Turán-Type Inequalities for Bessel, Modified Bessel and Krätzel Functions

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Abstract:

We establish Turán-type inequalities for Bessel functions, modified Bessel functions, Krätzel function and Beta function, by using a new form of Cauchy–Bunyakovsky–Schwarz inequality.

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1. Introduction:

In various fields of mathematics, the inequalities of the type

$$f_n(x)f_{n+2}(x) - [f_{n+1}(x)]^2 \leq 0, \quad (1)$$

where $n = 0,1,2, \dots..$, have great importance.

These inequalities are named by Karlin and Szegö [1] as Turán-type inequalities because the first of this type of inequality was introduced by Turán [2] in 1941, while studying the zeros of Legendre polynomials.

By using the classical recurrence relation [3, page 81]

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x), \qquad n = 0,1,2, \dots.. \quad (2)$$

$$P_{-1}(x) = 0, \qquad P_0(x) = 1$$

and the differential relation [3, page 83]

$$(1-x^2)P'_n(x) = nP_{n-1}(x) - nxP_n(x), \quad (3)$$

The Hungarian mathematician P. Turán discovered the following inequality:

$$\begin{vmatrix} P_n(x) & P_{n+1}(x) \\ P_{n+1}(x) & P_{n+2}(x) \end{vmatrix} \leq 0, \qquad -1 \leq x \leq 1, \quad (4)$$

where $P_n(x)$ is the Legendre polynomial of degree n. In (4) equality occurs only if $x = \pm 1$.

This classical result has been extended in several directions. Many authors have deduced analogous results for classical (orthogonal) polynomials and special functions. For example, Laforgia and Natalini [4] establish some Turán-type inequalities involving the special functions as gamma, Polygamma functions and Riemann's zeta function. Joshi and Bissu [5] presented some two-sided inequalities for the ratio of modified Bessel functions of first kind. They also deduced some Turán-type and Wronski type inequalities for Bessel and modified Bessel functions. Segura [6] presented bounds for ratios of modified Bessel functions and associated Turán-type inequalities. Baricz [7] presented Turán-type inequalities for regular Coulomb wave functions.

The Turán-type inequalities now have an extensive literature, because of their important applications in complex analysis, number theory, combinatorics, theory of mean – values, or statistics, control theory, information theory, economic theory and biophysics.

Bessel functions, modified Bessel functions and Krätzel function are frequently used in many fields of physics and applied mathematics. Many inequalities like two-sided inequalities, functional inequalities, Turán-type and Wronski type inequalities have been investigated in the literature. For example, Joshi and Nalwaya [8] presented some two-sided inequalities for modified Bessel functions and for their ratios. They also deduced some Turán-type and Wronski type inequalities for modified Bessel functions. Baricz, Jankov and Pogany [9] established Laguerre and Turán-type inequalities for Krätzel function.

In this paper our main aim is to prove Turán-type inequalities for Bessel functions, modified Bessel functions, Krätzel function and Beta function, by using our new form of the Cauchy–Bunyakovsky–Schwarz inequality.

It is well-known that the discrete version of the Cauchy–Schwarz inequality (see for instant; [10]) is given by the following relation

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \sum_{i=1}^{n} [a_i]^2 \sum_{i=1}^{n} [b_i]^2 \qquad (a_i, b_i \in \mathbb{R}), \qquad (5)$$

while its integral representation in the space of continuous real–valued functions $C([a, b], \mathbb{R})$, i.e. the Cauchy–Bunyakovsky–Schwarz (CBS) inequality [10] is given by

$$\left(\int_a^b [u(t)]^{1/2}\ [v(t)]^{1/2}\, dt\right)^2 \leq \left(\int_a^b u(t)\, dt\right)\left(\int_a^b v\,(t)\, dt\right), \qquad (6)$$

These inequalities play an important role in various branches of modern mathematics. To date, a large number of generalizations and refinements of the inequalities (5) and (6) have been investigated in the literature, for instance, [11-17].

