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On $(n, \mathfrak{D})$- quasi Operators

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ABSTRACT: Through the article, we will provide an entirely novel type of quasi operators base on the concept of $\mathfrak{D}$razin inverse in Operator theory which is named $(n, \mathfrak{D})$ -quasi operators on the Hilbert spaces, also we explain the sum, product, and other features of these classes are discussed. and restriction via some necessary conditions have been holds in order to obtain these operations, in addition that we investigate the scalar and power of this concept, moreover we submit prove of the Tensor product and direct product of $(n, \mathfrak{D})$-quasi operators.

Keywords: Functional analysis; $(n, \mathfrak{D})$-quasi operator; $\mathfrak{D}$-Operator; $\mathfrak{D}$razin operator.

1. INTRODUCTION

In recent years, many mathematicians have concentrated their efforts on a $\mathfrak{D}$-Operator is a particular type of bonded linear Operator, and it provides certain generalized versions of this type of Operator, like, such In 2018 the classes of $\mathfrak{D}$-Operator and another classes in same field of Operator theory has been introduced by M. Dana and R. Youssfi [1], submitted some fundamental characteristics of these Operators defined on Hilbert space H and also developed Some new kinds of Operators, namely $[\mathfrak{D}\mathcal{N}]$, $[n\mathfrak{D}N]$, $[\mathfrak{D}QN]$, and $[n\mathfrak{D}Q\mathcal{N}]$ upon a $\mathfrak{D}$razin invertible Operator, which respectively extend the concepts of $[\mathfrak{D}Q\mathcal{N}]$ and $[\mathfrak{D}\mathcal{N}]$ Operators.

Additionally, the relationships between these kinds on this source. In 2020 S. D. Mohsen, showed new properties of new types of Operator call (N, k)-Hyponormal Operator [2].

In 2020, Beinane and Sid [3], In keeping with our earlier work, such that their provided a number of Operators on H, including $\mathfrak{D}$-m-quasi-normal Operator, indicated $([\mathfrak{D}(QN)m]$, additionally provided power of n such as $\mathfrak{D}$-m-quasi normal operator, which are symbolizes as $[n\mathfrak{D}(QN)m]$, $\mathfrak{D}$razin inverse related to a $\mathfrak{D}$razin invertible Operator. Some modifications of $[\mathfrak{D}(QN)m]$ and $[n\mathfrak{D}QNm]$ characterizations are discussed. The relationships of inclusion among the several types of normal Operators are described. as well as introducing a new type of $\mathfrak{D}$-Operators and some of its core characteristics.

In 2021 [4], Wanjala and Nyongesa introduced the N quasi-$\mathfrak{D}$-Operator, studying the kind of It is introduced a quasi $\mathfrak{D}$-Operators functioning on the common H across the intricate plane. If $Q(Q^{*2}Q\mathfrak{D}2) = K(Q * Q\mathfrak{D})2Q$, where $\mathcal{K}$ is a bound Operator on H, where the Operator Q is referred to as a N quasi-$\mathfrak{D}$-operator, with gave class of Operators' fundamental behavior. In 2021 [4], Wanjala and Beatrice introduced the type of (p, qBQ) Operators acting on a complex H, when $\left(Q^{*2p}Q^{2q}\right) = (Q^{*p}Q^{*q})^2$ with equivalently, $\left[\left(Q^{*2p}Q^{2q}\right) - (Q^{*p}Q^{*q})^2\right] = 0$, an Operator if$Q \in B(H)$ is called to belong to class (p, qBQ), and look into the advantages this type has with regard to algebra, with examine the connection between this class and the (p, q)-power class (Q) Operators, moreover. A new class, the $[B\mathfrak{D}]$-Operator, was introduced, and its spectrum features and relationship to the $\mathfrak{D}$-Operator generalizations were explored, in order to emphasize this. In the year 2021, Eiman and Samira demonstrated some $\mathfrak{D}$-Operator properties, such that give the $\mathfrak{D}$-Operator where this type, a new class of $\mathfrak{D}$razin invertible Operator on H, also several features of the class of $\mathfrak{D}$-Operators are explored. As well as, that the unitary equivalent property and scalar product are preserved by the D-Operators, they demonstrated that the $\mathfrak{D}$-Operator in particular is not represented by the sum or product of two $\mathfrak{D}$-Operators, but is instead represented by the direct and tenser products [5]. In 2022 Samir Al Mohammady; Sid Ahmed Ould Beinane [6], a novel family of Operators on semi-Hilbertian spaces, that is, namely $(n,m)-\mathcal{A}$-normal, If $Q^n(Q^{\#\mathcal{A}})^m-(Q^{\#\mathcal{A}})^m Qn=\left[Qn,(Q^{\#\mathcal{A}})^m\right]=0$ when n and m are positive integers. where the spaces created by positive semi-definite Sesquilinear forms is introduced and studied. They are Assume A is a positively bounded Operators over H. and define H be a Hilbert space, also the idea of semi-norm.A was brought about by the semi internal multiply $<\mathcal{O},\mathcal{L}>_{\mathcal{A}}=<\mathcal{A}\mathcal{O},\mathcal{L}>,\mathcal{O},\mathcal{L}\in H$ becomes a semi-Hilbertian space the consequence of our work, in 2020 N. M. Atheab and S. D. Mohsen, introduced some imprtant classes of (N,k)-Hyponormal Operators, and in 2023 S.D. Mohsen discovered new types of Operators which is $(k,m)-n$-paranormal Operators such as proved some important theorems of this Operator, [7].

