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TWO-STAGE SHRINKAGE BAYESIAN ESTIMATORS FOR THE SHAPE PARAMETER OF PARETO DISTRIBUTION DEPENDENT ON KATTI'S REGIONS

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ABSTRACT: This study proposes a two-stage shrinkage Bayesian estimation of the shape parameter of Pareto distribution. Additional information from the past and considered presently in new estimation processes has been receiving considerable attention in the last few decades, especially when a sample unit is costly or difficult to obtain. The proposed two-stage pooling estimation procedure assumes that the prior knowledge of θ can take the form of an initial estimate θ_0 of θ. The expressions for bias, bias ratio, mean square error, expected sample size, and relative efficiency are derived based on the two regions of R_1 and R_2. Certain values of the constants are considered, and the R language is used for statistical programming. The numerical results and conclusion suggest that the proposed estimators have higher relative efficiency compared with the classical Bayesian estimator with respect to a guess value. The effective region of the estimator dependent on R_2 is better than that of the estimator dependent dependent on R_1.

Keywords: Pareto distribution; Bayes estimator; Shrinkage Estimation; Relative efficiency.

1. INTRODUCTION

Pareto distribution was originally used to describe a society's wealth distribution; it presents a trend in which asubstantial part of the money is held by an extremely small percentage of the population [1]. In probability theory and statistics, Pareto distribution is a continuous probability distribution named after the Italian economist Vilfredo Pareto. Outside of economic circles, the scheme is called Bradford distribution, in which a continuous random variable X of the Pareto distribution has the following probability density function (p.d.f.):

$$f(x;\beta,\theta) = \frac{\theta\beta^{\theta}}{x^{(\theta+1)}} \quad , \quad \beta,\theta > 0, x \geq \beta(1) \tag{1}$$

When the scale parameter (i.e., β) is equal to 1, we have

$$f(x;\theta) = \frac{\theta}{x^{(\theta+1)}} \quad , \quad x \geq 1; \theta > 0 \tag{2}$$

As for the cumulative distribution function (c.d.f.),

$$F(x;\theta) = 1 - x^{-\theta} \tag{3}$$

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where θ is the shape parameter.

Parameter estimation is an important topic in statistical inference because it helps to decide the complete description of life distributions that match suitably with the data. When prior knowledge about the unknown parameters is lacking, we use traditional estimation approaches, such as moments, maximum likelihood estimator, uniformly minimal variance unbiased estimator, and so on. When prior knowledge exists, it must be included in the estimation. The Bayes estimation method is useful in this situation, particularly if the prior knowledge about the unknown parameter θ is in the form of a prior distribution. When the prior knowledge is only available in the form of an initial guess value, i.e., θ_0 of θ, this guess can be used to improve the estimation procedure [2]. A guess value arises for any single reason, such as when we are estimating θ and

1. we are assuming that θ_0 is close to the true value of θ; or

2. we are concerned that θ_0 may be near the true value of θ, implying that a negative if $\theta_0 = \theta$, and we will be unaware of it.

In such cases, the guess value must be included in the estimation process.

Most of the time, the sample unit is costly or difficult to obtain. Therefore, using inexpensive estimates is a good alternative. In this study, we also propose a two-stage pooling estimation procedure. Thus far, Ref. [3] is the most outstanding method for estimating the mean of a normal distribution when the initial value μ_0 of the mean is given and the variance is known. If the mean of the first sample belongs to the area R, then it will be used as an approximation of μ. Otherwise, a second sampling will be drawn, and the pooled mean will be constructed as $\frac{n_1\bar{x}_1+n_2\bar{x}_2}{n_1+n_2}$ to calculate μ. Refs. [4] and [5] attempted to obtain the weights of θ_0 and $\hat{\theta}_1$ by using the constant k (i.e., 0 < k < 1). If $\hat{\theta}_1 \in$ R, then the estimate is $k\bar{x}_1 + (1-k)\mu_0$, where k is a constant specified by the experimenter according to his or her beliefin θ_0. A value of k around 0 indicates a strong belief that θ_0 is close to the true mean θ, whereas a value near 1 indicates that the double-stage shrinkage estimator is based primarily on the sample. Refs. [4] and [5] used the double-stage estimation of the mean by using prior knowledge following the works of Refs. [2] and [3] to conceptualize the double-stage shrinkage estimator.

Many researchers have studied the aforementioned estimate for different distribution proverbs [6-17]. This research deals with the proposal and study of the properties of the two-stage shrinkage Bayesian estimators of the shape parameter for the Pareto distribution that is dependent on two different two regions. In this case, prior information is given as the initial value θ_0.

Let $x_1, x_2, ..., x_n$ be a random sample size n from the Pareto distribution as defined in Eq. (2). The joint density for the sample $x_1, x_2, ..., x_n$ and parameter θ is expressed as

$$H(x_1, x_2, \ldots, x_n, \theta) = \prod_{i=1}^{n} f\left(\frac{xi}{\theta}\right) g(\theta)(4) \tag{4}$$

$g(\theta) \propto \sqrt{I(\theta)}$, where I($\theta$) is a fisher information.

In this study, Jeffery's non-informative prior is used as a prior distribution, i.e., $g(\theta) = \frac{1}{\theta^c}, c > 0$. The details are shown in Ref. [18]. The posterior distribution of θ can be obtained by combining the prior distribution and the likelihood in Eq. (4) . Along with Bayes theorem, the formula can be written as

$\pi(\theta|x_1, x_2, \ldots, x_n) = \frac{H(x_1,x_2,\ldots,x_n,\theta)}{\int_0^\infty H(x_1,x_2,\ldots,x_n,\theta)d\theta}$

$$Thus,\ \pi(\theta; x_1, x_2, \ldots, x_n) = \frac{[\sum_{i=1}^{n} Lnx_i]^{n-c+1}\theta^{n-c}e^{-\theta\sum_{i=1}^{n} Lnx_i}}{\Gamma(n-c+1)} \tag{5}$$

The Bayes estimator for the squared error loss function of the posterior mean is given by

$$\hat{\theta}_B = \int_0^\infty \theta\pi(\theta|x)d\theta \tag{6}$$

Substituting $\pi(\theta; x)$ of Eq. (5) in Eq. (6) gives

$$\hat{\theta}_B = \frac{n-c+1}{\sum_{i=1}^{n} Lnx_i} \qquad , c > 0 \tag{7}$$

Assume that X has a Pareto distribution and LnX has an exponential distribution. Thus, $\sum_{i=1}^{n} Lnx_i$ has a Gamma distribution with the parameter (n, θ), and $\frac{1}{\hat{\theta}_B}$ has a Gamma distribution with the parameter (n, (n-c+1) θ). Consequently, the distribution of $\hat{\theta}_B$ with an inverted Gamma distribution has the following p.d.f.:

$$f\left(\hat{\theta}_B\right) = \frac{((n-c+1)\theta)^n \left(\frac{1}{\hat{\theta}_B}\right)^{n-1} e^{\frac{-(n-c+1)\theta}{\hat{\theta}_B}}}{\Gamma(n)}, \hat{\theta}_B > 0 \tag{8}$$

2. SELECTION OF REGIONS

The estimator depends on the choice of weighted contraction constant K and a suitable test area denoted by R. In this study, two regions are derived and compared based on the mean-squared error. The first region R_1 was evaluated using the criterion used in Ref. [11].

$$R_1 = \left\{\left(\hat{\theta}_{B1} - \theta_0\right)^2 < MSE(\hat{\theta}_{B1}/\theta_0)\right\} \tag{9}$$

