

Certain Types of Regular and T_3 Spaces

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Abstract: We introduced certain types of separation axioms which are Z-regular space, ZT_3 space, semi-Z-regular space and semi- ZT_3 space, by using boundary points and semi boundary points. The relationships among them and relationships with each of T_3 , T_1 and semi- T_1 axioms have been given.

Keywords T_1 space, semi- T_1 space, regular space, T_3 , semi-regular space, semi- T_3 , Z-regular space, ZT_3 space, semi-regular space, semi- ZT_3 space.

1 Introduction

There are many types of the separation axioms: T_0 , T_1 , T_2 , regular, T_3 , semi- T_1 , semi- T_2 , semi-regular, and semi- T_3 . T_0 was introduced by A. Kolmogorov, T_1 was introduced by Frechet in 1923, T_2 was introduced by Hausdorff in 1914,[15]. Semi- T_0 , semi- T_1 , and semi- T_2 were introduced by S.N. Maheshwari and R. Prasad in 1975,[4]. Regular and T_3 space were introduced by V. Neumann in 1925,[15], and semi-regular and semi- T_3 space were introduced by C. Dorsett in 1982, [7]. In [12], Levine defined a set A in a topological space X to be semi-open if there exists an open set U such that $U \subseteq A \subseteq \bar{U}$. In [3], S. Gene Crossley and S. K.

Hildebrand defined semi-closed set as the complement of semi-open set, the union of all semi-open sets of X contained in A and the intersection of all semi-closed sets containing A , respectively. In [13], P.Das defined semi-boundary of A as $\bar{A}^s - A^{os}$. We shall use $\bar{A}^s$, A^{bs} , $\bar{A}$, A^b , and $\mathbb{Z}$ to the semi-closure of A , the semi-boundary of A , the closure of A , the boundary of A , and the integer numbers, respectively. We shall refer to the topological space as a space.

2 Preliminaries

Definition 2.1[[11]] A space (X, τ) be defined a T_1 space if and only if for any distinct points x and y in X ,

there exists open sets U, V such that $x \in U, y \notin U, x \notin V$ and $y \in V$.

Definition 2.2[5] A space (X, τ) is called a semi- T_1 space if and only if for any distinct points x, y in X , there are semi-open sets U and V such that $x \in U, y \notin U, x \notin V$ and $y \in V$.

Definition 2.3 [11] A space (X, τ) is called a regular space if and only if for every closed set F and every point $x \notin F$, there exists disjoint sets $U, V \in \tau, F \subseteq U$, and $x \in V$.

A regular T_1 space is called a T_3 space.

Definition 2.4 [7] A space (X, τ) be defined semi-regular space if and only if for every $x \in X$ and semi-closed set F , with $x \notin F$, there exists disjoint semi-open sets U and V such that $F \subseteq U, x \in V$.

A semi-regular semi- T_1 space is called a semi- T_3 space.

Theorem 2.5[2] If $f : X \rightarrow Y$ is a homeomorphism, then $f(A^b) = (f(A))^b$, for all $A \subset X$

Theorem 2.6[10] If $f : X \rightarrow Y$ be a semi-homeomorphism, then $f(A^{bs}) = (f(A))^{bs}$, for all $A \subset X$.

Theorem 2.7 [14] The product of semi-open(semi-closed) sets is a semi-open(semi-closed) set.

proposition 2.8 [11] In any space, $A^b = \phi$ iff A is a clopen set.

Theorem 2.9 [10] In a space X , A is a semi-clopen (semi-open and semi-closed) set iff $A^{bs} = \phi$.

Remark 2.10 [9] Every open (closed) set is semi-open (semi-closed) set.

Remark 2.11[3] Let X be a space and

A be a subset of X , then A is semi-closed set iff $A = \overline{A}^s$.

Remark 2.12[14] The concept of semi-open set and open set in cofinite space, discrete space and indiscrete space are the same.

Theorem 2.13[8] Let (X, τ) be a space (Y, τ_y) a subspace of (X, τ) and let $A \subseteq Y$, then:

$$\text{i. } \overline{A}^y = Y \cap \overline{A}$$

$$\text{ii. } A^{b_y} \subseteq Y \cap A^b$$

Where $\overline{A}^y$ and A^{b_y} are the closure of A in Y and boundary of A in Y , respectively.

Theorem 2.14[11] Let (X, τ) be a space and let $A, B \subseteq X$, then $(A \cap B)^b \subseteq A^b \cup B^b$.

