

ZT_1 spaces and semi- ZT_1 spaces

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Abstract

In this paper, two separation axioms have been introduced: ZT_1 space and semi- ZT_1 space. Relationships between them and with each of T_1 and semi- T_1 spaces have been given.

DOI : <http://dx.doi.org/10.31642/JoKMC/2018/080202>

Keywords: T_1 space, semi- T_1 space, ZT_1 space, semi- ZT_1 space.

1 Introduction

There are many types of the separation axioms: T_0 , T_1 , T_2 , regular, T_3 , semi- T_1 , semi- T_2 . T_0 was introduced by A. Kolmogorov, T_1 was introduced by Frechet in 1923, T_2 was introduced by Hausdorff in 1914,[10]. Semi- T_0 , semi- T_1 , and semi- T_2 were introduced by S.N. Maheshwari and R. Prasad in 1975,[6]. Regular and T_3 space were introduced by V. Neumann in 1925,[10]. In [7], Levine defined a set A in a topological space X to be semi-open if there exists an open set U such that $U \subseteq A \subseteq \bar{U}$. In [8], S. Gene Crossley and S. K. Hildebrand defined semi-closed set as the complement of semi-open set, the union

of all semi-open sets of X contained in A and the intersection of all semi-closed sets containing A , respectively. In [9], P.Das defined semi-boundary of A as $\bar{A}^s - A^{os}$. We shall use A^c , A^{os} , $\bar{A}^s$, A^{bs} , A° , $\bar{A}$, A^e , A^b , $(X \times Y, \tau_{pro})$, τ_y , A^{b_y} , $\mathbb{R}$, $\mathbb{Z}$ to the complement of A , the semi-interior, the semi-closure, the semi-boundary, the interior of A , the closure of A , the exterior of A , the boundary of A , the product space, the sub space topology, the boundary of A in a sub space, the real numbers, the integer numbers, respectively.

2 Preliminaries

Definition 2.1 [5] A topological space (X, τ) is called T_1 space if and only if for any distinct points x, y in X , there are open sets U, V such that $x \in U, y \notin U, x \notin V$ and $y \in V$.

Definition 2.2 [6] A topological space (X, τ) is called semi- T_1 space if and only if for any distinct points x, y in X , there are semi-open sets U, V such that $x \in U, y \notin U, x \notin V$ and $y \in V$.

Theorem 2.3 [4] Let (X, τ) and (Y, τ^*) be spaces, non empty subsets A and B of X and Y , respectively, then $(A \times B)^b = (A^b \times \overline{B}) \cup (\overline{A} \times B^b)$.

Theorem 2.4 [2] If $f : X \rightarrow Y$ is a homeomorphism, then $f(A^b) = (f(A))^b$, for all $A \subset X$

Theorem 2.5 [3] Let (X, τ) and (Y, τ^*) be spaces, $A \subset X$ and $B \subset Y$, then $(A \times B)^s \subset \overline{A}^s \times \overline{B}^s$.

Remark 2.6 [1] In a space X , $A^{bs} = \overline{A}^s \cap \overline{A^c}^s$.

Remark 2.7 [9] In a space X , $A^{\circ s} = A - A^{bs}$.

Theorem 2.8 [7] The product of semi-open sets is a semi-open set.

Theorem 2.9 [8] If $g : X \rightarrow Y$ is a semi-homeomorphism then for any subsets $A \subset X$ and $B \subseteq Y$:

- i. $\overline{g^{-1}(B)}^s = g^{-1}(\overline{B}^s)$
- ii. $(g^{-1}(B))^{\circ s} = g^{-1}(B^{\circ s})$
- iii. $\overline{g(A)}^s = g(\overline{A}^s)$
- iv. $(g(B))^{\circ s} = g(B^{\circ s})$

Remark 2.10 [5] In any space, $A^b = \phi$ iff A is a clopen set.

