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Fuzzy Soc-T-ABSO Sub-Modules

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ABSTRACT: This research analyses the correlation that exists between fuzzy T-ABSO and fuzzy socle. To answer this question, we define a new concept called, fuzzy socle T-ABSO submodules. Results showed many properties that satisfied the new notion. In addition, we applied some algebra operations and found the relationship between fuzzy T-ABSO and fuzzy socle -T-ABSO. We also investigated the structure of fuzzy socle T-ABSO sub- modules of a module under fuzzy direct sum, homomorphic image and intersection. Moreover, the relation between fuzzy socle T-ABSO sub-module and other types of fuzzy sub-module was presented. The implications of this study could be used to define new concepts that depend on fuzzy socle T-ABSO concepts.

Keywords: $\mathcal{F}$ -T-ABSO sub-module, Socle of $\mathcal{F}$-module.

1. INTRODUCTION

In 1965, Zadeh [1] presented a fuzzy subset A of X as a function from X to the unit interval [0,1]. The theory of fuzzy sets has been developed in many fields since then. Artificial intelligence, computer science, medicine, control engineering, decision theory, expert systems, logic and other fields can all benefit from this idea. The concept of fuzzy sets was first employed in algebra, which was one of the first branches of pure mathematics. In 1971, Rosenfeld [2] provided the fuzzy subgroupoid and fuzzy subgroups. The concepts of fuzzy modules and fuzzy submodules were first introduced by Negoita and Ralescu in 1975 [3]. Consequently, fuzzy finitely generated submodules, fuzzy quotient modules [4], radical of fuzzy submodules and primary fuzzy submodules [5, 6] were investigated and Saikia and Kalita [7] established fuzzy essential submodules and looked into their different properties.

All rings are commutative with identity in this article, and all modules are unitary. Prime submodules [8, 9] play an important turn in module theory over a commutative ring. Ali [10] established the notion of approximately submodule: Let M be an $\mathcal{R}$-module. A submodule H of M is said to be app-prime if $rx \in H$, where $r \in \mathcal{R}$ and $x \in M$, implies that either $x \in M + Soc(M)$ or $r \in [H + Soc(M) :_{\mathcal{R}} M]$, where $Soc(M)$ is the sum of all simple submodules of M'. Rabi [11], in (2001), generalised this concept to a prime fuzzy submodule: Let X be a $\mathcal{F}$-module of an $\mathcal{R}$-module M, A $\mathcal{F}$-submodule U of X is called prime if $r_b m_t \subseteq U$,with $r_b$is $\mathcal{F}$-singleton of $\mathcal{R}$ and $m_t$is a $\mathcal{F}$-singleton of X, implying that either $m_t \subseteq U$or $r_b \subseteq [U :_{\mathcal{R}} X]$ for each $t, b \in [0, 1]$'.

Wafaa presented the concept of T-Absorbing fuzzy submodule: Let X be a $\mathcal{F}$-module of an $\mathcal{R}$-module M, A $\mathcal{F}$-submodule U of X is called T-ABSO if $s_q r_b m_t \subseteq U$,where $s_q, r_b$are $\mathcal{F}$-singletons of $\mathcal{R}$ and $m_t$is a $\mathcal{F}$-singleton of X, implying that either $r_b m_t \subseteq U$ or $s_q m_t \subseteq U$or $s_q r_b \subseteq [U :_{\mathcal{R}} X]$ for each $t, q, b \in [0, 1]$'[12].

The concept of T-Absorbing fuzzy submodule is generalised in this article to fuzzy Socle T-Absorbing submodule. This article consists of two sections. In section one, we provide some basic definitions and properties that we needed in the next section. In section two, many basic properties, characterisations and outcomes of fuzzy Socle T-Absorbing submodule are studied.

Note: We denote fuzzy Socle T-Absorbing submodule by $\mathcal{F}$-Soc-T-ABSO submodule.

2. PRELIMINARIES

There are various definitions and characteristics in this section of $\mathcal{F}$-sets, $\mathcal{F}$-modules and prime $\mathcal{F}$-sub-modules.