Then, $$\left(\hat{\theta}_{B1} - \theta_0\right)^2 < \theta_0^2\left[\frac{(n_1-c+1)^2}{(n_1-1)(n_1-2)} - \frac{2(n_1-c+1)}{(n_1-1)} + 1\right]$$

and we further obtain $$R_1 = \left\{\theta_0\left(1-\sqrt{r}\right), \theta_0\left(1+\sqrt{r}\right)\right\} \tag{10}$$

Where $r = \frac{(n_1-c+1)^2}{(n_1-1)(n_1-2)} - \frac{2(n_1-c+1)}{(n_1-1)} + 1$

The second choice is region R_2, which was evaluated following the work of Ref. [3]. The region R is derived using the minimum mean square error (M.S.E) of $\tilde{\theta}_{BD}$ when $\theta = \theta_0$.

$$M.S.E\left(\tilde{\theta}_{BD}/\theta, R\right) = E(\tilde{\theta}_{BD} - \theta)^2$$

where $\tilde{\theta}_{BD}$ represents the Bayes double-stage shrinkage estimation.

$$M.S.E\left(\tilde{\theta}_{BD}/\theta, R\right) = \int_R (k(\hat{\theta}_{B1} - \theta_0) + \theta_0 - \theta)^2 f_1\left(\hat{\theta}_{B1}\right) d\left(\hat{\theta}_{B1}\right) + \int_0^\infty \int_0^\infty \left(\frac{n_1\hat{\theta}_{B1} + n_2\hat{\theta}_{B2}}{n_1+n_2} - \theta\right)^2 f_1\left(\hat{\theta}_{,B1}\right) f_2\left(\hat{\theta}_{,B2}\right) d\left(\hat{\theta}_{B1}\right) d\left(\hat{\theta}_{B2}\right) - \int_0^\infty \int_R \left(\frac{n_1\hat{\theta}_{B1} + n_2\hat{\theta}_{B2}}{n_1+n_2} - \theta\right)^2 f_1\left(\hat{\theta}_{B1}\right) f_2\left(\hat{\theta}_{B2}\right) d\left(\hat{\theta}_{B1}\right) d\left(\hat{\theta}_{B2}\right)$$

where $f_1\left(\hat{\theta}_{B1}\right) and f_2\left(\hat{\theta}_{B2}\right)$ are the p.d.f. of $\hat{\theta}_{B1} and \hat{\theta}_{B2}$, respectively.

Assume $\theta = \theta_0$. With some simplification,

$$= \left(k^2 - \frac{n_1^2}{(n_1+n_2)^2}\right)\int_R (\hat{\theta}_{B1} - \theta_0)^2 f_1\left(\hat{\theta}_{B1}\right) d\left(\hat{\theta}_{B1}\right) + \frac{n_1^2}{(n_1+n_2)^2} MSE(\hat{\theta}_{B1}/\theta_0) + \frac{n_2^2}{(n_1+n_2)^2} MSE(\hat{\theta}_{B2}/\theta_0) + \frac{2n_1n_2}{(n_1+n_2)^2} B(\hat{\theta}_{B1}/\theta_0) B(\hat{\theta}_{B2}/\theta_0) - \frac{n_2^2}{(n_1+n_2)^2} MSE(\hat{\theta}_{B2}/\theta_0) \int_R f_1\left(\hat{\theta}_{B1}\right) d\left(\hat{\theta}_{B1}\right) - \frac{2n_1n_2}{(n_1+n_2)^2} B(\hat{\theta}_{B2}/\theta_0) \int_R (\hat{\theta}_{B1} - \theta_0) f_1\left(\hat{\theta}_{B1}\right) d\left(\hat{\theta}_{B1}\right)$$

Then,

$$\frac{\partial MSE(\tilde{\theta}_{BD}/\theta, R)}{\partial R} = \left(k^2 - \frac{n_1^2}{(n_1+n_2)^2}\right)(\hat{\theta}_{B1} - \theta_0)^2 - \frac{n_2^2}{(n_1+n_2)^2} MSE(\hat{\theta}_{B2}/\theta_0) - \frac{2n_1n_2}{(n_1+n_2)^2} B(\hat{\theta}_{B2}/\theta_0)(\hat{\theta}_{B1} - \theta_0) = 0$$

The simple calculation leads to

$$R_2 = \{Max(0, \theta_0(1-Q)), \theta_0(1+Q)\}(11) \tag{11}$$

where $$Q = \frac{\sqrt{b} + n_1n_2(2-c)\sqrt{n_2-2}}{\left(N^2k^2 - n_1^2\right)(n_2-1)\sqrt{n_2-2}}$$

and $b = n_2^2\left(N^2k^2 - n_1^2\right)(n_2-1)\left[(n_2-c+1)^2 - 2(n_2-c+1)(n_2-2) + (n_2-2) + (n_2-1)(n_2-2)\right] + n_1^2n_2^2(2-c)^2(n_2-2)$

3. PROPOSED ESTIMATORS

The first proposed shrinkage estimator is denoted by

$$\tilde{\theta}_{BD1} = \begin{cases} k_1\hat{\theta}_{B1} + (1-k_1)\theta_0 & if\hat{\theta}_{B1} \in R_1 \\ \frac{n_1\hat{\theta}_{B1}+n_2\hat{\theta}_{B2}}{n_1+n_2} & otherwise, \end{cases} \tag{12}$$

where k_1 is a constant between 0 and 1, i.e., ($0 < k_1 < 1$), and R_1 is defined by Eq. (10).
The second proposed shrinkage estimator is defined as

$$\tilde{\theta}_{BD2} = \begin{cases} k_2\hat{\theta}_{B1} + (1-k_2)\theta_0 & if\hat{\theta}_{B1} \in R_1 \\ \frac{n_1\hat{\theta}_{B1}+n_2\hat{\theta}_{B2}}{n_1+n_2} & otherwise, \end{cases} \tag{13}$$

where k_2 is obtained using the M.S.E. of the estimator, as shown in Eq. (12), with respect to k_1.
Then, $\frac{\partial}{\partial K_1}MSE\left(\tilde{\theta}_{BD1}|\theta\right) = 0$.

$$Thus, k^* = \frac{(\mathbf{1}-\lambda)\left[\frac{(n_1-C+1)}{\Gamma(n_1)}\int_{R_j^*} w^{n_1-2}\exp^{-W}dW - \frac{\lambda}{\Gamma(n_1)}\int_{R_j^*} w^{n_1-2}\exp^{-W}dW\right]}{\left[\frac{(n_1-c+1)^2}{\Gamma(n_1)}\int_{R_j^*} w^{n_1-3}\exp^{-W}dW - -\frac{2(n_1-C+1)\lambda}{\Gamma(n_1)}\int_{R_j^*} w^{n_1-2}\exp^{-W}dW + \frac{\lambda^2}{\Gamma(n_1)}\int_{R_j^*} w^{n_1-1}\exp^{-W}dW\right]} \tag{14}$$

Theoretically, we cannot show whether k^* lies between 0 and 1. Therefore, the k value is a restriction.

$$k_2 = \begin{cases} 0 & k^* < 0, \\ k^* & 0 \le k^* \le 1, \\ 1 & k^* > 1, \end{cases} \tag{15}$$

The third proposed shrinkage estimator is defined as

$$\tilde{\theta}_{BD3} = \begin{cases} k_3\hat{\theta}_{B1} + (1-k_3)\theta_0 & if\hat{\theta}_{B1} \in R_2 \\ \frac{n_1\hat{\theta}_{B1}+n_2\hat{\theta}_{B2}}{n_1+n_2} & otherwise, \end{cases} \tag{16}$$

where R_2 is defined according to Eq. (11). When the R_2 region is used, the shrinkage factor is $k_3 \ge \frac{n_1}{N}$.
Thus, the equation leads to

$$k_3 = \frac{n_1}{N} + 10^{-q} \tag{17}$$

where q is a constant, i.e., $q=1,2,3$, and $N= n_1+n_2$.