Through this paper, we will investigate novel generalizations of new class of quasi normal Operators up on $\mathfrak{D}$razin inverse of Operators on Hilbert space such that we have $(n,\mathfrak{D})$-quasi and illustrate this concept via some examples have been submitted in this article with prove some properties and basic behavior of this class of this concept like scalar and restriction when are action on this operators, further to that we study the product and addition finite sets from $(n,\mathfrak{D})$-quasi, as well as, the tensor product and direct sum of finite sets of this concept has been given in this article this search consist of four sections, Section 1 has the introduction of work, but Section 2 deals with the basic concepts of this article in addition Section 3 including important results of properties on this concept, finally Section 4 has conclusions of this work.

2. BASIC DEFINITIONS

In this section of this article, we recall some basic concepts and fundamental of functional analysis in specific operator theory filed, also deals with the $\mathfrak{D}$razin Operator, as well as some Basic properties of these concepts have been recalled.

Definition 1. *[15] Let V be a vector space over the field F. An inner product (,) is a mapping from $V\times V$ into the field have some properties require*

1. *For all $a\in V$, we must has $(a,a)\geq 0$,*
2. *in case $(a,a)=0$ if and only if $a = 0$,*
3. *For each $a,b\in V$, holds $(a, b)=\overline{(b,a)}$,*
4. *For each $a,b, c\in V$, holds $(ia+jb,c)=i(a,c)+j(b,c)$, such as $i,j\in F$. Where the order pair V take with (,) namely an inner product space over the filed F.*

It is useful to recall in this part of article the Hilbert space with symbolize as, which is complete inner product space.

Definition 2. *[14] Hypothesis the operators $\mathcal{Q}:H\to H$ which is bounded operator, then has unique adjoint symbolize as $\mathcal{Q}^*$ and hold the condition $(\mathcal{Q}a\ ,b)=\ (a,\mathcal{Q}^*b)$ for any $a,b\in H$.*

From this definition many authors obtained some classes of operators in Hilbert space we will recall next,

Definition 3. *[14] Hypothesis the operators $\mathcal{Q}:H\to H$ which is bounded operator, then*

1. *$\mathcal{Q}$ is known as self-adjoint if require $\mathcal{Q}^*a=\mathcal{Q}a$ for each $a\in H$.*
2. *$\mathcal{Q}$ is known as normal if require $\mathcal{Q}^*\mathcal{Q}a=\mathcal{Q}\mathcal{Q}^*a$ for each $a\in H$.*
3. *$\mathcal{Q}$ is known as hyponormal if require $\mathcal{Q}^*\mathcal{Q}a\leq\mathcal{Q}\mathcal{Q}^*a$ for each $a\in H$.*
4. *$\mathcal{Q}$ is known as quasinormal if require $\mathcal{Q}\mathcal{Q}^*\mathcal{Q}a\leq\mathcal{Q}^*\mathcal{Q}\mathcal{Q}a$ for each $a\in H$.*

Definition 4. *[8] Assume that $\mathcal{Q}\in B(H)$, a $\mathfrak{D}$razin-inverse of Q when it exists is the Operator $Q\mathfrak{D}:H\to H$:*

1. *$QQ^{\mathfrak{D}}=Q^{\mathfrak{D}}Q$.*
2. *$Q^{\mathfrak{D}}QQ^{\mathfrak{D}}=Q^{\mathfrak{D}}$.*
3. *$Q^{k+1}Q^{\mathfrak{D}}=Q^k$. for several $k\geq 0$ which is integer.*

The goal of this paper is to generalize the normal operator by introducing a new kind of Operator, several authors presented generalizations of this concept, as in example [9–13]. Now, the following lemma will remind of several properties of $\mathfrak{D}$-Operator that that have been in [13, 14].