Definition 2.1. *[1]*
Let D be a set that isn't empty and I be the interval that is closed$[0,1]$*of real numbers. A $\mathcal{F}$-set B in D (a $\mathcal{F}$-subset of D) is a function from D into I.*

Definition 2.2. *[1]*
A $\mathcal{F}$-set B of a set D is said to be $\mathcal{F}$-constant if $B(x) = t$, $\forall x \in Dt \in [0,1]$

Definition 2.3. *[1]*
Let $x_t : D \to [0,1]$ be a $\mathcal{F}$-set in D, where $x \in D$, $t \in [0, 1]$ defined by:

$$x_t(y) = \begin{cases} t & if\, x = y \\ 0 & if\, x \neq y \end{cases}$$

for all $y \in D$. x_t is said to be a $\mathcal{F}$-singleton or $\mathcal{F}$-point in D.

Definition 2.4. *[13]*
Let B be a $\mathcal{F}$-set is D, for all $t \in [0, 1]$, the set $B_t = \{x \in D; B(x) \geq t\}$ is said to be a level subset of B.
Keep in mind that, B_t is a subset of D in the ordinary sense.

Definition 2.5. *[14]*
If C is a $\mathcal{F}$-module of an $\mathcal{R}$-module M, then the sub-module $C_t$of M is called the level sub-module of M where $t \in [0, 1]$.

Remark 2.6. *[15]*
Let A and B be two $\mathcal{F}$-sets in S, then:
1- $A = B$ if and only if $A(x) = B(x)$.
2- $A \subseteq B$ if and only if $A(x) \leq B(x)$.
3- $A = B$if and only if $A_t = B_t$.
If $A < B$ and there exist $x \in S$ such that $A(x) < B(x)$, then A is a proper $\mathcal{F}$-subset of B and written as $A < B$.
By part (2), we can deduce that $x_t \subseteq A$ if and only if $A(x) \geq t$.

Definition 2.7. *[15]*
If M be an $\mathcal{R}$-module. A $\mathcal{F}$-set X of M is called $\mathcal{F}$-module of an $\mathcal{R}$-module M if :
1- $X(x-y) \geq \min\{X(x), X(y), for all x, y \in M\}$.
2- $X(rx) \geq X(x) for all x \in M and r \in \mathcal{R}$.
3- $X(0) = 1$.

Definition 2.8. *[16]*
Let X and A be two $\mathcal{F}$-modules of $\mathcal{R}$-module M. A is said to be a $\mathcal{F}$-sub-module of X if $A \subseteq X$.

Proposition 2.9. *[13]*
Let A be a $\mathcal{F}$-set of an $\mathcal{R}$-module M. Then the level subset A_t, $t \in [0, 1]$ is a sub-module of M if A is a $\mathcal{F}$-sub-module of X where X is a $\mathcal{F}$-module of an $\mathcal{R}$-module M.
Now, we'll go over various $\mathcal{F}$-sub-module attributes that will be useful in the next section.

Lemma 2.10. *[15]*
If r_t be a $\mathcal{F}$-singleton of $\mathcal{R}$ and A be a $\mathcal{F}$-module of an $\mathcal{R}$-module M.Then for any $w \in M$

$$(r_tA)(w) = \begin{cases} \sup\{inf(t, A(x))\} : if\, w = rx\} & for\, some\, x \in M \\ 0 & otherwise \end{cases}$$

Definition 2.11. *[15]*

Let A and B be two $\mathcal{F}$-sub-modules of a $\mathcal{F}$-module X. The residual quotient of A and B denoted by $(A : B)$ is the $\mathcal{F}$-subset of $\mathcal{R}$ defined by:

$(A : B)(r) = sup\{t \in [0,1] : r_t B \subseteq A\}$*, for all* $r \in \mathcal{R}$*. That* $(A : B) = \{r_t : r_t B \subseteq A; r_t isa\mathcal{F} - singletonof\mathcal{R}\}$*. If* $B = \langle x_k \rangle$*, then* $(A : \langle x_k \rangle) = \{r_t : r_t x_k \subseteq A; r_t isa\mathcal{F} - singletonof\mathcal{R}\}$*.*

Lemma 2.12. *[11]*

Let A be a $\mathcal{F}$-sub-module of $\mathcal{F}$-module X, $(A_t : X_t) \geq (A : X)_t$,For all t ∈ [0, 1].

Also , we can prove that by Lemma 2.3.3.[9].