4. BIAS AND MEAN SQUARE ERROR OF THE SUGGESTED ESTIMATOR

The bias of the proposed estimators is defined as

$$Bias\left(\tilde{\theta}_{BDi}|\theta\right) = E\left(\tilde{\theta}_{BDi}\right) - \theta c, i = 1,2,3$$

$$= \int_0^\infty \int_{R_j}\left[K\left(\hat{\theta}_{B1}-\theta_0\right)+\theta_0\right] f_1\left(\hat{\theta}_{B1}\right) f_2\left(\hat{\theta}_{B2}\right) d\hat{\theta}_{B1} d\hat{\theta}_{B2} + \int_0^\infty \int_{R_j}\left[\frac{n_1\hat{\theta}_{B1}+n_2\hat{\theta}_{B2}}{n_1+n_2}\right] f_1\left(\hat{\theta}_{B1}\right) f_2\left(\hat{\theta}_{B2}\right) d\hat{\theta}_{B1} d\hat{\theta}_{B2} - \theta$$

where $j=1,2$

Then, by transforming $W = \frac{(n_1-c+1)\theta}{\hat{\theta}_{B1}} and M = \frac{(n_2-c+1)\theta}{\hat{\theta}_{B2}}, we\ obtain$

$$Bias\left(\tilde{\theta}_{BDi}|\theta\right) = \theta\left[\frac{k(n_1-c+1)}{\Gamma(n_1)}\int_{R_j^*} w^{n_1-2}\exp^{-W}dW + \frac{(1-k)\lambda}{\Gamma(n_1)}\int_{R_j^*} w^{n_1-1}\exp^{-W}dW + \frac{n_1(n_1-c+1)}{(n_1+n_2)(n_1-1)} + \frac{n_2(n_2-c+1)}{(n_1+n_2)(n_2-1)} - \frac{n_1(n_1-c+1)}{(n_1+n_2)\Gamma(n_1)}\int_{R_j^*} W^{n_1-2}\exp^{-W}dW + \frac{n_2(n_2-c+1)}{(n_1+n_2)(n_2-1)\Gamma(n_1)}\int_{R_j^*} W^{n_1-1}\exp^{-W}dW - 1\right] \tag{18}$$

where R_j^* is equal to R_j after the above transformation, i.e.,

$R_1^* = \left\{\frac{(n_1-c+1)}{\lambda(1+\sqrt{r})}, \frac{(n_1-c+1)}{\lambda(1-\sqrt{r})}\right\} and R_2^* = \left\{\frac{(n_1-c+1)}{\lambda(1+Q)}, \frac{(n_1-c+1)}{\max[0,\lambda(1-Q)]}\right\}$

We also define the bias ratio for the estimators $\tilde{\theta}_{BDi}$ as

$$BR\left(\tilde{\theta}_{BDi}\right)=\frac{Bias\left(\tilde{\theta}_{BDi}|\theta\right)}{\theta} \qquad ,i=1,2,3 \tag{19}$$

The M.S.E. is further defined as

$$MSE\left(\tilde{\theta}_{BDi}\right)=E(\tilde{\theta}_{BDi}-\theta)^2$$

$$=\int_{R_j}\left(k_1\left(\hat{\theta}_{B1}-\theta_0\right)+\theta_0-\theta\right)^2 f_1\left(\hat{\theta}_{B1}\right)d\left(\hat{\theta}_{B1}\right)+\int_0^\infty\int_0^\infty\left(\frac{n_1\hat{\theta}_{B1}+n_2\hat{\theta}_{B2}}{n_1+n_2}-\theta\right)^2 f_1\left(\hat{\theta}_{B1}\right)f_2\left(\hat{\theta}_{B2}\right)d\left(\hat{\theta}_{B1}\right)d\left(\hat{\theta}_{B2}\right)-\int_0^\infty\int_{R_j}\left(\frac{n_1\hat{\theta}_{B1}+n_2\hat{\theta}_{B2}}{n_1+n_2}-\theta\right)^2 f_1\left(\hat{\theta}_{B1}\right)f_2\left(\hat{\theta}_{B2}\right)d\left(\hat{\theta}_{B1}\right)d\left(\hat{\theta}_{B2}\right)$$

Furthermore, by transforming $W=\frac{(n_1-c+1)\theta}{\hat{\theta}_{B1}} and M=\frac{(n_2-c+1)\theta}{\hat{\theta}_{B2}}, we obtain$

$$\begin{aligned}MSE\left(\tilde{\theta}_{BDi}\right)=\theta^2\Bigg[&k^2\left(\frac{(n_1-c+1)^2}{\Gamma(n_1)}\int_{R_j^*}W^{n_1-3}\exp^{-W}dW-\right.\\&\left.\frac{2(n_1-c+1)\lambda}{\Gamma(n_1)}\int_{R_j^*}W^{n_1-2}\exp^{-w}dW+\frac{\lambda^2}{\Gamma(n_1)}\int_{R_j^*}W^{n_1-1}\exp^{-W}dW\right)-2k(1-\lambda)\\&\left(\frac{(n_1-c+1)}{\Gamma(n_1)}\int_{R_j^*}W^{n_1-2}\exp^{-W}dW-\frac{\lambda}{\Gamma(n_1)}\int_{R_j^*}W^{n_1-1}\exp^{-W}dW\right)+\\&\frac{(1-\lambda)^2}{\Gamma(n_1)}\int_{R_j^*}W^{n_1-1}\exp^{-W}dW+\frac{n_1^2(n_1-c+1)^2}{(n_1+n_2)^2(n_1-1)(n_1-2)}-\frac{2n_1^2(n_1-c+1)}{(n_1+n_2)^2(n_1-1)}+\\&\frac{n_1^2}{(n_1+n_2)^2}+\frac{2n_1n_2(2-c)^2}{(n_1+n_2)^2(n_1-1)(n_2-1)}+\frac{n_2^2(n_2-c+1)^2}{(n_1+n_2)^2(n_2-1)(n_2-2)}-\frac{2n_2^2(n_2-c+1)}{(n_1+n_2)^2(n_2-1)}-\\&\frac{n_1^2(n_1-c+1)^2}{(n_1+n_2)^2\Gamma(n_1)}\int_{R_j^*}W^{n_1-3}\exp^{-W}dW+\frac{n_2^2}{(n_1+n_2)^2}+\frac{2n_1^2(n_1-c+1)}{(n_1+n_2)^2\Gamma(n_1)}\int_{R_j^*}W^{n_1-2}\exp^{-W}dW-\\&\frac{n_1^2}{(n_1+n_2)^2\Gamma(n_1)}\int_{R_j^*}W^{n_1-1}\exp^{-W}dW-\frac{2n_1n_2(2-c)}{(n_1+n_2)^2(n_2-1)}\left(\frac{(n_1-c+1)}{\Gamma(n_1)}\int_{R_j^*}W^{n_1-2}\exp^{-W}dW-\frac{1}{\Gamma(n_1)}\int_{R_j^*}W^{n_1-1}\exp^{-W}dW\right)-\\&\frac{n_2^2(n_2-c+1)^2}{(n_1+n_2)^2(n_2-1)(n_2-2)}\left(\int_{R_j^*}W^{n_1-1}\exp^{-W}dW\right)+\frac{2n_2^2(n_2-c+1)}{(n_1+n_2)^2(n_2-1)}\left(\frac{1}{\Gamma(n_1)}\int_{R_j^*}W^{n_1-1}\exp^{-W}dW\right)-\\&\frac{n_2^2}{(n_1+n_2)^2\Gamma(n_1)}\int_{R_j^*}W^{n_1-1}\exp^{-W}dW\Bigg]\end{aligned} \tag{20}$$