Lemma 1. *Let $Q, s : H \to H$ are two $\mathfrak{D}$razin-invertible Operators:*

1. $(Q^*)^{\mathfrak{D}} = (Q^{\mathfrak{D}})^*$
2. $(Q^{-1} s \mathfrak{T})^{\mathfrak{D}} = Q^{-1} s^{\mathfrak{D}} \mathfrak{T}$
3. $(s^k)^{\mathfrak{D}} = (s^{\mathfrak{D}})^k$, *for* $k = 1, 2, \ldots$
4. *If* $Qs = sQ$ *then* $(\mathfrak{T} s)^{\mathfrak{D}} = \mathfrak{T}^{\mathfrak{D}} s^{\mathfrak{D}}$
5. *If* $Qs = sQ = 0$ *then* $(Q + s)^{\mathfrak{D}} = Q^{\mathfrak{D}} + s^{\mathfrak{D}}$

3. CERTAIN PROPERTIES OF (n , $\mathfrak{D}$)-QUASI OPERATORS

Here, we submit new types of Operator called $(n, \mathfrak{D})$-quasi with some charachrizations of its. This operator is considered one of the important generalizations of the operators as well as an expansion of previous studies in this field of mathematics.

Definition 5. *Supposing $\mathfrak{T} \in B(H)$ be $\mathfrak{D}$razin operator. Then $\mathfrak{T}$ to be call as $(n, \mathfrak{D})$-quasi when require* $\mathfrak{T}(mathcalT^{*2}(\mathfrak{T}^{\mathfrak{D}})^{2n} = (\mathfrak{T}^*\left(\mathfrak{T}^{\mathfrak{D}}\right)^n)^2 \mathfrak{T}$

The following issue proposition presents the most important generalizations for these operator and in this type of study

Proposition 1. *An Operator $\mathfrak{T} \in B(H)$, was $(n, \mathfrak{D})$- quasi, then:*

1. $\lambda \mathfrak{T} is (n, \mathfrak{D})$- *quasi,* λ *is any scalar.*
2. *The Power's of* $(n, \mathfrak{D})$- *quasi is again* $(n, \mathfrak{D})$-*quasi.*
3. $\mathfrak{T}_{|\lambda}$ *is* (n, D)- *quasi, where* $\lambda \subset H$.

Proof. 1. Let $\lambda \neq 0$, by Definition 5 we have:
$(\lambda\mathfrak{T})(\lambda\mathfrak{T})^{*2}((\lambda\mathfrak{T})^{\mathfrak{D}})^{2n} = (\lambda\mathfrak{T})\left(\lambda^2\mathfrak{T}^{*2}\right)\left(\lambda^{2n}\left(\mathfrak{T}^{\mathfrak{D}}\right)^{2n}\right)$, this implies to $(\lambda\mathfrak{T})(\lambda\mathfrak{T})^{*2}((\lambda\mathfrak{T})^{\mathfrak{D}})^{2n} = \lambda\lambda^2\lambda^{2n}\mathfrak{T}\left(\left(\mathfrak{T}^{*2}\right)\left(\mathfrak{T}^{\mathfrak{D}}\right)^{2n}\right)$, also one can use some fundamental rules to get $(\lambda\mathfrak{T})(\lambda\mathfrak{T})^{*2}((\lambda\mathfrak{T})^{\mathfrak{D}})^{2n} = \lambda\ \lambda^2\lambda^{2n}\mathfrak{T}\left(\left(\mathfrak{T}^{*2}\right)\left(\mathfrak{T}^{\mathfrak{D}}\right)^{2n}\right)$, then we will obtain about

$$\begin{aligned}(\lambda\mathfrak{T})(\lambda\mathfrak{T})^{*2}((\lambda\mathfrak{T})^{\mathfrak{D}})^{2n} &= \lambda\ \lambda^2\lambda^{2n}\left(\mathfrak{T}^*\left(\mathfrak{T}^{\mathfrak{D}}\right)^n\right)^2\ \mathfrak{T}, \text{ because } \lambda \text{ which is scalar belong to field}\\ &= \lambda^2 T^* \lambda^{2n}\left(\mathfrak{T}^{\mathfrak{D}}\right)^{2n}\lambda\mathfrak{T}\\ &= (\lambda\mathfrak{T}^*)^2\left(\lambda\ \mathfrak{T}^{\mathfrak{D}}\right)^{2n}(\lambda\ \mathfrak{T})\\ &= \left((\lambda\mathfrak{T}^*)(\lambda\mathfrak{T})^{\mathfrak{D}}\right)^2(\lambda\mathfrak{T})\end{aligned}$$