It follows that if , $X = A \oplus B$, for $A, B \leq X$ then $X_t = (A \oplus B)_t = A_t \oplus B_t$.

Definition 2.13. *[14]*

Let f be a mapping from a set M into a set N, let A be $\mathcal{F}$-set in M. The image of A denoted by f (A) is the $\mathcal{F}$-set in N defined by:

$$f(A)(y) = \begin{cases} \sup\left\{A(z) : z \in f^{-1}(y) \neq \varnothing\right\} & for \quad all \quad y \in N \\ 0 & otherwise \end{cases}$$

Note that, if f is a bijective mapping, then $f(A)(y) = A(f^{-1}(y))$

Proposition 2.14. *[17]*

Let A be a $\mathcal{F}$-sub-module of a $\mathcal{F}$-module X, then: f (A) is a $\mathcal{F}$-sub-module of Y.

Definition 2.15. *[18]*

A $\mathcal{F}$-subset K of a ring $\mathcal{R}$ is called a $\mathcal{F}$-ideal of $\mathcal{R}$, if $\forall\ x, y \in \mathcal{R}$:

1- $K(x - y) \geq min\{K(x), K(y)\}$.

2- $K(xy) \geq max\{K(x), K(y)\}$.

Definition 2.16. *[19]*

Let X be a $\mathcal{F}$-module of an $\mathcal{R}$-module M, let A be a $\mathcal{F}$-sub-module of X and K be a $\mathcal{F}$ -ideal of $\mathcal{R}$, the product KA of K and A is defined by:

$$KA(x) = \begin{cases} sup\{inf\{K(r_1), \ldots, K(r_n), A(x_1), \ldots, A(x_n)\}\} & forsomer_i \in \mathcal{R}, x_i \in M, n \in N \\ 0 & otherwise \end{cases}$$

Note that KA is a $\mathcal{F}$-sub-module of X, and $(KA)_t = K_t A_t$ $\forall t \in$ [0, 1].

Definition 2.17. *[11]*

Let X be a $\mathcal{F}$-module of an $\mathcal{R}$-module M, A $\mathcal{F}$-sub-module U of X is called completely prime if whenever $r_b m_t \subseteq U$,with $r_b \neq 0_1$ is $\mathcal{F}$-singleton of $\mathcal{R}$ and $m_t$is a $\mathcal{F}$-singleton of X, implies that $m_t \subseteq U$ for each $t, b \in [0,1]$.

Definition 2.18. *[15]*

Let A and B be two $\mathcal{F}$-sub-modules of an R-module M. The addition A +B is defined by:

$$(A + B)(x) = \sup\{\inf\{A(y), B(z)\} withx = y + z, forallx, y, z \in M\}.$$

Furthermore, A +B is a $\mathcal{F}$-sub-module of an $\mathcal{R}$-module M.

Corollary 2.19. *[16]*

If X is a $\mathcal{F}$-module of an $\mathcal{R}$-module M and $x_t \subseteq X$, then for all $\mathcal{F}$-singleton r_k of $\mathcal{R}$, $r_k x_t = (rx)_\lambda$, where $\lambda = min\{t, k\}$.

Proposition 2.20. *[15]*

Let A and B be two $\mathcal{F}$-sub-modules of a $\mathcal{F}$-module X of an $\mathcal{R}$-module M. Then $(A : B)$ is a $\mathcal{F}$-ideal of $\mathcal{R}$.

Proposition 2.21. *[20]*

Let $f : M \longrightarrow N$ be an $\mathcal{R}$-homomorphisim, then $f(Soc(M)) \subseteq Soc(N)$.

Definition 2.22. *[21, 22]*

Let X be a $\mathcal{F}$-module of an R-module M, X is called $\mathcal{F}$-simple if and only if X has no proper $\mathcal{F}$-sub-modules (in fact X is $\mathcal{F}$-simple if and only if X has only itself and 0_1).

Definition 2.23. *[23]*
A $\mathcal{F}$-module $\mathcal{X}$ is called semi-simple if $\mathcal{X}$ is a summation of simple $\mathcal{F}$-sub-modules of $\mathcal{X}$.
Moreover, the $\mathcal{F}$-socle of X is sum of simple $\mathcal{F}$-sub-module of $\mathcal{X}$ and denoted by $F - Soc(X)$. That is $\mathcal{X}$ is called semi-simple if $X = F - Soc(X)$.