5. RELATIVE EFFICIENCY AND EXPECTED SAMPLE SIZE

The expected sample size is defined as

$$\begin{aligned}E\left(n|R_j\right)&=n-(n-n_1)\,p(\hat{\theta}_{B1}\in R_j)\\&=n-\frac{n_2}{\Gamma(n_1)}\int_{R_j^*}W^{n_1-1}\exp^{-W}dW \quad ,j=1,2\end{aligned} \tag{21}$$

To study the properties of the estimators $\tilde{\theta}_{\mathbf{BD1}}$,$\tilde{\theta}_{\mathbf{BD2}}$,**and**$\tilde{\theta}_{\mathbf{BD3}}$, we compare the relative efficiencies of the estimators given above with the pooled Bayes estimator $\hat{\theta}_D$. For this purpose, the expression for $M.S.E(\hat{\theta}_D)$ IS(expression for 2)) so; f thee of having asmaller is obtain as

$$M.S.E\left(\hat{\theta}_D\right)=E\left(\hat{\theta}_D-\theta\right)^2=\int_0^\infty\int_0^\infty\left(\hat{\theta}_D-\theta\right)^2 f_1\left(\hat{\theta}_{B1}\right)f_2\left(\hat{\theta}_{B2}\right)d\left(\hat{\theta}_{B1}\right)d\left(\hat{\theta}_{B2}\right)$$

By evaluating the integral, we obtain

$$\begin{aligned}M.S.E\left(\hat{\theta}_D\right)=\theta^2\Bigg[&\frac{n_1^2(n_1-c+1)^2}{(n_1+n_2)^2(n_1-1)(n_1-2)}-\frac{2n_1^2(n_1-c+1)}{(n_1+n_2)^2(n_1-1)}+\frac{n_1^2}{(n_1+n_2)^2}+\frac{2n_1n_2(2-c)^2}{(n_1+n_2)^2(n_1-1)(n_2-1)}+\\&\frac{n_2^2(n_2-c+1)^2}{(n_1+n_2)^2(n_2-1)(n_2-2)}-\frac{2n_2^2(n_2-c+1)}{(n_1+n_2)^2(n_2-1)}+\frac{n_2^2}{(n_1+n_2)^2}\Bigg]\end{aligned} \tag{22}$$

Consequently, the relative efficiencies of the estimators $\tilde{\theta}_{BDi}; i=1,2,3$ with respect to $\hat{\theta}_D$ is given by

$$REE\left(\tilde{\theta}_{BDi}\right)=\frac{MSE\left(\hat{\theta}_D\right)}{MSE\left(\tilde{\theta}_{BDi}\right)};i=1,2,3 \tag{23}$$

The findings indicate that the equation for the relative efficiency of the estimators $\tilde{\theta}_{BDi}; i = 1,2,3$ with respect to the Bayesian pooled estimator $\hat{\theta}_{BD}$, bias ratio, and expected sample size are dependent on n_1, n_2, c, λ, α, and k_1. Here, we considered the following values of the constants: n_1=6,12,18; n_2=6,12,18; c=1,2,3; λ=0.25 (0.25) 1.75; k_1=0.1, 0.5; and α=0.01,0.05. R language is used for the statistical programming. The derived numerical results and conclusions are based on the above values.

6. NUMERICAL RESULTS

1. The suggested estimators perform better than the Bayesianpooled estimator in the neighborhood of λ =1.

2. The relative efficiency of the estimator $\tilde{\theta}_{BD1}$ with respect to Bayesian pooled estimator $\hat{\theta}_D$ is a decreasing function of k, as shown in Figure (1).

3. The relative efficiency of the estimators $\tilde{\theta}_{BD1} and \tilde{\theta}_{BD2}$ with respect to the Bayesian pooled estimator $\hat{\theta}_D$ are an increasing function of n_1, as shown in Figures (1) and (3).

4. The relative efficiency of the estimators $\tilde{\theta}_{BD1} and \tilde{\theta}_{BD2}$ with respect to the Bayesian pooled estimator $\hat{\theta}_D$ are a decreasing function of n_2, as shown in Figures (5) and (6).

5. The relative efficiency of the estimators $\tilde{\theta}_{BD1} and \tilde{\theta}_{BD2}$ with respect to the Bayesian pooled estimator $\hat{\theta}_D$ is an increasing function of c, as shown in Figures (2) and (4).

6. The estimates $\tilde{\theta}_{BD1}$ at k = 0.1 and $\tilde{\theta}_{BD2}$ with respect to the Bayesian pooled estimator $\hat{\theta}_D$ give usapproximately similar behavior results in for the relative efficiency, as shown in Figure (5).

7. The relative efficiency of the estimator $\tilde{\theta}_{BD2}$ with respect to the Bayesian pooled estimator $\hat{\theta}_D$ is higher than the estimator $\tilde{\theta}_{BD1}$ when k = 0.5, as shown in Figure (6).

8. The relative efficiency of the estimator $\tilde{\theta}_{BD3}$ with respect to the Bayesian pooled estimator $\hat{\theta}_D$ is an increasing function of q, i.e., the relative efficiency is higher when q = 3, as shown in Figures (7), (8), and (10).

9. The relative efficiency of the estimator $\tilde{\theta}_{BD3}$ with respect to the Bayesian pooled estimator $\hat{\theta}_D$ is a decreasing function of c, as shown in Figure (9).

10. The relative efficiency of the estimator $\tilde{\theta}_{BD3}$ with respect to the Bayesian pooled estimator $\hat{\theta}_D$ is an increasing function of n_1, as shown in Figures (7) and (8).

11. The relative efficiency of the estimator $\tilde{\theta}_{BD3}$ with respect to the Bayesian pooled estimator $\hat{\theta}_D$ is a decreasing function of n_2, as shown Figures (7) and (8).

12. The bias ratio of the estimators$\tilde{\theta}_{BD1}, \tilde{\theta}_{BD2}, and \tilde{\theta}_{BD3}$ is reasonable in the neighborhood of λ=1, and it isslightly increasing when (λ >1) , as shown in Figures (11) to (16).

13. The expected sample size is concave when $\lambda = 1$ with respect to R_1, as shown in Figure (17).

14. The expected sample size is concave when $(1 < \lambda)$ at q = 1 with respect to R_2 as shown in Figure (18).

7. CONCLUSIONS

- In general, the double-stage shrinkage Bayesian estimators proposed in this work perform better than the Bayesian pooled estimator, both under R_1 and R_2.

- The effective region of the estimator $\tilde{\theta}_{BD3}$ at q = 3 is wider compared with those of the estimators $\tilde{\theta}_{BD1} and \tilde{\theta}_{BD2}$, as shown in Figures (5), (6), (8), and (9).