By this reason, λ is an $(n, \mathfrak{D})$ –quasi:

2. By using the mathematical induction, the result is holds for:

   (i) $r = 1$, so one have: $\left(\mathfrak{T}^*(\mathfrak{T}^{\mathfrak{D}})^n\right)^2\mathfrak{T} = \mathfrak{T}\left(\mathfrak{T}^{*2}\left(\mathfrak{T}^D\right)^{2n}\right)$

   (ii) $r = p$, so become: $\left(\left(\mathfrak{T}^*\left(\mathfrak{T}^{\mathfrak{D}}\right)^n\right)^2\mathfrak{T}\right)^p = \left(\mathfrak{T}\left(\mathfrak{T}^{*2}\left(\mathfrak{T}^D\right)^{2n}\right)^p\right)$.
   Now, prove for $r = p + 1$, exactly we will have
   $\left(\mathfrak{T}\left(\mathfrak{T}^{*2}\left(\mathfrak{T}^{\mathfrak{D}}\right)^{2n}\right)\right)^p\left(\mathfrak{T}\left(\mathfrak{T}^{*2}\left(\mathfrak{T}^{\mathfrak{D}}\right)^{2n}\right)\right) = \left(\mathfrak{T}\left(\mathfrak{T}^{*2}\left(\mathfrak{T}^{\mathfrak{D}}\right)\right)^{2n}\right)^{p+1}$, by a) and b), one have the next step:
   $= \left(\left(\mathfrak{T}^*\left(\mathfrak{T}^{\mathfrak{D}}\right)^n\right)^2\mathfrak{T}\right)^p\left(\left(\mathfrak{T}^*\left(\mathfrak{T}^{\mathfrak{D}}\right)^n\right)^2\mathfrak{T}\right) = \left(\left(\mathfrak{T}^*\left(\mathfrak{T}^{\mathfrak{D}}\right)^n\right)^2\mathfrak{T}\right)^{p+1}$, And by result, proof is required.

3. Assumption λ is a closed invariant subset of H:

$$(T_{/\lambda})\left((T_{/\lambda})^{*2}\left(T^{\mathcal{D}}{}_{\lambda}\right)^{2n}\right) = (T_{/\lambda})\left((T_{/\lambda})^{*}(T_{/\lambda})^{*}\left(T^{\mathcal{D}}{}_{\lambda}\right)^{n}\left(T^{\mathcal{D}}{}_{\lambda}\right)^{n}\right) \text{ from some basic rule we have}$$

$$\left[(T_{|\lambda})(T_{|\lambda})^{*2}(T^{\mathcal{D}}{}_{|\lambda})^{2n}\right] = (T_{/\lambda})(T^{*}{}_{\lambda})(T^{*}{}_{/\lambda})\left(\left(T^{\mathcal{D}}\right)^{n}{}_{/\lambda}\right)\left(\left(T^{\mathcal{D}}\right)^{n}{}_{/\lambda}\right) \text{ and we can obtain the next step}$$

$$\begin{aligned}
\left[(T_{|\lambda})(T_{|\lambda})^{*2}(T^{\mathcal{D}}{}_{|\lambda})^{2n}\right] &= (T_{/\lambda})(T^{*}{}_{/\lambda})^{2}\left(\left(T^{\mathcal{D}}\right)^{n}{}_{/\lambda}\right)^{2} \\
&= (T_{/\lambda})\left(T^{*2}{}_{/\lambda}\right)\left(\left(T^{\mathcal{D}}\right)^{2n}{}_{/\lambda}\right) \text{ After these steps, we have} \\
&= \left(T\left(T^{*2}\left(T^{\mathcal{D}}\right)^{2n}\right)\right)/\lambda \\
&= \left(\left(T^{*}\left(T^{\mathcal{D}}\right)^{n}\right)^{2}T\right)^{2}/\lambda \text{ since } T \text{ is is } (n,\mathcal{D})\text{-quasi, we must have} \\
&= \left(\left(T^{*}\left(T^{\mathcal{D}}\right)^{n}\right)/\lambda\right)\left(\left(T^{*}\left(T^{\mathcal{D}}\right)^{n}\right)/\lambda\right)(T_{/\lambda}) \\
&= \left(T^{*}{}_{/\lambda}\left(T^{\mathcal{D}}\right)^{n}{}_{/\lambda}\right)\left(T^{*}{}_{/\lambda}(T^{\mathcal{D}})^{n}{}_{/\lambda}\right)(T_{/\lambda}),
\end{aligned}$$

