle theorem and Kannan fixed point theorem in a partial b-rectangular metric space. In order to obtain that we need to give some definitions and information in a partial b-rectangular metric space.

Definition 8.

Let $\langle a_n \rangle$ be a sequence in (S, p_{br}) , we say that $\langle a_n \rangle$ converges to $a \in S$, if $\forall \varepsilon > 0, \exists n_0(\varepsilon) \in \mathbb{N}$ such that $|p_{br}(a_n, a) - p_{br}(a, a)| < \varepsilon, \forall n > n_0(\varepsilon)$.

Definition 9.

Let $\langle a_n \rangle$ be a sequence in (S, p_{br}) , we say that $\langle a_n \rangle$ Cauchy sequence, if $\forall \varepsilon > 0, \exists n_0(\varepsilon) \in \mathbb{N}$ such that $p_{br}(a_n, a_m) < \varepsilon, \forall n, m > n_0(\varepsilon)$, in other words $\lim_{n, m \rightarrow \infty} p_{br}(a_n, a_m)$ exists and finite.

Definition 10.

A partial b-rectangular metric space (S, p_{br}) is said to be complete if every Cauchy sequence in S is convergent, in other words $\lim_{n, m \rightarrow \infty} p_{br}(a_n, a_m) = \lim_{n \rightarrow \infty} p_{br}(a_n, a) = p_{br}(a, a)$.

Example 11.

Let $A = \left\{ \frac{1}{n+1} : n \in \mathbb{N} \right\}, B = \{0, 1\}$ and $A \cup B = S, \beta \geq 0$. Define $p_{br}: S \times S \rightarrow [0, \infty)$ as follows:

$$p_{br}(a, b) = \begin{cases} \beta, & \text{if } a = b \text{ or } a, b \in B \\ \beta + \frac{1}{n+1}, & \text{if } a \in A \text{ and } b \in B \text{ or } b \in A \text{ and } a \in B \\ \beta + \frac{1}{2}, & \text{otherwise} \end{cases}$$

We see that the properties (PBR1), (PBR2) and (PBR3) are verified for all $a, b \in S$.

To proof (PBR4) we have:

$$\begin{aligned} p_{br}(a, b) &= \beta + \frac{1}{n+1} \leq k \left[\left(\beta + \frac{1}{n+1} \right) + \frac{1}{2} + \frac{1}{n+1} \right] + 2\beta - 2\beta \\ &= k \left[\left(\beta + \frac{1}{n+1} \right) + \left(\beta + \frac{1}{2} \right) + \left(\beta + \frac{1}{n+1} \right) \right] - \beta - \beta \\ &= k[p_{br}(a, c) + p_{br}(c, d) + p_{br}(d, b)] - p_{br}(c, c) - p_{br}(d, d) \\ &\forall a, b \in S \text{ and for } c \neq d \in S - \{a, b\}, \text{ for some } k \in \mathbb{R}, k \geq 1. \end{aligned}$$

So, (S, p_{br}) is a partial b-rectangular metric space.

In (S, p_{br}) the convergent sequence may have more than one convergence point and may not be Cauchy sequence.

Consider the following sequence

$$a_n = \begin{cases} \frac{1}{n+1}, & n \text{ is odd} \\ 1, & n \text{ is even} \end{cases} \text{ in } S$$

$$\lim_{n \rightarrow \infty} p_{br}(a_n, 0) = \begin{cases} \lim_{n \rightarrow \infty} p_{br}\left(\frac{1}{n+1}, 0\right), & n \text{ is odd} \\ \lim_{n \rightarrow \infty} p_{br}(1, 0), & n \text{ is even} \end{cases} = \begin{cases} \beta \\ \beta \end{cases}$$

$$\lim_{n \rightarrow \infty} p_{br}(a_n, 1) = \begin{cases} \lim_{n \rightarrow \infty} p_{br}\left(\frac{1}{n+1}, 1\right), & n \text{ is odd} \\ \lim_{n \rightarrow \infty} p_{br}(1, 1), & n \text{ is even} \end{cases} = \begin{cases} \beta \\ \beta \end{cases}$$

So, $\langle a_n \rangle$ convergence to two different points 0 and 1.