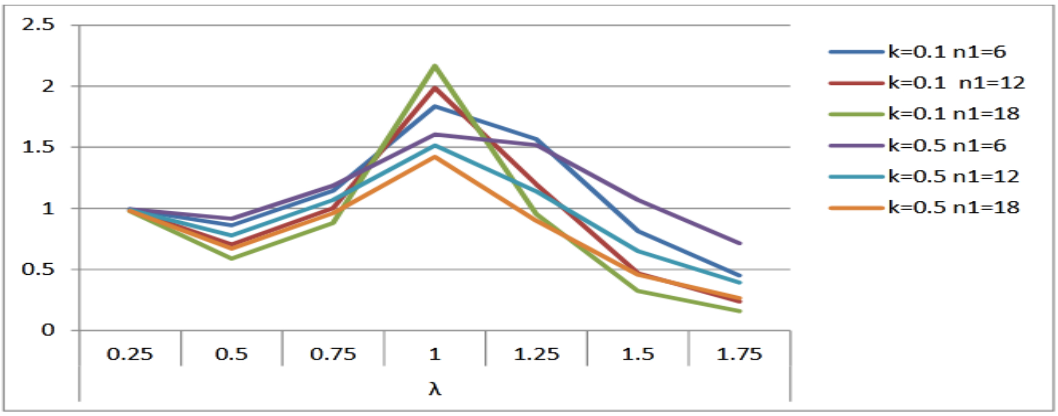

FIGURE 1. **Relative efficiency of estimator θBD1 when c=1**

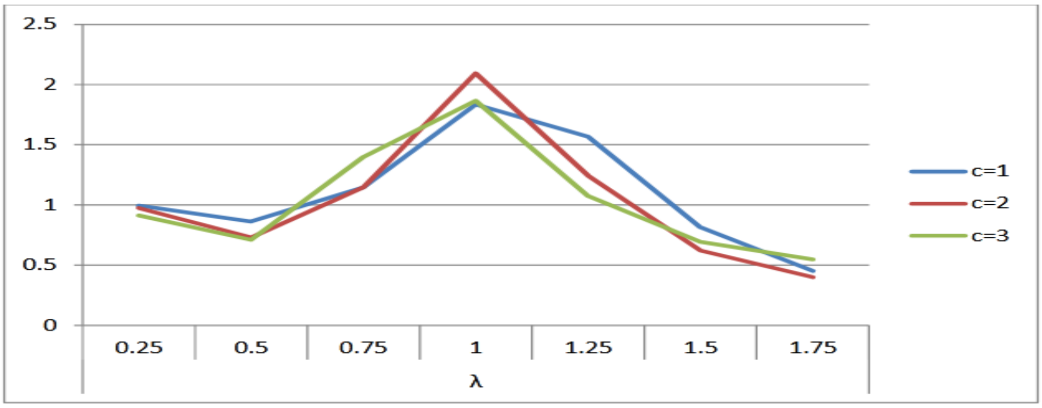

FIGURE 2. **Relative efficiency of estimator θBD1 when, k=0.1, n1=6, n2=6**

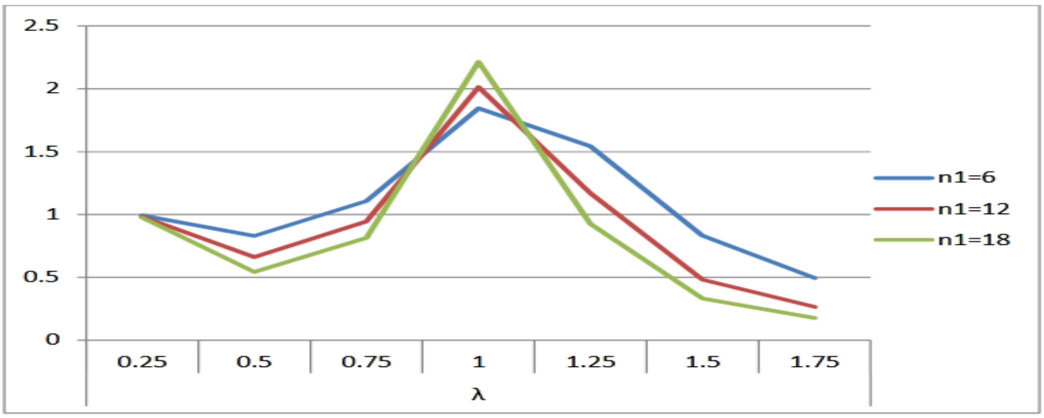

FIGURE 3. **Relative efficiency of estimator $\tilde{\theta}_{BD2}$ when, c=1, n_2=6**