Remark 2.15 [10] In a space X , $A^{bs} \subset A^b$

Theorem 2.16[11],[6] The product space of T_1 (semi- T_1) spaces is T_1 (semi- T_1) space.

Theorem 2.17[3] A semi-topological property is a topological property.

3 Main Results

Definition 3.1 A space (X, τ) be defined a Z-regular space if and only if for every closed set F and every point $x \notin F$, there exists disjoint open sets U and V such that $F \subseteq U, x \in V$, and $U^b \cap V^b = \phi$.

A Z-regular T_1 space is called a ZT_3 space.

Note that if (X, τ) is Z-regular space then it is regular space and if X is ZT_3 space then it is a T_3 space and Z-regular space.

Example 3.2 Let $X = \{a, b, c\}, \tau = \{X, \phi, \{a\}, \{b, c\}, \{a, b, c\}\}$, (X, τ) is a **Z-regular** space:

Consider the closed set $\{b, c\}$ and $a \notin \{b, c\}$, $\{a\}$ and $\{b, c\}$ are disjoint open sets, $\{b, c\} \subseteq \{b, c\}, a \in \{a\}$ and $\{a\}^b \cap \{b, c\}^b = \phi$.

The closed sets $\{a\}$ with each of b and c , there exist disjoint open sets $\{a\}$ and $\{b, c\}$ such that $\{a\} \subseteq \{a\}, b \in \{b, c\}$ and $c \in \{b, c\}$, respectively, and $\{a\}^b \cap \{b, c\}^b = \phi$. Hence, it is **Z-regular** space.

Note that (X, τ) is not ZT_3 space for it is not T_1 space.

Example 3.3 The discrete space of more than one point, (X, τ_d) , is **Z-regular** space.

Example 3.4 The discrete space of more than one point, (X, τ_d) , is ZT_3 space.

Example 3.5 An infinite Cofinite space (X, τ_c) is not **Z-regular** space since there are no disjoint open sets in the cofinite space except X and ϕ .

Example 3.6 An infinite Cofinite space (X, τ_c) is not ZT_3 since it is not **Z-regular**.

Example 3.7 The indiscrete space of more than one point, (X, τ_{ind}) is **Z-regular** space, but not ZT_3 space.

Example 3.8 The Usual space, $(\mathbb{R}, \tau_u)$, is ZT_3 space.

Theorem 3.9 The property of being **Z-regular** be a topological property.

Proof: Suppose that $X \cong Y$, where X is **Z-regular** space. Then there exists a homeomorphism $f: X \rightarrow Y$.

To show that Y be a **Z-regular** space,

let $F \subset Y$ be a closed set and $a \in Y$ with $a \notin F$.

$f^{-1}(F)$ is closed set in X . Then there exists $b \in X$ such that $f(b) = a$.

Since $a \notin F$ then $b \notin f^{-1}(F)$. Since X is a **Z-regular** space then there exist disjoint open sets U and V such that $f^{-1}(F) \subset U, b \in V$, and $U^b \cap V^b = \phi$. So $f(f^{-1}(F)) = F \subseteq f(U)$, and $f(b) = a \in f(V)$.

$f(U)$ and $f(V)$ are open in Y , and $f(U \cap V) = f(U) \cap f(V) = f(\phi) = \phi$.

Now, $(f(U))^b \cap (f(V))^b = f(U^b) \cap f(V^b)$, [Theorem 2.5)]

$$= f(U^b \cap V^b)$$

$$= f(\phi) = \phi.$$

Hence (Y, τ^*) is **Z-regular** space. □

Theorem 3.10 The property of being ZT_3 is a topological property.

Proof: Since **Z-regular** space and T_1 space is a topological properties then ZT_3 is a topological property. □

Theorem 3.11 An open subspace of a **Z-regular** space be a **Z-regular** space.

Proof: Suppose that (Y, τ_y) be an open subspace of (X, τ) , where X be a **Z-regular** space. To show that (Y, τ_y) is **Z-regular** space, let F be a closed set in Y , and $a \in Y$ with $a \notin F$.

$\overline{F}^y = \overline{F} \cap Y$ [Theorem 2.13 (i)] Also, $F = \overline{F}^y$ for F be a closed in Y , and so $F = \overline{F} \cap Y$. Since $a \notin F$, then $a \notin \overline{F} \cap Y$, and so $a \notin \overline{F}$.

Note that $\overline{F}$ is a closed set in X , and $a \notin \overline{F}$, There exist disjoint sets $U, V \in \tau, \overline{F} \subseteq U, a \in V$, and $U^b \cap V^b = \phi$, for X be a **Z-regular** space.