Also, $\lim_{n \rightarrow \infty} p_{br}(a_{2n}, a_{2n+2}) = \beta + \frac{1}{2}$ and

$$\lim_{n \rightarrow \infty} p_{br}(a_{2n}, 1) = \beta$$

It means the limit does not exist, so $\langle a_n \rangle$ is not Cauchy sequence.

Definition 12.

Let $T: (S, p_{br}) \rightarrow (S, p_{br})$ be a mapping, we say that $a_0 \in S$ is a fixed point of T if $T(a_0) = a_0, \forall a_0 \in S$.

Definition 13.

Let $T: (S, p_{br}) \rightarrow (S, p_{br})$ be a mapping, then T is called a contraction mapping if $p_{br}(T(a), T(b)) \leq \theta p_{br}(a, b)$, where $0 \leq \theta < 1, \forall a, b \in S$.

In the following theorem we prove Banach contraction principle (Banach type of fixed Point) in partial b-rectangular metric space

Theorem 14. (Banach Type of fixed Point)

Let (S, p_{br}) be a complete partial b-rectangular metric space with a real number $k > 1$, let

$T: (S, p_{br}) \rightarrow (S, p_{br})$ be a mapping such that $p_{br}(T(a), T(b)) \leq \theta p_{br}(a, b)$, where $0 \leq \theta < \frac{1}{k}$, $\forall a, b \in S$, then there exists a unique fixed point $a_0 \in S$ of T and the self-distance between two points equals zero.

Proof: Firstly to prove if T has a fixed point $a_0 \in S$, then a_0 is unique. We shall prove that if $a_0 \in S$ is a fixed point of T , i.e. $T(a_0) = a_0$, then the self-distance between two points equals zero.

Suppose that $a_0 \in S$ is a fixed point of T , i.e. $T(a_0) = a_0$

Consider $p_{br}(a_0, a_0) = p_{br}(T(a_0), T(a_0)) \leq \theta p_{br}(a_0, a_0) < \frac{1}{k} p_{br}(a_0, a_0) < p_{br}(a_0, a_0)$

Which is a contradiction. Thus, we must have the self-distance between two points equals zero.

Now, to prove a_0 is a unique, suppose there exist two fixed points of T in S say a_0 and b_0 such that $a_0 \neq b_0$, i.e. $T(a_0) = a_0 \neq b_0 = T(b_0)$

Consider $p_{br}(a_0, b_0) = p_{br}(T(a_0), T(b_0)) \leq \theta p_{br}(a_0, b_0) < \frac{1}{k} p_{br}(a_0, b_0) < p_{br}(a_0, b_0)$

Which is a contradiction. Thus, we must have the self-distance between two points equals zero, i.e. $a_0 = b_0$.

Next, to prove the existence of fixed point, let $\langle a_n \rangle$ be a sequence in S such that $T(a_n) = a_{n+1}$, $n = 0, 1, 2, \dots$, if $a_{n+1} = a_n$ then T has a fixed point which is a_n .

Assume that $a_{n+1} \neq a_n$, $n = 0, 1, 2, \dots$, put $p_{brn} = p_{br}(a_n, a_{n+1})$

Consider $p_{brn} = p_{br}(a_n, a_{n+1}) = p_{br}(T(a_{n-1}), T(a_n)) \leq \theta p_{br}(a_{n-1}, a_n) = \theta p_{brn-1}$

And

$\theta p_{brn-1} = \theta p_{br}(a_{n-1}, a_n) = \theta p_{br}(T(a_{n-2}), T(a_{n-1})) \leq \theta^2 p_{br}(a_{n-2}, a_{n-1}) = \theta^2 p_{brn-2}$

After repetition of this process we get:

$p_{brn} \leq \theta p_{brn-1} \leq \theta^2 p_{brn-2} \leq \dots \leq \theta^n p_{br0}$ i.e. $p_{brn} \leq \theta^n p_{br0}$

Again put $p_{brn}^* = p_{br}(a_n, a_{n+2})$

Consider

$p_{brn}^* = p_{br}(a_n, a_{n+2}) = p_{br}(T(a_{n-1}), T(a_{n+1})) \leq \theta p_{br}(a_{n-1}, a_{n+1}) = \theta p_{brn-1}^*$

After repetition of this process we get:

$p_{brn}^* \leq \theta p_{brn-1}^* \leq \theta^2 p_{brn-2}^* \leq \dots \leq \theta^n p_{br0}^*$ i.e. $p_{brn}^* \leq \theta^n p_{br0}^*$

To prove that $\langle a_n \rangle$ is a Cauchy sequence considered $p_{br}(a_n, a_m)$, $m = n + q$, $q > 0$, we have two cases:

Case 1: If $q = 2p + 1$ (odd) we have:

$p_{br}(a_n, a_{n+2p+1}) \leq k[p_{br}(a_n, a_{n+1}) + p_{br}(a_{n+1}, a_{n+2}) + p_{br}(a_{n+2}, a_{n+2p+1})] - p_{br}(a_{n+1}, a_{n+1})$
 $\leq k p_{br}(a_n, a_{n+1}) + k p_{br}(a_{n+1}, a_{n+2}) + k^2 p_{br}(a_{n+2}, a_{n+3}) + k^2 p_{br}(a_{n+3}, a_{n+4}) + k^3 p_{br}(a_{n+4}, a_{n+5}) + k^3 p_{br}(a_{n+5}, a_{n+6}) + \dots + k^p p_{br}(a_{n+2p}, a_{n+2p+1})$

$- p_{br}(a_{n+1}, a_{n+1}) - p_{br}(a_{n+2}, a_{n+2}) - p_{br}(a_{n+3}, a_{n+3}) - p_{br}(a_{n+4}, a_{n+4}) - p_{br}(a_{n+5}, a_{n+5}) - p_{br}(a_{n+6}, a_{n+6}) - \dots - p_{br}(a_{n+2p}, a_{n+2p})$
 $\leq k p_{br}(a_n, a_{n+1}) + k p_{br}(a_{n+1}, a_{n+2}) + k^2 p_{br}(a_{n+2}, a_{n+3}) + k^2 p_{br}(a_{n+3}, a_{n+4}) + k^3 p_{br}(a_{n+4}, a_{n+5}) + k^3 p_{br}(a_{n+5}, a_{n+6}) + \dots + k^p p_{br}(a_{n+2p}, a_{n+2p+1})$

Since $p_{brn}^* = p_{br}(a_n, a_{n+2})$, we have:

$p_{br}(a_n, a_{n+2p+1}) = k p_{brn} + k p_{brn+1} + k^2 p_{brn+2} + k^2 p_{brn+3} + k^3 p_{brn+4} + k^3 p_{brn+5} + \dots + k^p p_{brn+2p}$

Since $p_{brn} \leq \theta^n p_{br0}$, we have:

$p_{br}(a_n, a_{n+2p+1}) \leq k \theta^n p_{br0} + k \theta^{n+1} p_{br0} + k^2 \theta^{n+2} p_{br0} + k^2 \theta^{n+3} p_{br0} + k^3 \theta^{n+4} p_{br0} + k^3 \theta^{n+5} p_{br0} + \dots + k^p \theta^{n+2p-1} p_{br0} + k^p \theta^{n+2p} p_{br0}$
 $= [(1 + k \theta^2 + k^2 \theta^4 + \dots + k^{p-1} \theta^{2p-2}) + \theta(1 + k \theta^2 + k^2 \theta^4 + \dots + k^{p-1} \theta^{2p-2})] k \theta^n p_{br0}$

$\leq \frac{k \theta^n p_{br0} (1 + \theta)}{1 - k \theta^2}$ (since $k > 1$ and $0 \leq \theta < \frac{1}{k}$ so $k \theta^2 < 1$)