Recently, in [18], we have proved inequalities for some classical integral transforms, by using the following new form of the Cauchy–Bunyakovsky–Schwarz inequality:

$$\left(\int_a^b [g(t)]\,[h(t)]^{\alpha}\,dt\right)^2 \leq \left(\int_a^b [g(t)]^{\gamma}\,[h(t)]^{\alpha+\beta} dt\right)\left(\int_a^b [g(t)]^{2-\gamma}\,[h(t)]^{\alpha-\beta} dt\right), \quad (7)$$

where g and h are two non-negative functions of a real variable and α, β and γ belong to a set S of real numbers, such that the involved integrals in (7) exist.

In this paper, by applying the above form of the Cauchy–Bunyakovsky–Schwarz inequality, we investigate about Turán-type inequalities satisfied by several special functions such as Bessel functions, modified Bessel functions, Krätzel function and Beta function.

2. The Results:

In this section, we prove Turán-type inequalities for Bessel functions, modified Bessel functions, Krätzel function and Beta function.

2.1 Turán-type inequalities for first kind of Bessel functions:

Let us consider the second- order linear ordinary Bessel's differential equation:

$$\boldsymbol{x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + \left(x^2 - \upsilon^2\right)y(x) = 0} \quad (8)$$

where υ is called the order of the Bessel equation.

The following relation is known as the Poisson integral representation [19, p. 223] of the first kind of Bessel function for $x > 0$ and $\upsilon > -1/2$:

$$J_\upsilon(x) = J(\upsilon; x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\ \Gamma(\upsilon+n+1)}\left(\frac{x}{2}\right)^{\upsilon+2n}$$

$$= \frac{\left(\frac{x}{2}\right)^{\upsilon}}{\sqrt{\pi}\ \Gamma\left(\upsilon+\frac{1}{2}\right)} \int_{-1}^{1}(1-t^2)^{\upsilon-\frac{1}{2}}\cos(xt)\,dt. \quad (9)$$

Now if $g(t) = \cos(xt)$ and $h(t) = (1-t^2)^{\upsilon-1/2}$ are substituted in inequality (7) for $[a, b] = [-1,1]$ and $\gamma = 1$, the following inequality is derived

$$\left(\int_{-1}^{1}(1-t^2)^{\alpha(\upsilon-1/2)}\cos(xt)\,dt\right)^2 \leq \left(\int_{-1}^{1}(1-t^2)^{(\alpha+\beta)(\upsilon-1/2)}\cos(xt)\,dt\right)\left(\int_{-1}^{1}(1-t^2)^{(\alpha-\beta)(\upsilon-1/2)}\cos(xt)\,dt\right). \quad (10)$$

By applying (9) in inequality (10), the following inequality with respect to the variable υ, will eventually be obtained

$$\left[J\left\{\alpha\left(\upsilon-\frac{1}{2}\right)+\frac{1}{2}\,;x\right\}\right]^2 \leq \frac{\Gamma\left[(\alpha+\beta)\left(\upsilon-\frac{1}{2}\right)+1\right]\,\Gamma\left[(\alpha-\beta)\left(\upsilon-\frac{1}{2}\right)+1\right]}{\left[\Gamma\left\{\alpha\left(\upsilon-\frac{1}{2}\right)+1\right\}\right]^2} \times J\left\{(\alpha+\beta)\left(\upsilon-\frac{1}{2}\right)+\frac{1}{2}\,;x\right\} J\left\{(\alpha-\beta)\left(\upsilon-\frac{1}{2}\right)+\frac{1}{2}\,;x\right\} \qquad (11)$$

If we put $\nu=\alpha\left(\upsilon-\frac{1}{2}\right)+\frac{1}{2}$ and $\mu=\beta\left(\upsilon-\frac{1}{2}\right)$ in inequality (11), we obtain the Turán-type inequality for Bessel functions of the first kind as follows:

$$[J_\nu(x)]^2 \leq \frac{\Gamma\left(\nu-\mu+\frac{1}{2}\right)\,\Gamma\left(\nu+\mu+\frac{1}{2}\right)}{\left[\Gamma\left(\nu+\frac{1}{2}\right)\right]^2} \times J_{\nu-\mu}(x)\,J_{\nu+\mu}(x) \qquad (12)$$

provided that $x>0$ and $\left(\nu+\frac{1}{2}\right)>|\mu|$.