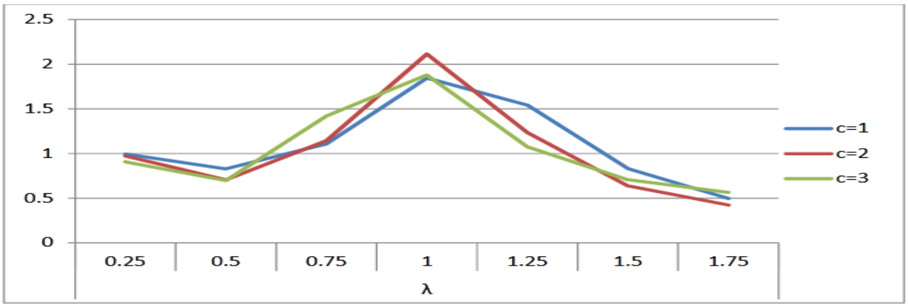

FIGURE 4. **Relative efficiency of estimator θBD2 when, n1=6 , n2=6**

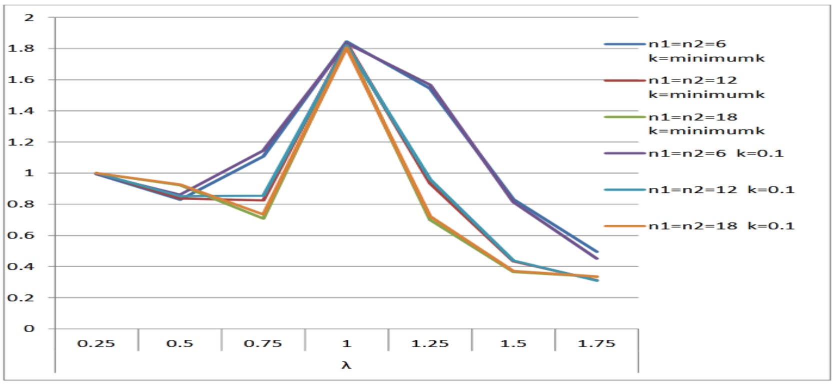

FIGURE 5. **Relative efficiency of estimators θBD1 and θBD2 when c=1**

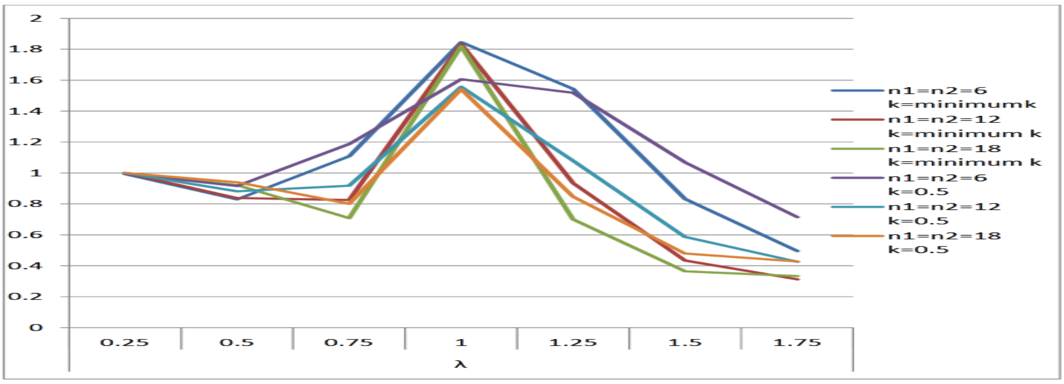

FIGURE 6. **Relative efficiency of estimators θBD1 and θBD2 when c=1**

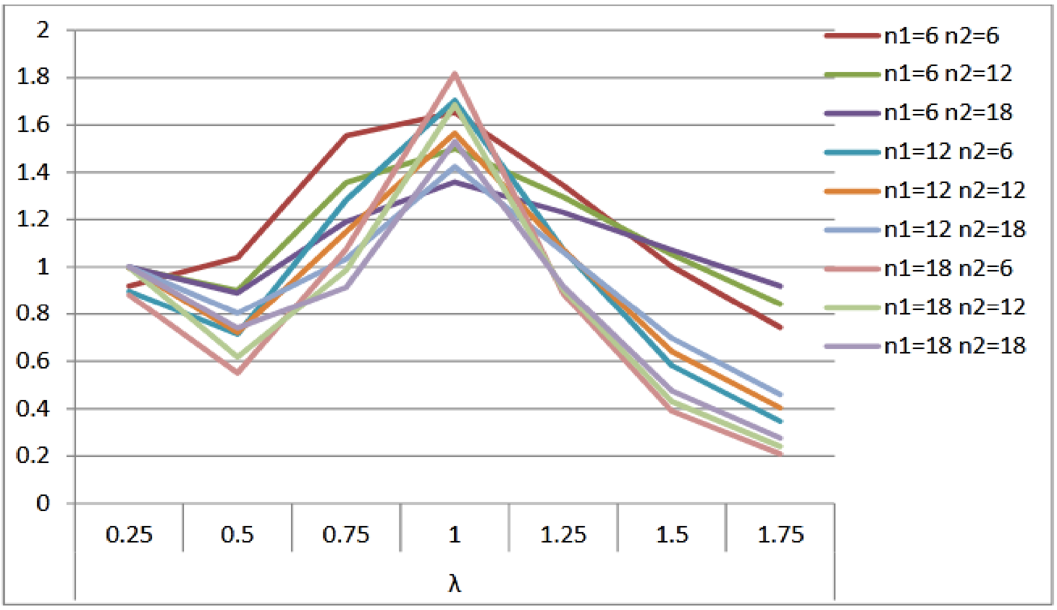

FIGURE 7. **Relative efficiency of estimator θBD3 for q=1, c=1**

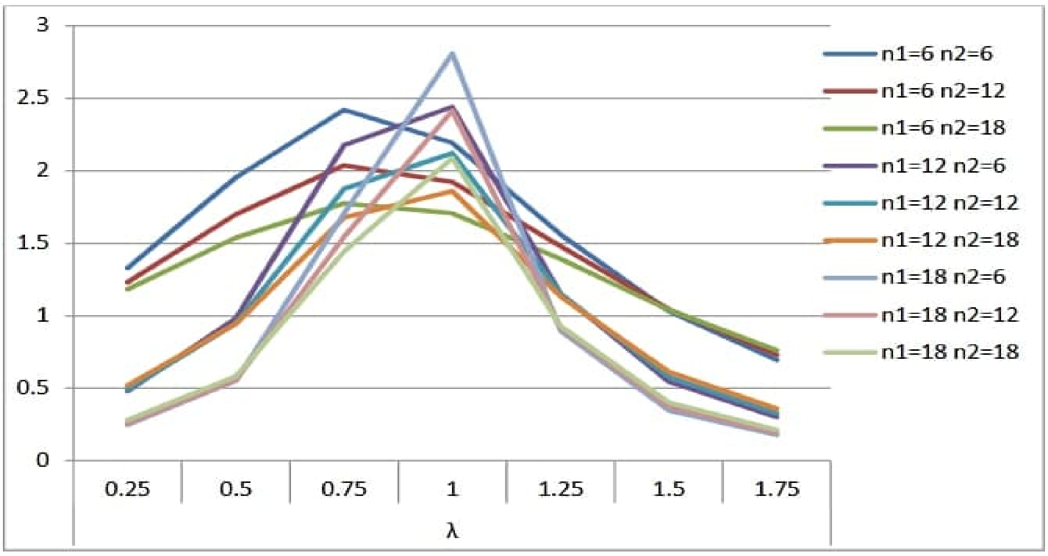

FIGURE 8. **Relative efficiency of estimator $\tilde{\theta}_{BD3}$ for q=3, c=1**

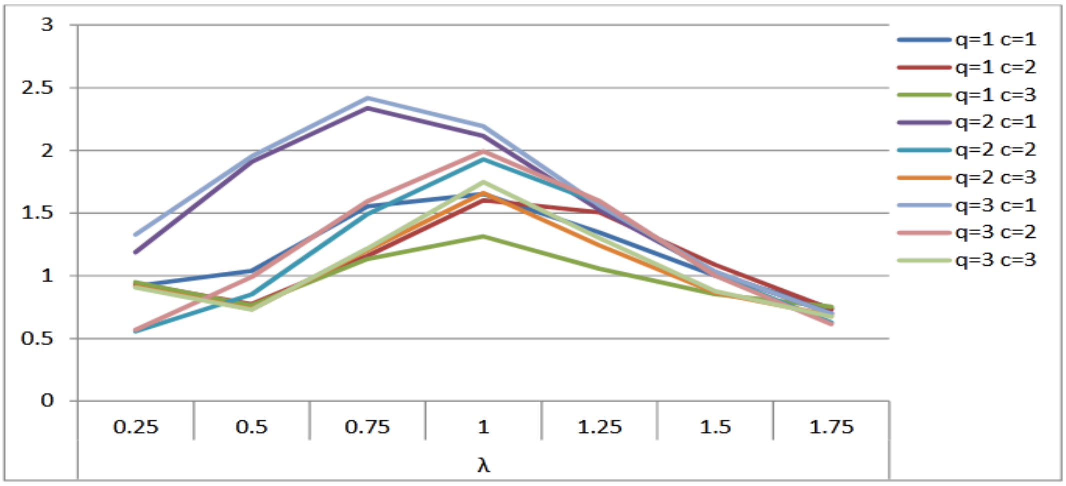

FIGURE 9. Relative efficiency of estimator θBD3 for n1=6, n2=6

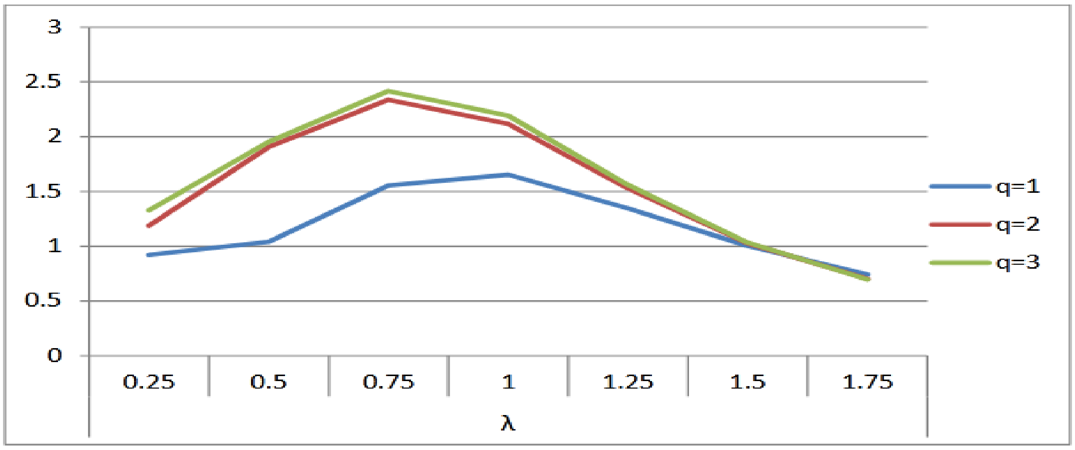

FIGURE 10. Relative efficiency of estimator $\tilde{\theta}_{BD3}$ for c=1, n_1=6, n_2=6