finally, one can have the next step from proof of this theorem about restriction

$$\begin{aligned}
&= \left((T_{/\lambda})^{*}\left((T_{/\lambda})^{\mathcal{D}}\right)^{n}\right)\left((T_{/\lambda})^{*}\left((T_{/\lambda})^{\mathcal{D}}\right)^{n}\right)(T_{/\lambda}) \\
&= \left((T_{/\lambda})^{*}\left((T_{/\lambda})^{\mathcal{D}}\right)^{n}\right)^{2}(T_{/\lambda})^{*}.
\end{aligned}$$

Therefore $T_{|\lambda}$ is $(n,\mathcal{D})$-quasi.

This theory specializes in studying the space of $(n,\mathcal{D})$ quasi operator , and through it we discovered that its closed from space of all operator define on H.

Theorem 1. *$\mathcal{A} = \{T, T : H \to H$ is $(n,\mathcal{D})$-quasi on Hilbert space $H\} \subseteq B(H)$, then $\mathcal{A}$ is closed subset of $B(H)$.*

Proof. We must prove $\mathcal{A}$ contains all its limit points, such that if each sequence of $(n,\mathcal{D})$-quasi $T_n \to T$ in $\mathcal{A}$, then $T \in \mathcal{A}$, now since $T_n \to T$, thus $T\left(T^{*2}\left(T^{\mathcal{D}}\right)^{2n}\right) = \left(T^{*}\left(T^{\mathcal{D}}\right)^{n}\right)^{2}T$, $T_n{}^{*2}\left(T_n^{\mathcal{D}}\right)^{2n}T_n \to \left(T^{*}\left(T^{\mathcal{D}}\right)^{n}\right)^{2}T$, but T_n is $(n,\mathcal{D})$-quasi, its has $T_n\left(T_n{}^{*2}\left(T_n{}^{\mathcal{D}}\right)^{2n}\right) \to \left(T^{*}\left(T^{\mathcal{D}}\right)^{n}\right)^{2}T$, also

$$\left\|T\left(T^{*2}(TT^{\mathcal{D}})^{2n}\right) - T_n\left(T_n{}^{*2}\left(T_n{}^{\mathcal{D}}\right)^{2n}\right) + T_n\left(T_n{}^{*2}\left(T_n{}^{\mathcal{D}}\right)^{2n}\right) - \left(T^{*}\left(T^{\mathcal{D}}\right)^{n}\right)^{2}T\right\| \leqslant \left\|T(T^{*2}(T^{\mathcal{D}})^{2n} - T_n\left(T_n{}^{*2}\left(T_n{}^{\mathcal{D}}\right)^{2n}\right)\right\|$$

$+\left\|T_n\left(T_n{}^{*2}\left(T_n{}^{\mathcal{D}}\right)^{2n}\right) - \left(T^{*}\left(T^{\mathcal{D}}\right)^{n}\right)^{2}T\right\| \to 0$, when $n \to \infty$. So $T \in \mathcal{A}$, therefore; $\mathcal{A}$ is closed subset of $B(H)$.

Now, its clear that the addition of two $(n,\mathcal{D})$-quasi is not required to be an operator of type $(n,\mathcal{D})$. In order to prove that one can see the identity Operator I is $(n,\mathcal{D})$-quasi, but the addition $2I$ of this is not necessary to be $(n,\mathcal{D})$-quasi. However, the following theorem outlines the requirements for this truth to be true.

Theorem 2. *If $T, s \in B(H)$ are two $(n,\mathcal{D})$-quasi on Hilbert space H. If $T^{*2}(s^{D})^{2n} = 0 = s^{*2}(T^{D})^{2n} = Ts^{*2}(s^{\mathcal{D}})^{2n} = sT^{*2}(T^{\mathcal{D}})^{2n}$, then $T + s$ is $(n,\mathcal{D})$- quasi.*