Since $a \in V$ and $a \in Y$ then $a \in V \cap Y$, and so $\overline{F} \subseteq U$ then $\overline{F} \cap Y = F \subseteq U \cap Y$.

Note that $(U \cap Y) \in \tau_y$ and $(V \cap Y) \in \tau_y$.

$(U \cap Y) \cap (V \cap Y) = (U \cap V) \cap Y = \phi \cap Y = \phi$

Now,

$(U \cap Y)^{b_y} \subseteq Y \cap (U \cap Y)^b$, [Theorem 2.13 (ii)]

2.14] $\subseteq Y \cap (U^b \cup Y^b)$, [Theorem

2.8] $= (Y \cap U^b) \cup (Y \cap Y^b)$
 $= (Y \cap U^b) \cup \phi$, [Proposition

2.8] $= Y \cap U^b$

Similarity, $(V \cap Y)^{b_y} \subseteq Y \cap V^b$
 So, $(U \cap Y)^{b_y} \cap (V \cap Y)^{b_y} \subseteq (Y \cap U^b) \cap (Y \cap V^b)$
 $= (U^b \cap V^b) \cap Y$

$Y = \phi \cap Y = \phi$.

Hence, (Y, τ_y) is a **Z**-regular space.

□ Note 3.17 :

Theorem 3.12 An open subspace of a ZT_3 space is a ZT_3 space.

Proof: Since an open subspace of a **Z**-regular and T_1 spaces is a **Z**-regular T_1 space then an open subspace of a ZT_3 space is a ZT_3 space.

Remark 3.13 The continuous image of **Z**-regular space needs not be a **Z**-regular space :

$f : (\mathbb{Z}, \tau_d) \rightarrow (\mathbb{R}, \tau_c)$, $f(x) = x$ is continuous function and, $(\mathbb{Z}, \tau_d)$ is **Z**-regular space but $f(\mathbb{Z}) = \mathbb{Z}$ with the relative cofinite topology is not **Z**-regular space.

Remark 3.14 The continuous image of ZT_3 space needs not be a ZT_3 space.

Definition 3.15 A space (X, τ) be defined a semi-**Z**-regular space if and only if for every $x \in X$ and semi-closed set F , with $x \notin F$, there are disjoint semi-open sets U and V such that $F \subseteq U$, $x \in V$, and $U^{bs} \cap V^{bs} = \phi$.

A semi-**Z**-regular semi- T_1 space is called a semi- ZT_3 space.

Example 3.16 Let $X = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}, \{b, c\}, \}$. (X, τ) is a semi-**Z**-regular space:

Consider semi-closed set $\{b, c\}$ and $a \notin \{b, c\}$, then there are disjoint semi-open sets $\{a\}$ and $\{b, c\}$ such that $\{b, c\} \subseteq \{b, c\}$, $a \in \{a\}$ and $\{a\}^{bs} \cap \{b, c\}^{bs} = \phi$.

The semi-closed sets $\{a\}$ with each of b and c , there exist disjoint semi-open sets $\{a\}$ and $\{b, c\}$ such that $\{a\} \subseteq \{a\}$, $b \in \{b, c\}$ and $c \in \{b, c\}$, respectively, and $\{a\}^{bs} \cap \{b, c\}^{bs} = \phi$. Hence, (X, τ) is semi-**Z**-regular space.

Note that (X, τ) is not semi- ZT_3 space for it is not semi- T_1 space.

i. Every semi-**Z**-regular space is semi-regular space

ii. Every semi- ZT_3 space is a semi- T_3 space

iii. Every semi- ZT_3 space is a semi-**Z**-regular space, but the converse is not true in general for the above example is semi-**Z**-regular space but not semi- ZT_3 space

iv. Every semi- T_3 space is a semi-regular space

Example 3.18 The discrete space of more than one point, (X, τ_d) , is semi-**Z**-regular space:

Since the concepts of open set and semi-open set are the same in the discrete spaces, [Remark 2.12], then the concept of closed and semi closed are the same in discrete space. We have the result as in [Example 3.3].

Example 3.19 The discrete space of more than one point, (X, τ_d) , is semi- ZT_3 space for it is semi-**Z**-regular and semi- T_1 space.

Example 3.20 Let X be an infinite

set. The Cofinite space (X, τ_c) is not semi-Z-regular space for there are no disjoint semi-open sets in the cofinite space except X and ϕ .

Example 3.21 Let X be an infinite set. The Cofinite space (X, τ_c) is not semi- ZT_3 space.