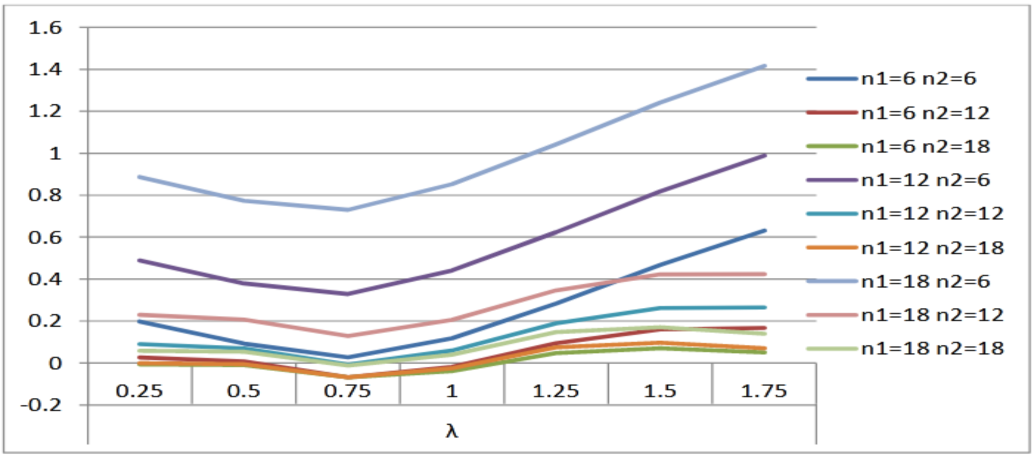

FIGURE 11. Bias Ratio of estimator $\tilde{\theta}_{BD1}$ for c=1, k=0.1

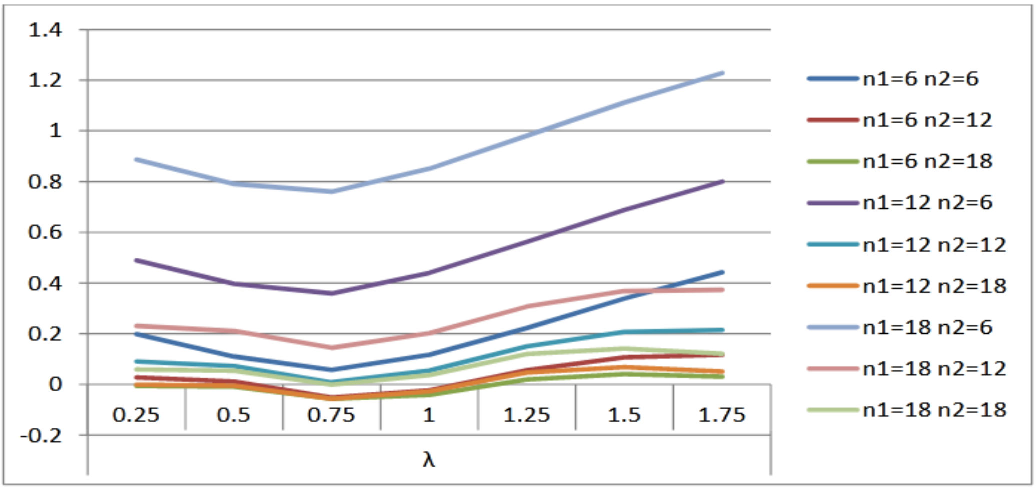

FIGURE 12. **Bias Ratio of estimator θBD1 for c=1, k=0.5**

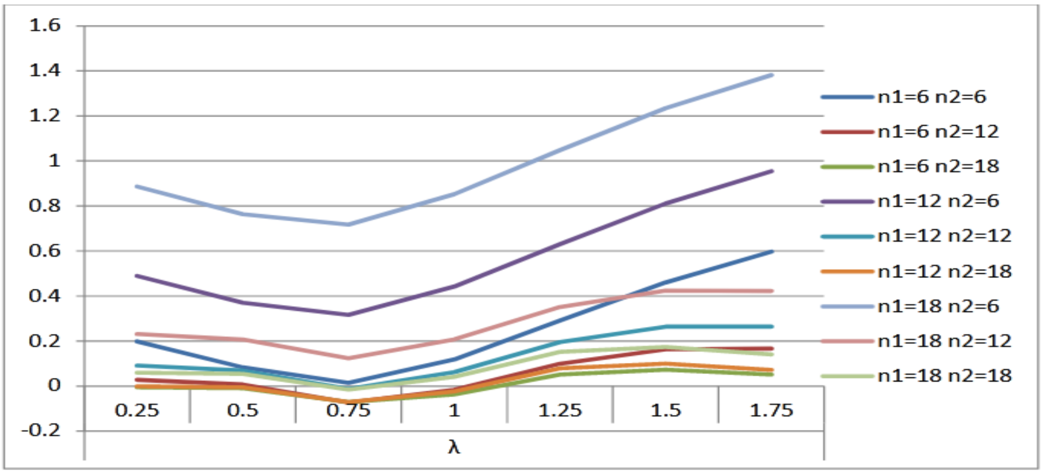

FIGURE 13. **Bias Ratio of estimator $\tilde{\theta}_{BD2}$ for c=1**

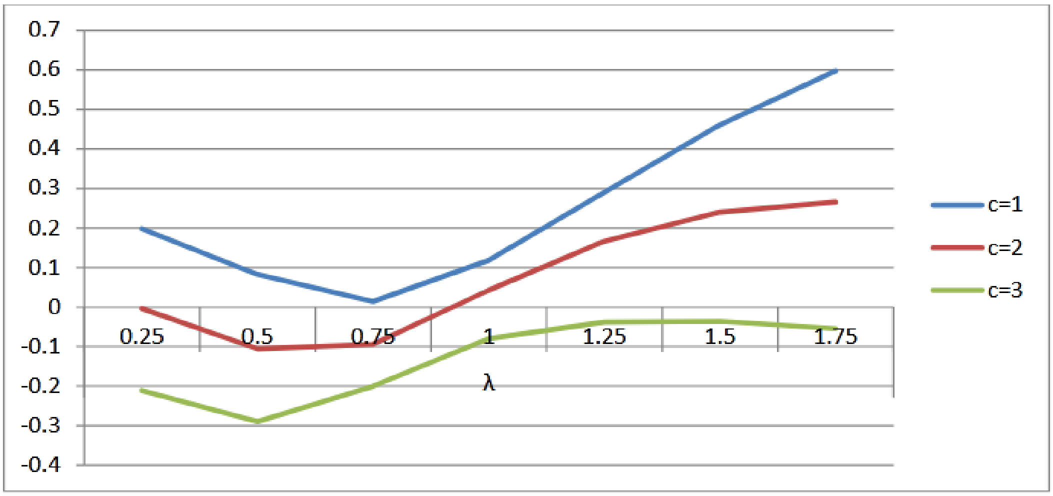

FIGURE 14. **Bias Ratio of estimator $\tilde{\theta}_{BD2}$ for n_1=6, n_2=6**

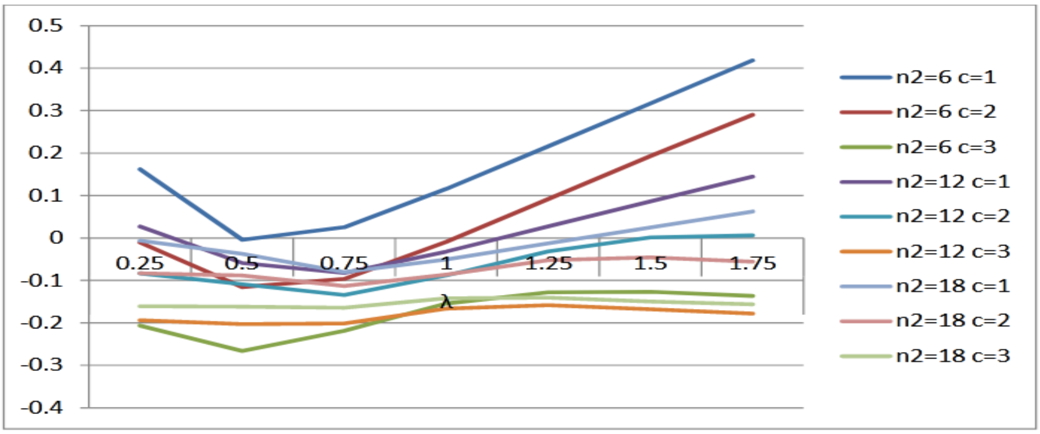