2.2 Turán-type inequalities for various kinds of modified Bessel functions:

Consider the following differential equation

$$\boldsymbol{x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-\left(x^2+\upsilon^2\right)y(x)=0} \qquad (13)$$

which differs from (8) only in the coefficient of y. The particular solution of (13) is the function I_υ, which is called the modified Bessel function of the first kind [19] of order υ, can be represented by the following relations for $x>0$ and $\upsilon>-1/2$:

$$I_\upsilon(x)=I(\upsilon;x)=\frac{\left(\frac{x}{2}\right)^\upsilon}{\sqrt{\pi}\;\Gamma\left(\upsilon+\frac{1}{2}\right)}\int_{-1}^{1}(1-t^2)^{\upsilon-\frac{1}{2}}\cosh(xt)\,dt. \qquad (14)$$

The another solution of the differential equation (13) is function K_υ, which is called the modified Bessel function of the second kind of order υ.

The specific integral representation of modified Bessel function of the second kind for $x>0$ and $\upsilon>-1/2$ is given by

$$K_\upsilon(x)=K(\upsilon;x)=\frac{\sqrt{\pi}\,\left(\frac{x}{2}\right)^\upsilon}{\Gamma\left(\upsilon+\frac{1}{2}\right)}\int_{1}^{\infty}e^{-xt}\,(t^2-1)^{\upsilon-1/2}\,dt. \qquad (15)$$

Now, applying inequality (7) for $g(t)=\cosh(xt)$, $h(t)=(1-t^2)^{\upsilon-1/2}$, $[a,b]=[-1,1]$ and $\gamma=1$, results in:

$$\left(\int_{-1}^{1}(1-t^2)^{\alpha(\upsilon-1/2)}\cosh(xt)\,dt\right)^2 \leq \left(\int_{-1}^{1}(1-t^2)^{(\alpha+\beta)(\upsilon-1/2)}\cosh(xt)\,dt\right)\left(\int_{-1}^{1}(1-t^2)^{(\alpha-\beta)(\upsilon-1/2)}\cosh(xt)\,dt\right)$$

Corresponding to definition (14), the following result will eventually be obtained

$$\left[I\left\{\alpha\left(\upsilon-\frac{1}{2}\right)+\frac{1}{2};x\right\}\right]^2 \le \frac{\Gamma\left[(\alpha+\beta)\left(\upsilon-\frac{1}{2}\right)+1\right]\,\Gamma\left[(\alpha-\beta)\left(\upsilon-\frac{1}{2}\right)+1\right]}{\left[\Gamma\left\{\alpha\left(\upsilon-\frac{1}{2}\right)+1\right\}\right]^2} \times I\left\{(\alpha+\beta)\left(\upsilon-\frac{1}{2}\right)+\frac{1}{2};x\right\} I\left\{(\alpha-\beta)\left(\upsilon-\frac{1}{2}\right)+\frac{1}{2};x\right\} \quad (16)$$

If we replace $\nu=\alpha\left(\upsilon-\frac{1}{2}\right)+\frac{1}{2}$ and $\mu=\beta\left(\upsilon-\frac{1}{2}\right)$ in inequality (16), we obtain the Turán-type inequality for modified Bessel function of the first kind as follows:

$$[I_\nu(x)]^2 \le \frac{\Gamma\left(\nu-\mu+\frac{1}{2}\right)\Gamma\left(\nu+\mu+\frac{1}{2}\right)}{\left[\Gamma\left(\nu+\frac{1}{2}\right)\right]^2} \times I_{\nu-\mu}(x)\, I_{\nu+\mu}(x) \quad (17)$$

provided that $x>0$ and $\left(\nu+\frac{1}{2}\right)>|\mu|$.

By a similar approach, substituting $g(t)=e^{-xt}$ and $h(t)=(t^2-1)^{\upsilon-1/2}$ in inequality (7) for $[a,b]\to[1,\infty)$, we obtain

$$\left(\int_1^\infty e^{-xt}\,(t^2-1)^{\alpha(\upsilon-1/2)}dt\right)^2 \le \left(\int_1^\infty e^{-\gamma xt}\,(t^2-1)^{(\alpha+\beta)(\upsilon-1/2)}dt\right)\left(\int_1^\infty e^{-(2-\gamma)xt}\,(t^2-1)^{(\alpha-\beta)(\upsilon-1/2)}dt\right)$$