Theorem 2.11 [8] In any space:

- i. A is semi-open set iff $A = A^{\circ s}$
- ii. A is semi-closed set iff $A = \overline{A}^s$

Theorem 2.12 If $A \subset X$, $B \subset Y$ and $f : X \rightarrow Y$ be a semi-homeomorphism, then:

- i. $f(A^{bs}) = [f(A)]^{bs}$
- ii. $f^{-1}(A^{bs}) = [f^{-1}(A)]^{bs}$

proof: i. Since f is a semi-homeomorphism, then $f(\overline{A}^s) = \overline{[f(A)]^s}$ and $f(A^{\circ s}) = [f(A)]^{\circ s}$, for all $A \subset X$, [Theorem 2.9 (iii, iv)].

$A^{bs} = \overline{A}^s - A^{\circ s}$ implies $f(A^{bs}) = f(\overline{A}^s - A^{\circ s}) = f(\overline{A}^s) - f(A^{\circ s}) = \overline{[f(A)]^s} - [f(A)]^{\circ s} = [f(A)]^{bs}$.

ii. Since f is a semi-homeomorphism, then $f^{-1}(\overline{B}^s) = \overline{[f^{-1}(B)]^s}$ and $f^{-1}(B^{\circ s}) = [f^{-1}(B)]^{\circ s}$, for all $B \subset Y$, [Theorem 2.9 (i, ii)].

$B^{bs} = \overline{B}^s - B^{\circ s}$ implies $f^{-1}(B^{bs}) = f^{-1}(\overline{B}^s - B^{\circ s}) = f^{-1}(\overline{B}^s) - f^{-1}(B^{\circ s}) = \overline{[f^{-1}(B)]^s} - [f^{-1}(B)]^{\circ s} = [f^{-1}(B)]^{bs}$.

Theorem 2.13 In a space X , A is a semi-clopen (semi-open and semi-closed) set iff $A^{bs} = \phi$.

proof: $\Rightarrow$) Since A is semi-clopen, then $\overline{A}^s = A$ and $A^{\circ s} = A$, [Theorem 2.11 (i, ii)], so $A^{bs} = \overline{A}^s - A^{\circ s} = A - A = \phi$

$\Leftarrow$) Since $A^{\circ s} = A - A^{bs}$, [Remark 2.7], and $A^{bs} = \phi$, then $A^{\circ s} = A$, so A is a semi-open set, [Theorem 2.11 (i)].

$A^{bs} = \overline{A}^s - A^{\circ s} = \phi$ implies $\overline{A}^s \subseteq A^{\circ s}$. Since $A^{\circ s} \subseteq A \subseteq \overline{A}^s$, then $A^{\circ s} = A = \overline{A}^s$ and A is semi-closed and semi-open, [Theorem 2.11 (ii)].

Remark 2.14 $A^{bs} \subset A^b$, in any space.

Remark 2.15 [5] In any space X , $A \cap A^b = \phi$ iff A is an open set.

Theorem 2.16 [5] $(A \cup B)^b \subset A^b \cup B^b$ and $(A \cap B)^b \subset A^b \cup B^b$, in any space.

Theorem 2.17 [4] If Y a subspace of X then $A^{b_y} \subset Y \cap A^b$, for any subset A of Y .

3 Main Results

Definition 3.1 A space (X, τ) is called ZT_1 space if and only if for any distinct points x, y in X , there exists $U, V \in \tau$ such that $x \in U, y \notin U, x \notin V, y \in V$, and $U^b \cap V^b = \phi$.

Example 3.2 The usual space $(\mathbb{R}, \tau_u)$ is ZT_1 space:

Let $x, y \in \mathbb{R}$ with $x < y$ and let $d = d(x, y) = |x - y|$. put $U = (x - 1, x + \frac{d}{4})$, $V = (y - \frac{d}{2}, y + 1)$.

Note that $U, V \in \tau_u$, $x \in U$, $y \notin U$ and $y \in V$, $x \notin V$.