Proof. By Definition 4

$$\begin{aligned}
(T+S)\left((T+S)^{*2}\left((T+S)^{\mathcal{D}}\right)\right)^{2n} &= (T+S)(T+S)^{*}(T+S)^{*}\left((T+S)^{\mathcal{D}}\right)^{n}\left((T+S)^{\mathcal{D}}\right)^{n} \\
&= (T+S)(T^{*}+S^{*})(T^{*}+S^{*})\left(T^{\mathcal{D}}+S^{\mathcal{D}}\right)^{n}\left(T^{\mathcal{D}}+S^{\mathcal{D}}\right)^{n}, \text{One can have as follow step} \\
&= (T+S)\left(\left(T^{*2}+S^{*2}\right)\left(\left(T^{\mathcal{D}}\right)^{2n}+\left(S^{\mathcal{D}}\right)^{2n}\right)\right)
\end{aligned}$$

we can use the usual multiplication (composition) of the operators in order to get the down formula

$$\begin{aligned}
&= (T+S)\left(T^{*2}\left(T^{\mathcal{D}}\right)^{2n} + T^{*2}\left(S^{\mathcal{D}}\right)^{2n} + S^{*2}\left(T^{\mathcal{D}}\right)^{2n} + S^{*2}\left(S^{\mathcal{D}}\right)^{2n}\right) \\
&= (T+S)\left(T^{*2}\left(T^{\mathcal{D}}\right)^{2n} + S^{*2}\left(S^{\mathcal{D}}\right)^{2n}\right), \text{ Now, we will get the next step of this prove}
\end{aligned}$$

$$TT^{*2}\left(T^{\mathcal{D}}\right)^{2n} + TS^{*2}\left(S^{\mathcal{D}}\right)^{2n} + ST^{*2}\left(T^{\mathcal{D}}\right)^{2n} + SS^{*2}\left(S^{\mathcal{D}}\right)^{2n} = \left(T^{*}\left(T^{\mathcal{D}}\right)^{n}\right)^{2}T + \left(S^{*}\left(S^{\mathcal{D}}\right)^{n}\right)^{2}S$$

Hence, finally we have the guarantee result $\mathfrak{T}+s$ is $(n,\mathfrak{D})$- quasi.

Theorem 3. *If $\mathfrak{T}, s \in B(H)$ are two $(n,\mathfrak{D})$- quasi then $\mathfrak{T}s$ should be $(n,\mathfrak{D})$- quasi, when the following requires are holds:*

1. $S\mathfrak{T}^{*2} = \mathfrak{T}^{*2}S$.
2. $\mathfrak{T}S = S\mathfrak{T}$.
3. $S^{*2}\left(\mathfrak{T}^{\mathfrak{D}}\right)^{2n} = \left(\mathfrak{T}^{\mathfrak{D}}\right)^{2n}S^{*2}$.
4. $\left(\mathfrak{T}^{\mathfrak{D}}\right)^{2n}S = S\left(\mathfrak{T}^{\mathfrak{D}}\right)^{2n}$

Proof.

$$\begin{aligned}
(\mathfrak{T}S)\left((\mathfrak{T}S)^{*2}\left((\mathfrak{T}S)^{\mathfrak{D}}\right)^{2n}\right) &= (\mathfrak{T}S)\left((S\mathfrak{T})^{*2}\left((\mathfrak{T}S)^{\mathfrak{D}}\right)^{2n}\right)\\
&= (\mathfrak{T}S)\left(\left(\mathfrak{T}^{*2}S^{*2}\right)\left(\mathfrak{T}^{\mathfrak{D}}S^{\mathfrak{D}}\right)^{2n}\right), \text{ one can use basic details such has}\\
&= \mathfrak{T}\left(S\mathfrak{T}^{*2}\right)\left(S^{*2}\left(\mathfrak{T}^{\mathfrak{D}}\right)^{2n}\right)\left(S^{\mathfrak{D}}\right)^{2n}, \text{ form hypothesis, one get}\\
&= \mathfrak{T}\left(\mathfrak{T}^{*2}S\right)\left(S^{*2}\left(\mathfrak{T}^{\mathfrak{D}}\right)^{2n}\right)\left(S^{\mathfrak{D}}\right)^{2n}\\
&= \mathfrak{T}\left(\mathfrak{T}^{*2}S\right)\left(\left(\mathfrak{T}^{\mathfrak{D}}\right)^{2n}S^{*2}\right)\left(S^{\mathfrak{D}}\right)^{2n}, \text{from assumption appear here}\\
&= \mathfrak{T}\mathfrak{T}^{*2}\left(S\left(\mathfrak{T}^{\mathfrak{D}}\right)^{2n}\right)\left(S^{*2}\left(S^{\mathfrak{D}}\right)^{2n}\right)\\
&= \mathfrak{T}\mathfrak{T}^{*2}\left(\left(\mathfrak{T}^{\mathfrak{D}}\right)^{2n}S\right)\left(S^{*2}\left(S^{\mathfrak{D}}\right)^{2n}\right)\\
&= \left(\mathfrak{T}\mathfrak{T}^{*2}\left(\mathfrak{T}^{\mathfrak{D}}\right)^{2n}\right)\left(SS^{*2}\left(S^{\mathfrak{D}}\right)^{2n}\right)
\end{aligned}$$