Corresponding to definition (15), the following result after simplification eventually yields

$$\left[K\left\{\alpha\left(\upsilon-\frac{1}{2}\right)+\frac{1}{2};\ x\right\}\right]^2 \le \frac{1}{[\gamma]^{(\alpha+\beta)(\upsilon-1/2)+\frac{1}{2}}\ [2-\gamma]^{(\alpha-\beta)(\upsilon-1/2)+\frac{1}{2}}} \times \frac{\Gamma\left[(\alpha+\beta)\left(\upsilon-\frac{1}{2}\right)+1\right]\,\Gamma\left[(\alpha-\beta)\left(\upsilon-\frac{1}{2}\right)+1\right]}{\left[\Gamma\left\{\alpha\left(\upsilon-\frac{1}{2}\right)+1\right\}\right]^2} \times K\left\{(\alpha+\beta)\left(\upsilon-\frac{1}{2}\right)+\frac{1}{2};\gamma x\right\} K\left\{(\alpha-\beta)\left(\upsilon-\frac{1}{2}\right)+\frac{1}{2};(2-\gamma)x\right\} \quad (18)$$

Replacing $\nu=\alpha\left(\upsilon-\frac{1}{2}\right)+\frac{1}{2}$ and $\mu=\beta\left(\upsilon-\frac{1}{2}\right)$, the inequality (18) is transformed to

$$[K(\nu;\ x)]^2 \le \frac{1}{[\gamma]^{\nu+\mu}\ [2-\gamma]^{\nu-\mu}} \times \frac{\Gamma\left(\nu+\mu+\frac{1}{2}\right)\Gamma\left(\nu-\mu+\frac{1}{2}\right)}{\left[\Gamma\left(\nu+\frac{1}{2}\right)\right]^2} \times K[(\nu+\mu);\gamma x]\,K[(\nu-\mu);(2-\gamma)x] \quad (19)$$

provided that $x>0$, $\gamma\in(0,2)$ and $\left(\nu+\frac{1}{2}\right)>|\mu|$.

For $\gamma=1$, one can obtain the Turán-type inequality for modified Bessel function of the second kind as follows:

$$[K_\nu(x)]^2 \le \frac{\Gamma\left(\nu-\mu+\frac{1}{2}\right)\Gamma\left(\nu+\mu+\frac{1}{2}\right)}{\left[\Gamma\left(\nu+\frac{1}{2}\right)\right]^2} \times K_{\nu-\mu}(x)\, K_{\nu+\mu}(x) \quad (20)$$

provided that $x>0$ and $\left(\nu+\frac{1}{2}\right)>|\mu|$.

2.3 Turán-type inequalities for Krätzel functions:

The special function $Z_\rho^\upsilon(x)$ is defined for $x > 0, \rho \in \mathbb{R}$ and $\upsilon \in \mathbb{C}$, being such that

$Re(\upsilon) < 0$ when $\rho \leq 0$, by the following integral representation

$$Z_\rho^\upsilon(x) = \int_0^\infty t^{\upsilon-1} e^{\left(-t^\rho - \frac{x}{t}\right)} dt \qquad (x > 0), \qquad (21)$$

where $\mathbb{R}$ and $\mathbb{C}$ are sets of real and complex numbers, respectively.

For $\rho \geqq 1$ the function (21) was introduced by Kr ä tzel [20] as a kernel of the integral transform

$$\left(K_\upsilon^\rho f\right)(x) = \int_0^\infty Z_\rho^\upsilon(x)\, f(t)\, dt \qquad (x > 0), \qquad (22)$$

which is known as Krätzel function.

In particular, when $\rho = 1$ and $x = \frac{t^2}{4}$, then in accordance with [21, 7.12(23)],

$$Z_1^\upsilon\left(\frac{t^2}{4}\right) = 2\left(\frac{t}{2}\right)^\upsilon K_\upsilon(t) \qquad (23)$$

where $K_\upsilon(t)$ is the MacDonald or Hankel function (Modified Bessel function of the second kind); see for instant [21, Section 7.2.2].