Now, $U^o = U$, $U^c = \mathbb{R} \setminus (x - 1, x + \frac{d}{4})$ then $U^e = \mathbb{R} \setminus [x - 1, x + \frac{d}{4}]$ so $U^b = \{x - 1, x + \frac{d}{4}\}$ similarly, $V^b = \{y - \frac{d}{2}, y + 1\}$, so $U^b \cap V^b = \phi$. Then $(\mathbb{R}, \tau_u)$ is ZT_1 space.

Example 3.3 The discrete space of more than one point, (X, τ_d) , is ZT_1 space:

For any $a, b \in X, a \neq b$, $U = \{a\}$, $V = \{b\}$ are open sets, $a \in U, b \notin U$, $b \in V$ and $a \notin V$. Since $U^b = \phi$ and $V^b = \phi$, then $U^b \cap V^b = \phi$. Hence (X, τ_d) is ZT_1 space.

Example 3.4 Let X be infinite set. The Co-finite space (X, τ_c) is ZT_1 space :

$\forall x, y \in X, x \neq y$, $U = X \setminus \{y\}$ and

$V = X \setminus \{x\}$ are open sets such that $x \in U, y \notin U$ and $y \in V, x \notin V$. Now $U^o = X \setminus \{y\}$, $U^c = \{y\}$, so $U^e = \phi$. Then $U^b = \{y\}$ similarly, $V^b = \{x\}$ then $U^b \cap V^b = \phi$.

Hence (X, τ_c) is ZT_1 space.

Example 3.5 The indiscrete space of more than one point, (X, τ_{ind}) is not ZT_1 .

Example 3.6 Let $X = \{a, b, c, d\}, \tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$. (X, τ) is not ZT_1 space, for $a, c \in X, a \neq c$ cannot find open set contains c but not a .

Theorem 3.7 The property of being ZT_1 is a topological property.

Proof: Suppose that $X \cong Y$, where X is ZT_1 space. Then there exists a homeomorphism $f : X \rightarrow Y$. For any $y_1, y_2 \in Y, y_1 \neq y_2$ we have $f^{-1}(y_1), f^{-1}(y_2) \in X$ and $f^{-1}(y_1) \neq f^{-1}(y_2)$ for f is one to one. Then there exist $U, V \in \tau$, $f^{-1}(y_1) \in U, f^{-1}(y_2) \notin U, f^{-1}(y_2) \in V, f^{-1}(y_1) \notin V$ and $U^b \cap V^b = \phi$. Now,

$y_1 \in f(U)$, $y_2 \notin f(U)$ and $y_2 \in f(V)$, $y_1 \notin f(V)$. Since f is open function, then $f(U)$ and $f(V)$ are open in Y . Now, $(f(U))^b \cap (f(V))^b = f(U^b) \cap f(V^b)$, [Theorem 2.4], $= f(U^b \cap V^b) = f(\phi) = \phi$. Hence (Y, τ^*) is ZT_1 space. $\square$

Theorem 3.8 If X and Y are ZT_1 spaces, then $X \times Y$ is also ZT_1 space.

Proof: Let (X, τ) and (Y, τ^*) be ZT_1 spaces. To prove $(X \times Y, \tau_{pro})$ is also ZT_1 space, let $(x_1, y_1), (x_2, y_2) \in X \times Y, (x_1, y_1) \neq (x_2, y_2)$. Suppose that $x_1 \neq x_2$. Since X is a ZT_1 space, $x_1 \neq x_2$, then there exist $U_1, U_2 \in \tau$ such that $x_1 \in U_1, x_2 \notin U_1, x_2 \in U_2, x_1 \notin U_2$ and $U_1^b \cap U_2^b = \phi$.

$U_1 \times Y$ and $U_2 \times Y$ are open sets in $X \times Y$

and $(x_1, y_1) \in U_1 \times Y, (x_2, y_2) \notin U_1 \times Y$,
 $(x_2, y_2) \in U_2 \times Y, (x_1, y_1) \notin U_2 \times Y$.