Since $\mathfrak{T}$ and s are $(n,\mathfrak{D})$- quasi, so we have, the next action:

$$\begin{aligned}
&= \left(\mathfrak{T}^{*}\left(\mathfrak{T}^{\mathfrak{D}}\right)^{n}\right)^{2}\mathfrak{D}\left(S^{*}\left(S^{\mathfrak{D}}\right)^{n}\right)^{2}S, \text{ from some basic calculus.}\\
&= \mathfrak{T}^{*2}S^{*2}\left(\left(\mathfrak{T}^{\mathfrak{D}}\right)^{n}\right)^{2}\left(\left(S^{\mathfrak{D}}\right)^{n}\right)^{2}(\mathfrak{T}S)\\
&= (\mathfrak{T}S)^{*2}\left(\left(\mathfrak{T}^{\mathfrak{D}}S^{\mathfrak{D}}\right)^{n}\right)^{2}(\mathfrak{T}S)\\
&= (\mathfrak{T}S)^{*2}\left(\left((\mathfrak{T}S)^{\mathfrak{D}}\right)^{n}\right)^{2}(\mathfrak{T}S)\\
&= \left((\mathfrak{T}S)^{*}\left((\mathfrak{T}S)^{\mathfrak{D}}\right)^{n}\right)^{2}(\mathfrak{T}S).
\end{aligned}$$

Hence, $\mathfrak{T}s$ is $(n,\mathfrak{D})$- quasi. Further characteristics of the $(n,\mathfrak{D})$-quasi are then provided by the following theorem, which is generalizations for some theorems is this filed of article

Theorem 4. *Let $\mathfrak{T}_1, \mathfrak{T}_2, \ldots, \mathfrak{T}_n$ are $(n,\mathfrak{D})$-quasi Operators specify on the Hilbert space H, then*

1. $\mathfrak{T}_1 \oplus \ldots \oplus \mathfrak{T}_n$ *to be $(n,\mathfrak{D})$-quasi Operator.*
2. $\mathfrak{T}_1 \otimes \ldots \otimes \mathfrak{T}_n$ *to be $(n,\mathfrak{D})$-quasi Operator.*

Proof.

-

$$\begin{aligned}
&(\mathfrak{T}_1 \oplus \ldots \oplus \mathfrak{T}_n)(\mathfrak{T}_1 \oplus \ldots \oplus \mathfrak{T}_n)^{*2}\left((\mathfrak{T}_1 \oplus \ldots \oplus \mathfrak{T}_n)^{\mathfrak{D}}\right)^{2n}\\
&= (\mathfrak{T}_1 \oplus \ldots \oplus \mathfrak{T}_n)\left(\mathfrak{T}_1^{*^2} \oplus \ldots . \oplus \mathfrak{T}_n^{*^2}\right)\left(\left(\mathfrak{T}_1^{\mathfrak{D}}\right)^{2n} \oplus \ldots \oplus \left(\mathfrak{T}_n^{\mathfrak{D}}\right)^{2n}\right)\\
&= \mathfrak{T}_1\mathfrak{T}_1^{*^2}\left(\mathfrak{T}_1^{\mathfrak{D}}\right)^{2n} \oplus \ldots \oplus \mathfrak{T}_n\mathfrak{T}_n^{*^2}\left(\mathfrak{T}_n^{\mathfrak{D}}\right)^{2n}\\
&= \left(\mathfrak{T}_1^{*}\left(\mathfrak{T}_1^{\mathfrak{D}}\right)^{n}\right)^{2}\mathfrak{T}_1 \oplus \ldots \oplus \left(\mathfrak{T}_n^{*}\left(\mathfrak{T}_n^{\mathfrak{D}}\right)^{n}\right)^{2}\mathfrak{T}_n\\
&= \left(\mathfrak{T}_1^{*}\left(\mathfrak{T}_1^{\mathfrak{D}}\right)^{n} \oplus \ldots \oplus \mathfrak{T}_n^{*}\left(\mathfrak{T}_n^{\mathfrak{D}}\right)^{n}\right)^{2}(\mathfrak{T}_1 \oplus \ldots \oplus \mathfrak{T}_n)
\end{aligned}$$

Therefore; $\mathfrak{T}_1 \oplus \ldots \oplus \mathfrak{T}_n$ is $(n, \mathfrak{D})$-quasi Operator.