Now if $g(t) = e^{-t^\rho - \frac{x}{t}}$ and $h(t) = t^{\upsilon-1}$ are substituted in inequality (7) for $[a, b] \to [0, \infty)$ and $\gamma = 1$, the following inequality is derived

$$\left(\int_0^\infty t^{\alpha(\upsilon-1)}\, e^{-t^\rho - \frac{x}{t}}\, dt\right)^2 \leq \left(\int_0^\infty t^{(\alpha+\beta)(\upsilon-1)}\, e^{-t^\rho - \frac{x}{t}}\, dt\right)\left(\int_0^\infty t^{(\alpha-\beta)(\upsilon-1)}\, e^{-t^\rho - \frac{x}{t}}\, dt\right)$$

By the definition (21), the following inequality for Krätzel function eventually yields:

$$\left[Z_\rho^{\alpha(\upsilon-1)+1}(x)\right]^2 \leq Z_\rho^{(\alpha+\beta)(\upsilon-1)+1}(x)\, Z_\rho^{(\alpha-\beta)(\upsilon-1)+1}(x), \qquad (24)$$

provided that $\alpha, \beta, \upsilon, \rho \in \mathbb{R}$ and $x > 0$.

In particular, for $\nu = \alpha(\upsilon - 1) + 1$ and $\mu = \beta(\upsilon - 1)$, it reduces to the Turán-type inequality for Krätzel function as follows:

$$\left[Z_\rho^\nu(x)\right]^2 \leq Z_\rho^{\nu-\mu}(x)\, Z_\rho^{\nu+\mu}(x) \qquad (25)$$

provided that $\nu, \mu, \rho \in \mathbb{R}$ and $x > 0$.

If we put $\rho = 1$ and $x = \frac{t^2}{4}$ in inequality (25), then we have

$$\left[Z_1^\nu\left(\frac{t^2}{4}\right)\right]^2 \leq Z_1^{\nu-\mu}\left(\frac{t^2}{4}\right) Z_1^{\nu+\mu}\left(\frac{t^2}{4}\right) \qquad (26)$$

In accordance with the relation (23), one can obtain the Turán-type inequality for MacDonald or Hankel function after simplification as follows:

$$[K_\nu(t)]^2 \leq K_{\nu-\mu}(t)\, K_{\nu+\mu}(t) \qquad (27)$$

provided that $\nu, \mu \in \mathbb{R}$ and $t > 0$.

2.4 Turán-type inequalities for the Beta function:

If one considers the integral representation of the Beta function

$$\mathbf{B}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)} \qquad (x,y > 0) \qquad (28)$$

and then replaces $g(t) = t^{x-1}$ and $h(t) = (1-t)^{y-1}$ in inequality (7) for $[a, b] = [0, 1]$, the following inequality would be derived

$$\left(\int_0^1 t^{x-1}(1-t)^{\alpha(y-1)}dt\right)^2 \le \left(\int_0^1 t^{\gamma(x-1)}(1-t)^{(\alpha+\beta)(y-1)}dt\right)\left(\int_0^1 t^{(2-\gamma)(x-1)}(1-t)^{(\alpha-\beta)(y-1)}dt\right)$$

By (28), this is simplified to:

$$[\mathbf{B}\{x\,,\alpha(y-1)+1\}]^2 \le \mathbf{B}[\gamma(x-1)+1\,,(\alpha+\beta)(y-1)+1] \times \mathbf{B}[(2-\gamma)(x-1)+1\,,(\alpha-\beta)(y-1)+1] \qquad (29)$$

Replacing $\nu = \alpha(y-1)+1$ and $\mu = \beta(y-1)$, the inequality (29) is transformed to

$$[\mathbf{B}(x\,,\nu)]^2 \le \mathbf{B}[\gamma(x-1)+1, (\nu+\mu)]\,\mathbf{B}[(2-\gamma)(x-1)+1, (\nu-\mu)] \qquad (30)$$

provided that $x > 0$, $\gamma(x-1)+1 > 0$, $(2-\gamma)(x-1)+1 > 0$ and $\nu > |\mu|$.

For $\gamma = 1$, the inequality (30) is transformed to the Turán-type inequality for Beta function as follows:

$$[\mathbf{B}(x\,,\nu)]^2 \le \mathbf{B}[x, \nu-\mu]\,\mathbf{B}[x, \nu+\mu] \qquad (31)$$

provided that $x > 0$ and $\nu > |\mu|$.

Competing interests:

The authors declare that they have no competing interests.

Authors' contributions:

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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