Definition 2.24. *[11]*
Let X be a $\mathcal{F}$-module of an $\mathcal{R}$-module M, X is said to be faithful if $F - annX = 0_1$. Where $F - annX = \{r_t : r_t x_l = 0_1; forall x_l \subseteq X and r_t bea \mathcal{F} - singleton of \mathcal{R}\}$.

Definition 2.25. *[12]*
Let X be a $\mathcal{F}$-module of an $\mathcal{R}$-module M, X is said to be cancellative if whenever $r_t x_l = r_t y_d \ forall\ x_l, y_d \subseteq X and r_t bea \mathcal{F} - singleton of \mathcal{R}$ then $x_l = y_d$.

Definition 2.26. *[12]*
A proper $\mathcal{F}$-sub-module U of a $\mathcal{F}$-module X of an $\mathcal{R}$-module M is called T-ABSO $\mathcal{F}$-sub-module of X if $s_q r_b m_t \subseteq U$,where r_b, s_q are $\mathcal{F}$-singletons of $\mathcal{R}$ and m_t is a $\mathcal{F}$-singleton of X implies that either $s_q r_b \subseteq [U :_{\mathcal{R}} X]$ or $r_b m_t \subseteq U$ or $s_q m_t \subseteq U$for each $t, b, q \in [0, 1]$.

Definition 2.27. *[11]*
A $\mathcal{F}$-sub-module N of a $\mathcal{F}$-module X of an $\mathcal{R}$-module M is called weakly pure $\mathcal{F}$-sub-module of X if for any $\mathcal{F}$-singleton $r_b$of $\mathcal{R}$, implies that $r_b N = r_b X \cap N$ with $b \in [0, 1]$.

Lemma 2.28. *[24]*
let X be a $\mathcal{F}$-module of an $\mathcal{R}$-module M and let A,B and C are $\mathcal{F}$-sub-modules of X such that $C \subseteq B$. Then $C + (B \bigcap A) = (C + A) \bigcap B$.

Proposition 2.29. *[25]*
If M be a faithful multiplication $\mathcal{R}$-module, then $Soc(\mathcal{R})M = Soc(M)$

Definition 2.30. *[26]*
Let X be a $\mathcal{F}$-module of an $\mathcal{R}$- module M. X is called a multiplication $\mathcal{F}$-module if and only if for each $\mathcal{F}$-sub-module A of X ,there exists a $\mathcal{F}$-ideal K of $\mathcal{R}$ such that A = KX.

Proposition 2.31. *[24, 27]*
A $\mathcal{F}$-module X of an $\mathcal{R}$-module M is multiplication if and only if every non-empty $\mathcal{F}$ -sub-module A of X such that $A = (A :_R X)X$.
Keep in mind that a sub-module V of $\mathcal{R}$-module M is called an essential if $H \cap V \neq 0$.
For non-trivial sub-module H of M [28].

The following definition was introduced by Rabi [11] : Let X be a $\mathcal{F}$-module of an $\mathcal{R}$-module M. A $\mathcal{F}$-sub-module A of X is called an essential if $A \cap B \neq 0_1$, for nontrivial $\mathcal{F}$-sub-module B of X.

Finally, (shortly fuzzy set, fuzzy sub-module, fuzzy ideal, fuzzy module, fuzzy T-ABSO and fuzzy singleton is $\mathcal{F}$-set, $\mathcal{F}$-sub-module,$\mathcal{F}$-ideal , $\mathcal{F}$-module, $\mathcal{F}$-T-ABSO and $\mathcal{F}$-singleton)."

3. FUZZY SOC-T-ABSO SUB-MODULES

In this section, we offer the concept of a $\mathcal{F}$-Soc-T-ABSO sub-module as a generalisation of ordinary concept (Soc-T-ABSO Sub-module). Some characterisations of $\mathcal{F}$-Soc-T-ABSO sub-module are introduced.