Example 3.22 The indiscrete space of more than one point (X, τ_{ind}) is semi-Z-regular space but not semi- ZT_3 space.

Theorem 3.23 The property of being semi-Z-regular is a semi-topological property.

Proof: Suppose that X is semi-homomorphic to Y , where X is semi-Z-regular space. Then $f : X \rightarrow Y$ be a semi-homeomorphism .

To show that Y is semi-Z-regular space, let $F \subset Y$ be a semi-closed set and $a \in Y$ with $a \notin F$.

Then there are $b \in X$, $f(b) = a$. Since f is irresolute function then $f^{-1}(F)$ is semi-closed set.

Since $a \notin F$, then $b \notin f^{-1}(F)$. Since X is a semi-Z-regular space, then there are disjoint semi-open sets U and V such that $f^{-1}(F) \subseteq U$, $b \in V$, and $U^{bs} \cap V^{bs} = \phi$. So $f(f^{-1}(F)) = F \subseteq f(U)$, and $f(b) = a \in f(V)$.

Since f is pre-semi-open function, then $f(U)$ and $f(V)$ are semi-open in Y . Since f is injective then $f(U \cap V) = f(U) \cap f(V) = f(\phi) = \phi$.

Now, $(f(U))^{bs} \cap (f(V))^{bs} = f(U^{bs}) \cap f(V^{bs})$, [Theorem 2.6],

$$\begin{aligned} &= f(U^{bs} \cap V^{bs}) \\ &= f(\phi) = \phi. \end{aligned}$$

Hence, (Y, τ^*) is a semi-Z-regular space.

□

Theorem 3.24 The property of being a semi- ZT_3 space is a semi-topological property.

Proof: Suppose that X is semi-homomorphic to Y , where X is semi- T_1 space. Then $f : X \rightarrow Y$ be a semi-homeomorphism. For any $y_1, y_2 \in Y$, $y_1 \neq y_2$, we have $f^{-1}(y_1), f^{-1}(y_2) \in X$ and $f^{-1}(y_1) \neq f^{-1}(y_2)$ for f is one to one. Since X is a semi- T_1 space, then there exists semi-open sets U and V such that $f^{-1}(y_1) \in U, f^{-1}(y_2) \notin U$, $f^{-1}(y_2) \in V$, $f^{-1}(y_1) \notin V$, we have $y_1 \in f(U)$, $y_2 \notin f(U)$, $y_2 \in f(V)$, and $y_1 \notin f(V)$. $f(U)$ and $f(V)$ are semi-open sets in Y for f be pre-semi-open. Then Y is a semi- T_1 space and Y semi-Z-regular space is a semi-topological property, then Y is a semi- ZT_3 space. Hence, the property of being a semi- ZT_3 space is a semi-topological property.

□

Theorem 3.25 The property of being a semi-Z-regular space is a topological property, [Remark 2.17].

Theorem 3.26 The property of being a semi- ZT_3 space is a topological property, [Remark 2.17].

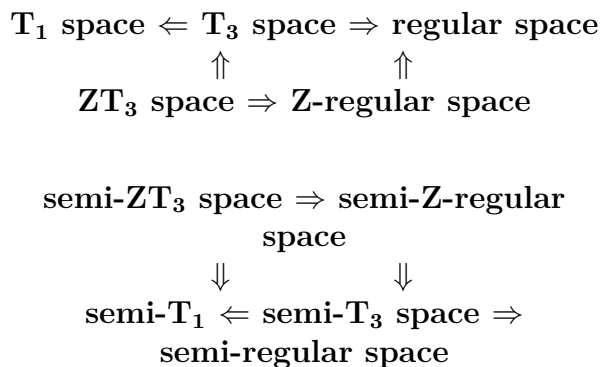
Remark 3.27 The semi-continuous image of a semi-Z-regular space needs not be a semi-Z-regular space as shown in the following example :

$f : (\mathbb{Z}, \tau_d) \rightarrow (\mathbb{R}, \tau_c)$, $f(x) = x$ for all $x \in \mathbb{Z}$ is a semi-continuous function and, $(\mathbb{Z}, \tau_d)$ is semi-Z-regular space, but $f(\mathbb{Z}) = \mathbb{Z}$ with the relative cofinite topology is not semi-Z-regular space.

Remark 3.28 The semi-continuous image of a semi- ZT_3 space needs not be a semi- ZT_3 space.

4 Conclusion

After we have been introduced a Z-regular space, semi-Z-regular space, ZT_3 space and a semi- ZT_3 space, we have the following diagram.



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