Case 2: If $q = 2p$ (even), by using the same way as the proof of the first case with $\theta^n p_{br0}^* \geq p_{brn}^*$ we get:

$p_{br}(a_n, a_{n+2p+1}) \leq \frac{k \theta^n p_{br0} (1 + \theta) + \theta^{n-2} p_{br0}^*}{1 - k \theta^2}$

From case 1 and case 2 we have $p_{br}(a_n, a_m) = 0$, as $n, m \rightarrow \infty$, $m = n + q$, $q > 0$

Hence $\langle a_n \rangle$ is a Cauchy sequence in (S, p_{br}) .

From completeness property of (S, p_{br}) we have $a_0 \in S$ such that $a_n \rightarrow a_0$ as $n \rightarrow \infty$

To prove $a_0 \in S$ is a fixed point of T

Consider, T satisfies $p_{br}(T(a), T(b)) \leq \theta p_{br}(a, b)$ we have:

$p_{br}(a_0, T(a_0)) \leq k[p_{br}(a_0, a_n) + p_{br}(a_n, a_{n+1}) + p_{br}(a_{n+1}, T(a_0))] - p_{br}(a_n, a_n) - p_{br}(a_{n+1}, a_{n+1})$
 $\leq k[p_{br}(a_0, a_n) + p_{br}(a_n, a_{n+1}) + p_{br}(T(a_n), T(a_0))]$

$\leq k[p_{br}(a_0, a_n) + p_{br}(a_n, a_{n+1}) + \theta p_{br}(a_n, a_0)]$

Since $p_{br}(a_n, a_m) = 0$, $\langle a_n \rangle$ converges to $a_0 \in S$ we obtain:

$p_{br}(a_0, T(a_0)) = 0 \Rightarrow a_0 = T(a_0)$

Hence T has a fixed point which is $a_0 \in S$.

In the following theorem we prove Kannan fixed point theorem in partial b-rectangular metric space

Theorem 15. (Kannan Type of fixed Point)

Let (S, p_{br}) be a complete partial b-rectangular metric space with a real number $k \geq 1$, let $T: (S, p_{br}) \rightarrow (S, p_{br})$ be a mapping such that $p_{br}(T(a), T(b)) \leq \theta[p_{br}(a, T(a)) +$

$p_{br}(b, T(b))]$ where $0 \leq \theta < \frac{1}{2k}$, $\forall a, b \in S$, then there exists a unique fixed point $a_0 \in S$ of T and the self-distance between two points equals zero.

Proof: Firstly to prove if T has a fixed point $a_0 \in S$, then a_0 is unique. We shall prove that if $a_0 \in S$ is a fixed point of T , i.e. $T(a_0) = a_0$, then the self-distance between two points equals zero.

Suppose that $a_0 \in S$ is a fixed point of T , i.e. $T(a_0) = a_0$

$$\begin{aligned} \text{Consider } p_{br}(a_0, a_0) &= p_{br}(T(a_0), T(a_0)) \\ &\leq \theta[p_{br}(a_0, T(a_0)) + p_{br}(a_0, T(a_0))] \\ &= 2\theta p_{br}(a_0, a_0) < \frac{1}{k} p_{br}(a_0, a_0) < p_{br}(a_0, a_0) \end{aligned}$$

Which is a contradiction. Thus, we must have the self-distance between two points equals zero.

Now, to prove a_0 is a unique, suppose there exist two fixed points of T in S say a_0 and b_0 such that $a_0 \neq b_0$, i.e. $T(a_0) = a_0 \neq b_0 = T(b_0)$, then we have the self-distance between two points equals zero.

$$\begin{aligned} \text{Consider } p_{br}(a_0, b_0) &= p_{br}(T(a_0), T(b_0)) \\ &\leq \theta[p_{br}(a_0, T(a_0)) + p_{br}(b_0, T(b_0))] \\ &= \theta[p_{br}(a_0, a_0) + p_{br}(b_0, b_0)] = 0 \end{aligned}$$

Which is contradiction. So, we must have the self-distance between two points equals zero, i.e. $a_0 = b_0$.