FIGURE 15. **Bias Ratio of estimator $\tilde{\theta}_{BD3}$ for n_1=6, q=1**

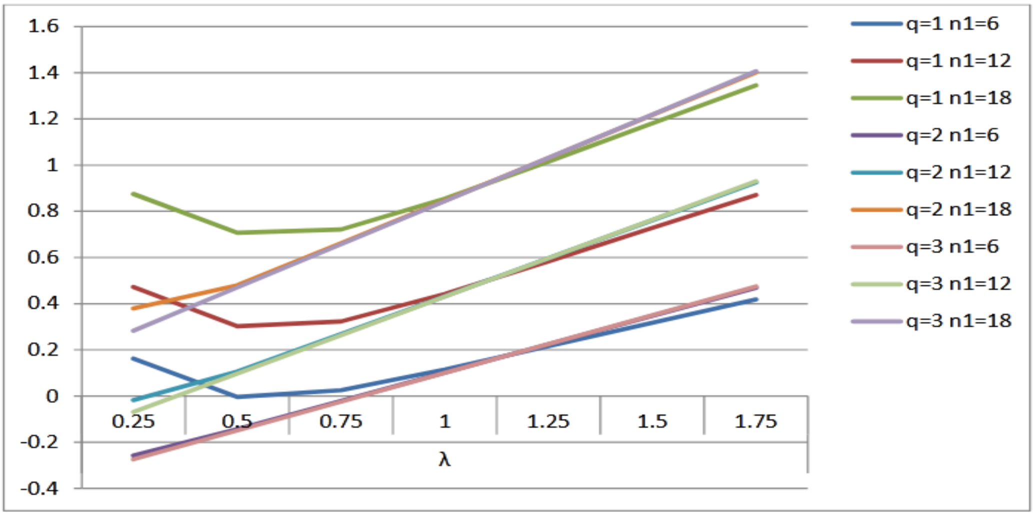

FIGURE 16. **Bias Ratio of estimator $\tilde{\theta}_{BD3}$ for n_2=6, c=1**

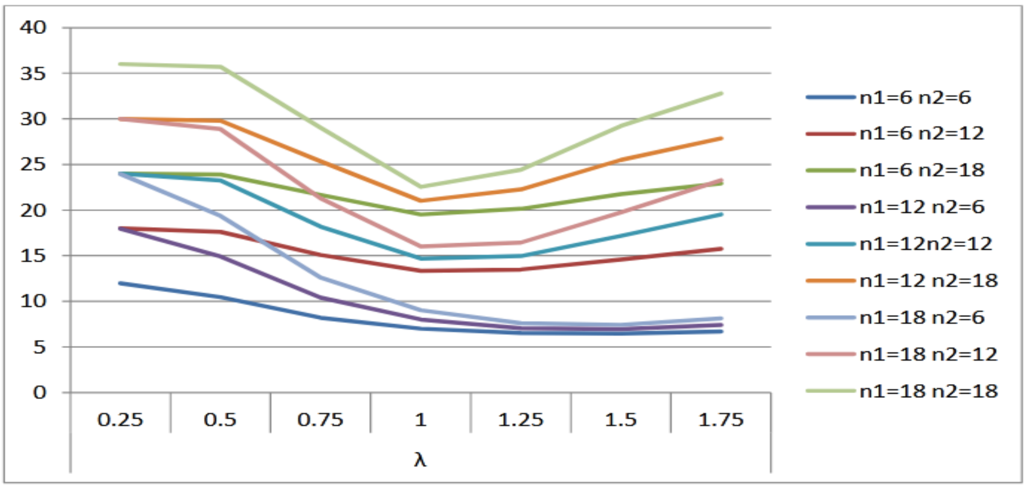

FIGURE 17. **Expected sample size of R_1**

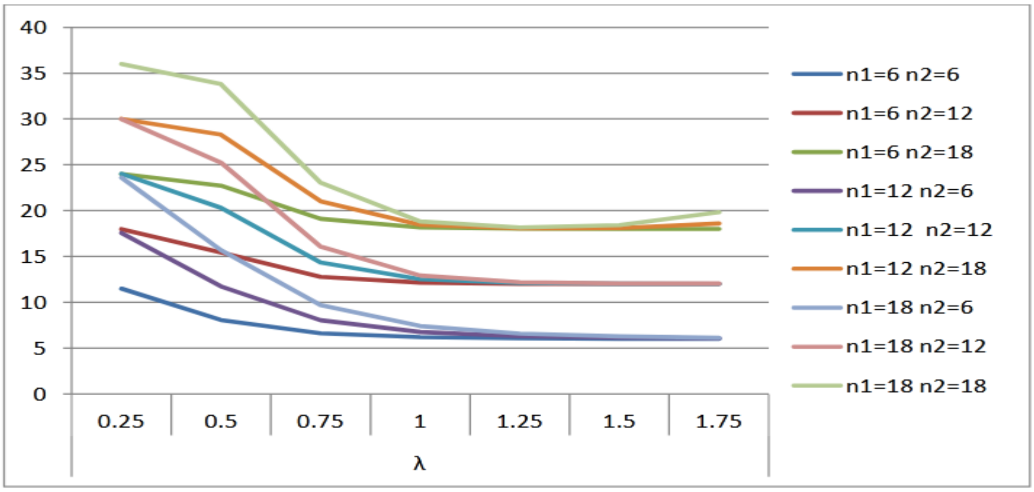

FIGURE 18. **Expected sample size of R_2**

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CONFLICTS OF INTEREST

The authors declare no conflict of interest.

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