- Another recommendation of this theorem proven the tensor product on this classes of operators, Let $x_1, x_2, \ldots x_n \in H$, then we start as follow

$$
\begin{aligned}
&(\mathfrak{T}_1 \ldots \mathfrak{T}_n)(\mathfrak{T}_1 \otimes \ldots \otimes \mathfrak{T}_n)^{*2}\Big((\mathfrak{T}_1 \otimes \ldots \otimes \mathfrak{T}_n)^{\mathfrak{D}}\Big)^{2n}\ (x_1 \otimes \ldots \otimes x_n)\\
&= (\mathfrak{T}_1 \otimes \ldots . \otimes \mathfrak{T}_n)\left(\mathfrak{T}_1^{*^2} \otimes \ldots \otimes \mathfrak{T}_n^{*^2}\right)\left(\left(\mathfrak{T}_1^{\mathfrak{D}}\right)^{2n} \otimes \ldots . \otimes \left(\mathfrak{T}_n^{\mathfrak{D}}\right)^{2n}\right)(x_1 \otimes x_2 \otimes \ldots \otimes x_n)\\
&= \mathfrak{T}_1 \mathfrak{T}_1^{*^2}\left(\mathfrak{T}_1^{\mathfrak{D}}\right)^{2n}(x_1) \otimes \ldots \otimes \mathfrak{T}_n \mathfrak{T}_n^{*^2}\left(\mathfrak{T}_n^{\mathfrak{D}}\right)^{2n}(x_n), \text{through some form of compensation}\\
&= \mathfrak{T}_1\left(\mathfrak{T}_1^{*}\left(\mathfrak{T}_1^{\mathfrak{D}}\right)^{n}\right)^{2}(x_1) \otimes \ldots \otimes \mathfrak{T}_n\left(\mathfrak{T}_n^{*}\left(\mathfrak{T}_n^{\mathfrak{D}}\right)^{n}\right)^{2}(x_n)\\
&= \left[\left(\mathfrak{T}_1^{*}\left(\mathfrak{T}_1^{\mathfrak{D}}\right)\right)^{2n}(x_1) \otimes \ldots \otimes \left(\mathfrak{T}_n^{*}\left(\mathfrak{T}_n^{\mathfrak{D}}\right)\right)^{2n}(x_n)\right](\mathfrak{T}_1(x_1) \otimes \ldots \otimes \mathfrak{T}_n(x_n))\\
&= \left(\mathfrak{T}_1^{*}(\mathfrak{T}_1^{\mathfrak{D}})\right)^{n}(x_1) \otimes \ldots \otimes \left(\mathfrak{T}_n^{*}(\mathfrak{T}_n^{\mathfrak{D}})\right)^{n}(x_2)^2\,(\mathfrak{T}_1(x_1) \otimes \ldots . \otimes \mathfrak{T}_n(x_2))\\
&= \left(\mathfrak{T}_1^{*}\left(\mathfrak{T}_1^{\mathfrak{D}}\right)\right)^{n} \otimes \ldots \otimes \left(\mathfrak{T}_n^{*}(\mathfrak{T}_n^{\mathfrak{D}})\right)^{n}(x_1)^2\,(\mathfrak{T}_1(x_1) \otimes \ldots \otimes \mathfrak{T}_n(x_1))(x_1 \otimes x_2 \otimes \ldots \otimes x_n)
\end{aligned}
$$

4. CONCLUSION

The objective of this article's conclusion is to define new definitions of the $(n, \mathfrak{D})$-quasi and various operations connected to the $(n, \mathfrak{D})$-quasi, examine into some of the characteristics of this category of Operators. Due to the fact that the requirements we stated are weaker than their circumstances, we also learn that this sort of our work is a generalization of their work. a significant relationship between the Operator we researched and the $\mathfrak{D}$-Operator already exists, and this manuscript novel suggests that several fundamental properties of the hyponormal Operator might not be met by the $(n, \mathfrak{D})$-quasi. One can achieve it by adding some conditions, as in the case of the property of the sum and the product

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CONFLICTS OF INTEREST

The authors declare no conflict of interest.

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