Next, to prove the existence of fixed point, let $\langle a_n \rangle$ be a sequence in S such that $T(a_n) = a_{n+1}$, $n = 0, 1, 2, \dots$, if $a_{n+1} = a_n$ then T has a fixed point which is a_n .

Assume that $a_{n+1} \neq a_n$, $n = 0, 1, 2, \dots$, put $p_{brn} = p_{br}(a_n, a_{n+1})$

$$\begin{aligned} \text{Consider } p_{brn} &= p_{br}(a_n, a_{n+1}) \\ &= p_{br}(T(a_{n-1}), T(a_n)) \\ &\leq \theta[p_{br}(a_{n-1}, T(a_{n-1})) + p_{br}(a_n, T(a_n))] \\ &= \theta[p_{br}(a_{n-1}, a_n) + p_{br}(a_n, a_{n+1})] \\ &= \theta[p_{brn-1} + p_{brn}] \end{aligned}$$

After repetition of this process we get:

$$p_{brn} \leq \left(\frac{\theta k}{1-\theta k}\right)^n p_{br0} \quad (\text{since } k \geq 1 \text{ and } 0 \leq \theta < \frac{1}{2k} \text{ so } \theta k < 1)$$

Thus, $\lim_{n \rightarrow \infty} p_{brn} = \lim_{n \rightarrow \infty} p_{br}(a_n, a_{n+1}) = 0$, it means for any $\varepsilon > 0$ we can find $n_0(\varepsilon) \in \mathbb{N}$ such that $p_{brn} < \frac{\varepsilon}{2}$, $\forall n > n_0(\varepsilon)$.

To prove that $\langle a_n \rangle$ is a Cauchy sequence in S

$$\begin{aligned} \text{Consider } p_{br}(a_n, a_m) &= p_{br}(T(a_{n-1}), T(a_{m-1})) \\ &\leq \theta[p_{br}(a_{n-1}, T(a_{n-1})) + p_{br}(a_{m-1}, T(a_{m-1}))] \\ &= \theta[p_{br}(a_{n-1}, a_n) + p_{br}(a_{m-1}, a_m)] \\ &= \theta[p_{brn-1} + p_{brm-1}] \end{aligned}$$

Since $p_{brn} = p_{br}(a_n, a_{n+1}) < \frac{\varepsilon}{2}$, $\forall n > n_0(\varepsilon)$, so

$$p_{brm} = p_{br}(a_m, a_{m+1}) < \frac{\varepsilon}{2}, \forall m > n_0(\varepsilon)$$

Therefore, $p_{br}(a_n, a_m) < \varepsilon$, $\forall n, m > n_0(\varepsilon)$

Hence $\langle a_n \rangle$ is a Cauchy sequence in (S, p_{br}) and $\lim_{n, m \rightarrow \infty} p_{br}(a_n, a_m) = 0$

From completeness property of (S, p_{br}) we have $a_0 \in S$ such that $a_n \rightarrow a_0$ as $n \rightarrow \infty$

$$0 = \lim_{n, m \rightarrow \infty} p_{br}(a_n, a_m) = \lim_{n \rightarrow \infty} p_{br}(a_n, a_0) = p_{br}(a_0, a_0).$$