Now, $(U_1 \times Y)^b \cap (U_2 \times Y)^b = \phi$
 $(U_1 \times Y)^b \cap (U_2 \times Y)^b = ((U_1^b \times \bar{Y}) \cup (\bar{U}_1 \times Y^b))$
 $\cap ((U_2^b \times \bar{Y}) \cup (\bar{U}_2 \times Y^b)), [\text{Theorem 2.3}]$
 $= ((U_1^b \times \bar{Y}) \cup (\bar{U}_1 \times \phi))$
 $\cap ((U_2^b \times \bar{Y}) \cup (\bar{U}_2 \times \phi)), [\text{Remark 2.10}]$
 $= ((U_1^b \times Y) \cup \phi) \cap$
 $((U_2^b \times Y) \cup \phi)$
 $= (U_1^b \times Y) \cap (U_2^b \times$
 $Y) = (U_1^b \cap U_2^b) \times (Y \cap Y) = \phi \times Y = \phi$
Hence $X \times Y$ is a ZT_1 space.

□

Theorem 3.9 An open subspace of a ZT_1 space is a ZT_1 space.

Proof: Let (X, τ) be a ZT_1 space and (Y, τ_y) be an open subspace of (X, τ) . For any $x, y \in Y, x \neq y$ we have $x, y \in X$. Since (X, τ) is ZT_1 space then, there exists open sets U and V such that $x \in U, y \notin U, y \in V, x \notin V$, and $U^b \cap V^b = \phi$. Note that

$x \in (U \cap Y), y \notin (U \cap Y) \in \tau_y, y \in$
 $(V \cap Y), x \notin (V \cap Y) \in \tau_y$. Now,
 $(U \cap Y)^{b_y} \subset Y \cap (U \cap Y)^b, [\text{Theorem 2.17}]$
 $\subset Y \cap (U^b \cup Y^b), [\text{Theorem 2.16}]$
 $] = (Y \cap U^b) \cup (Y \cap Y^b), [\text{Remark}$
2.15]

$$= (Y \cap U^b) \cup \phi$$

$$= (Y \cap U^b)$$

Similarity, $(V \cap Y)^{b_y} \subset (Y \cap V^b)$
So, $(U \cap Y)^{b_y} \cap (V \cap Y)^{b_y} \subset (Y \cap U^b) \cap (Y \cap V^b)$
 $= (U^b \cap V^b) \cap$

$$Y = \phi \cap Y = \phi.$$

Hence, (Y, τ_y) is a ZT_1 space.

□

Remark 3.10 The continuous image of ZT_1 needs not be a ZT_1 space :

$f : (\mathbb{Z}, \tau_d) \rightarrow (\mathbb{R}, \tau_{ind}), f(x) = x$ is continuous function and, $(\mathbb{Z}, \tau_d)$ is ZT_1 space

but $f(\mathbb{Z}) = \mathbb{Z}$ with the relative indiscrete topology is not ZT_1 space.

Definition 3.11 A space (X, τ) be defined a semi- ZT_1 space if and only if for any distinct points x, y in X , there exist semi-open sets U and $V, x \in U, y \notin U, x \notin V, y \in V$, and $U^{bs} \cap V^{bs} = \phi$.

Example 3.12 The usual space $(\mathbb{R}, \tau_u)$ is semi- ZT_1 space:

Let $x, y \in \mathbb{R}$ with $x < y$ and let $d = d(x, y) = |x - y|$. Put $U = (x - 1, x + \frac{d}{4}), V = (y - \frac{d}{2}, y + 1)$ are semi-open sets such that $x \in U, y \notin U, y \in V$, and $x \notin V$.

Since U is semi-clopen, then $U^{bs} = \phi$, [Theorem 2.13].

similarity, $V^{bs} = \phi$, so $U^{bs} \cap V^{bs} = \phi$. Then $(\mathbb{R}, \tau_u)$ is a semi- ZT_1 space.

Example 3.13 The discrete space of more than one point, (X, τ_d) , is a semi- ZT_1 space.

Example 3.14 Let X be infinite set. The Co-finite space (X, τ_c) , is a semi- ZT_1 space.