To prove $a_0 \in S$ is a fixed point of T

$$\begin{aligned} p_{br}(a_0, T(a_0)) &\leq k[p_{br}(a_0, a_n) + p_{br}(a_n, a_{n+1}) + p_{br}(a_{n+1}, T(a_0))] - p_{br}(a_n, a_n) - p_{br}(a_{n+1}, a_{n+1}) \\ &\leq k[p_{br}(a_0, a_n) + p_{br}(a_n, a_{n+1}) + p_{br}(T(a_n), T(a_0))] \\ &\leq k[[p_{br}(a_0, a_n) + p_{br}(a_n, a_{n+1}) + \theta[p_{br}(a_n, T(a_n)) + p_{br}(a_0, T(a_0))]]] \\ &= k[[p_{br}(a_0, a_n) + p_{br}(a_n, a_{n+1}) + \theta[p_{br}(a_n, a_{n+1}) + p_{br}(a_0, T(a_0))]]] \end{aligned}$$

$$\text{Since } \lim_{n \rightarrow \infty} p_{br}(a_n, a_0) = 0 \text{ and } \lim_{n \rightarrow \infty} p_{brn} = \lim_{n \rightarrow \infty} p_{br}(a_n, a_{n+1}) = 0, \text{ we obtain}$$

$$p_{br}(a_0, T(a_0)) = 0 \Rightarrow a_0 = T(a_0)$$

Hence T has a fixed point which is $a_0 \in S$.

References

- [1] Fréchet M. (1906). Sur quelques points du calcul fonctionnel. Rendiconti del circolo Matematico di Palermo. 22: 1-74.
- [2] Shukla S. (2014). Partial b-metric space and fixed point theorems. Mediterr. J. Math. 11: 703-711.
- [3] Roshan J, Parvaneh V, Kadelburg Z and Hussain N. (2016). New fixed point results in b-rectangular metric spaces*. Nonlinear Analysis. Modeling and Control. 21(5): 614-634.
- [4] Banach S. (1922). Sur les opérations dans les ensembles abstraction et leur application aux equations integrals. Fund Math. 2: 133-181.
- [5] Kannan R. (1969). Some results on fixed points ||. Am. Math. Mon. 76:405-408.
- [6] Pacurar M. (2010). A fixed point result for Φ -contractions on b-metric spaces without the boundedness assumption. Fasc. Math. 43: 127-137.
- [7] Altun I, Sola F and Simsek H. (2010). Generalized contractions on partial metric space. Topology and its Applications. 157: 2778-2785.
- [8] Anuradha G and Pragati G. (2015). Quasi-Partial b- metric space and some related fixed point theorems. Fixed Point Theory and Application. 18: 2-12.

- [9] Singh D, Ghauhan V and Wangkeeree R. (2017). Geraghty type generalized F-Continuous and related application in partial metric space. Int J Anal. 1-14.
- [10] Mitrovic Z.D., Radenovic S, (2017). A common fixed point theorem of Jungck in rectangular b-metric space. Acta Math Hungarica. 153(2): 401-407.
- [11] Parvaneh V and Kadelburg Z. (2018). Extended Partial b-metric space and some fixed point results. Faculty Sci Math. 32(8): 2837-2850.
- [12] Yaowaluck K. (2018). Contraction on some fixed point theorem in $b_v(s)$ - metric space. WCE London UK. 1: 1-6.
- [13] Deepak S, Varsha C, Poom K and Vishal J. (2018). Some applications of fixed point results for generalized two classes of Boyd-Wong's F-Continuous in partial metric spaces. Math Sci Springer. 12(2): 111-127.
- [14] Asim M., Imdad M. and Shukla S. (2021). Fixed point results for Geraghty-weak contractions in ordered partial rectangular b-metric spaces. Afirika Matematika. 32: 811-827.

الفضاء المتري الجزئي من النمط b المستطيل مع بعض النتائج

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الملخص

قدمنا في هذا البحث تعميماً جديداً للفضاء المتري يدعى الفضاء المتري الجزئي من النمط b المستطيل. كذلك وضعنا العلاقة بين هذا التعميم والتعميمات الأخرى مثل الفضاء المتري من النمط b المستطيل. علاوة على ذلك قمنا ببرهان مبدأ بناخ الانكماش للنقطة الصامدة ومبرهنة كانان للنقطة الصامدة في الفضاء المتري الجزئي من النمط b المستطيل. بالإضافة الى ذلك ناقشنا بعض التعاريف والنتائج المتعلقة بالفضاء المتري الجزئي من النمط b المستطيل.