Example 3.15 The indiscrete space of more than one point, (X, τ_{ind}) , is not a semi- ZT_1 space.

Example 3.16 Let $X = \{a, b, c, d\}, \tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$. (X, τ) is a semi- ZT_1 space.

Example 3.17 Let $X = \{a, b\}, \tau = \{X, \phi, \{a\}\}$
 $a, b \in X, a \neq b. a \in \{a\}$ but can not find semi-open set containing b but not a so (X, τ) is not semi- ZT_1 space.

Theorem 3.18 The property of being semi- ZT_1 space is a semi-topological property.

Proof: Suppose that X is semi-homomorphic to Y , where X is a semi- ZT_1 space. Then there exists a semi-homeomorphism $f : X \rightarrow Y$. For any $y_1, y_2 \in Y, y_1 \neq y_2$ we have $f^{-1}(y_1), f^{-1}(y_2) \in X$ and $f^{-1}(y_1) \neq f^{-1}(y_2)$ for f is one to one. Since X is semi- ZT_1 space then there exist U, V semi-open sets such that $f^{-1}(y_1) \in U, f^{-1}(y_2) \notin U, f^{-1}(y_2) \in V, f^{-1}(y_1) \notin V$ and $U^{bs} \cap V^{bs} = \phi$. Now, $y_1 \in f(U), y_2 \notin f(U)$ and $y_2 \in f(V), y_1 \notin f(V)$. Since f pre-semi-open then $f(U)$ and $f(V)$ are semi-open in Y . Now, $(f(U))^{bs} \cap (f(V))^{bs} = f(U^{bs}) \cap f(V^{bs})$, [Theorem 2.12 (i)], $= f(U^{bs} \cap V^{bs}) = f(\phi) = \phi$. Hence (Y, τ^*) is semi- ZT_1 space. $\square$

Theorem 3.19 The product space of two semi- ZT_1 spaces is semi- ZT_1 space.

Proof: Let (X, τ) and (Y, τ^*) be semi- ZT_1 spaces. To prove $(X \times Y, \tau_{pro})$ is semi- ZT_1 space, let $(x_1, y_1), (x_2, y_2) \in X \times Y, (x_1, y_1) \neq (x_2, y_2)$. Suppose that $x_1 \neq x_2$. Since X is a semi- ZT_1 space, $x_1 \neq x_2$, then there exist semi-open sets U_1, U_2 such that

$x_1 \in U_1, x_2 \notin U_1, x_2 \in U_2, x_1 \notin U_2$ and $U_1^{bs} \cap U_2^{bs} = \phi$.

$U_1 \times Y, U_2 \times Y$ are semi-open sets in $X \times Y$, [Theorem 2.8], such that $(x_1, y_1) \in U_1 \times Y, (x_2, y_2) \notin U_1 \times Y, (x_2, y_2) \in U_2 \times Y$ and $(x_1, y_1) \notin U_2 \times Y$

Now,

$(U_1 \times Y)^{bs} = (\overline{U_1 \times Y})^s \cap ((\overline{U_1 \times Y})^{cs})$, [Remark 2.6]

$$\begin{aligned} &= (\overline{U_1 \times Y})^s \cap ((\overline{U_1^c \times Y} \cup (X \times Y^c))^s) \\ &= (\overline{U_1 \times Y})^s \cap ((\overline{U_1^c \times Y} \cup (X \times \phi))^s) \\ &= (\overline{U_1 \times Y})^s \cap (\overline{U_1^c \times Y})^s \subset \\ &(\overline{U_1^s} \times \overline{Y^s}) \cap (\overline{U_1^{cs}} \times \overline{Y^s}), [\text{Theorem 2.5}] \\ &= (\overline{U_1^s} \times Y) \cap (\overline{U_1^{cs}} \times Y) = (\overline{U_1^s} \cap \overline{U_1^{cs}}) \times (Y \cap Y) \\ &= (\overline{U_1^s} \cap \overline{U_1^{cs}}) \times Y = U_1^{bs} \times Y. \end{aligned}$$

Similarity, $(U_2 \times Y)^{bs} \subset U_2^{bs} \times Y$.

So $(U_1 \times Y)^{bs} \cap (U_2 \times Y)^{bs} \subset (U_1^{bs} \times Y) \cap (U_2^{bs} \times Y) = (U_1^{bs} \cap U_2^{bs}) \times Y = \phi \times Y = \phi$. Hence the product space of two semi- ZT_1 spaces is a semi- ZT_1 space. $\square$

Remark 3.20 The semi continuous image of semi- ZT_1 needs not be a semi- ZT_1 space. $f : (\mathbb{Z}, \tau_d) \rightarrow (\mathbb{R}, \tau_{ind}), f(x) = x$ is semi continuous function and, $(\mathbb{Z}, \tau_d)$ is semi- ZT_1 space but $f(\mathbb{Z}) = \mathbb{Z}$ with the relative indiscrete topology is not a semi- ZT_1 space.

Theorem 3.21 The property of being semi- ZT_1 is not hereditary property For Example 3.16 is semi- ZT_1 space but $Y = \{c, d\}$ as a subspace of (X, τ) with the indiscrete topology is not semi- ZT_1 space.

Theorem 3.22 Let (X, τ) be a ZT_1 space then it is a semi- ZT_1 space.

Proof: Let x and y are distinct points in X , where X is a ZT_1 space, then there are $U, V \in \tau$ such that $x \in U, y \notin U, x \notin V, y \in V$, and $U^b \cap V^b = \phi$. U and V are semi-open sets for every open set is semi-open set and $U^{bs} \subset U^b, V^{bs} \subset V^b$, [Remark 2.14], and $U^b \cap V^b = \phi$, then $U^{bs} \cap V^{bs} = \phi$. Hence, (X, τ) is a semi- ZT_1 space.

Remark 3.23 The converse of Theorem 3.22 is not true in general, for Example 3.16 is semi- ZT_1 space but not ZT_1 space.

Remark 3.21 Let (X, τ) be a space and if X is a ZT_1 space then it is a T_1 space and if X is a semi- ZT_1 space then it is a semi- T_1 space.

4 Conclusion

After we have been introduced a ZT_1 space and a semi- ZT_1 space, we have the following diagram.

$$\begin{array}{ccc} ZT_1 \text{ space} & \Rightarrow & \text{semi-}ZT_1 \text{ space} \\ \downarrow & & \downarrow \\ T_1 \text{ space} & \Rightarrow & \text{semi-}T_1 \text{ space} \end{array}$$

References

- [1] B. Ahmad , M. Khan , T. Noiri. A note on semi-frontier, *Indian J. pure appl. Math.* 22(1), 61-62 (1991)
- [2] C. C. Adams and R. D. Franzosa. Introduction to Topology Pure and Applied, *Pearson Prentice Hall Upper Saddle River.* (2008)
- [3] C. Dorsett. Product spaces and semi-separation axioms, *Periodica Mathematica Hungarica* 13(1), 39-45 (1982)
- [4] J. Dugundji. Topology Allyn and Bacon, *Inc. Boston.* (1966)
- [5] J. N. Sharma. General topology, *Krishan Prakashan Meerut UP* (1977)
- [6] M. Caldas. A Separation Axioms Between Semi-T0 and Semi-T1, *Pro Mathematica* 11, 21-22, (1997)
- [7] N. Levine. Semi-open sets and semi-continuity in topological space, *The American Mathematical Monthly.* 70(1), 36-41 (1963)
- [8] S. G. Crossley and S. K. Hildebrand. Semi-topological properties, *Fundamenta Mathematicae.* 74(3), 233-254(1972)
- [9] T. M. Nour. A note on semi-application of semi-open sets, *Internat. J. Math. and Math. Sci.* 21(1), 205-207 (1998)
- [10] https://en.wikipedia.org/wiki/Axiom_of_regularity

Received Dec.29, 2019. Accepted for publication May 3,2021