Definition 3.1. *A proper $\mathcal{F}$-sub-module U of a $\mathcal{F}$-module X of an $\mathcal{R}$-module M is called a $\mathcal{F}$-Socle T-Absorbing (for short $\mathcal{F}$-Soc-T-ABSO) sub-module of X if whenever $s_q r_b m_t \subseteq U$,where r_b, s_q are $\mathcal{F}$-singletons of $\mathcal{R}$ and m_t is a $\mathcal{F}$-singleton of X implies that either $s_q r_b \subseteq [U + \mathcal{F} - Soc(X) :_{\mathcal{R}} X]$ or $r_b m_t \subseteq U + \mathcal{F} - Soc(X)$ or $s_q m_t \subseteq U + \mathcal{F} - Soc(X)$ for each $t, b, q \in [0, 1]$.*

Furthermore, A proper $\mathcal{F}$-ideal L of $\mathcal{R}$ is called a $\mathcal{F}$-Socle T-Absorbing (for short $\mathcal{F}$-Soc-T-ABSO) ideal of $\mathcal{R}$ if whenever $s_q r_b c_h \subseteq U$,where r_b, s_q and c_h are $\mathcal{F}$-singletons of $\mathcal{R}$ implies that either $s_q r_b \subseteq L + \mathcal{F} - Soc(\mathcal{R})$ or $r_b c_h \subseteq L + \mathcal{F} - Soc(\mathcal{R})$ or $s_q c_h \subseteq L + \mathcal{F} - Soc(\mathcal{R})$ for each $h, b, q \in [0, 1]$.

While, a sub-module H of $\mathcal{R}$-module M is called Socle T- Absorbing (for shortSoc-T-ABSO) sub-module of M if whenever $srm \in H$, where $r, s \in \mathcal{R}$ and $m \in H$ implies that either $sm \in H + Soc(M)$ or $rm \in H + Soc(M)$ or $rs \in [H + Soc(M) :_{\mathcal{R}} M]$.

We will adopt the definition of a $\mathcal{F}$-socle of X in this research as follows:

such that : $\mathcal{F} - Soc(X) : M \to [0,1]$

$$\mathcal{F} - Soc(X)(m) = \begin{cases} 1 & if \quad m \in Soc(M) \\ h & if \quad m \notin Soc(M) \end{cases} \quad with 0 < h < 1$$

Lemma 3.2. *for any $\mathcal{F}$-module X for each t*$\in (0,1]$ *with* $(\mathcal{F} - Soc(X))_t \neq X_t$ $(\mathcal{F} - Soc(X))_t = Soc(X_t)$

Lemma 3.3. *Let X be a $\mathcal{F}$-module of an $\mathcal{R}$-module M with X(m)=1 for each $m \in M$, if U is a $\mathcal{F}$-sub-module of X defined by* $U : M \to [0,1]$ *such that :*

$$U(m) = \begin{cases} 1 & if \quad m \in H \\ c & if \quad m \notin H \end{cases} \quad with\ 0 < c < 1$$

Where H is a sub-module of M. Then U is a $\mathcal{F}$-Soc-T-ABSO sub-module of X if and only if H is Soc-T-ABSO sub-module of M.

Proof. First of all, we must define $U + \mathcal{F} - Soc(X)$.

$(U + \mathcal{F} - Soc(X))(m) = sup\{\min(U(y), \mathcal{F} - Soc(X)(z)), y + z = m\}$

So, we have

$$(U + \mathcal{F} - Soc(X))(m) = \begin{cases} 1 & if \quad m \in H + Soc(M) \\ d & if \quad m \notin H + Soc(M) \end{cases} \quad with \quad d = max\{k, h\}$$

Now,

Suppose H is T-ABSO sub-module of M, to prove that U is a $\mathcal{F}$-Soc-T-ABSO sub-module of X. Assume $s_q r_b x_k \subseteq U$, for $x_k \subseteq X$ and $s_q, r_b \subseteq \mathcal{R}$, where $q, k, b \in [0,1]$ that is either $srx \in U$ or $srx \notin U$

either $s_q x_k \subseteq U + \mathcal{F} - Soc(X)$ or $r_b x_k \subseteq U + \mathcal{F} - Soc(X)$ $s_q r_b \subseteq [U + \mathcal{F} - Soc(X) :_{\mathcal{R}} X]$

1) If $srx \in H$, then either $sx \in H + Soc(M)$ or $rx \in H + Soc(M)$ or $sr \in (H + Soc(M) : M)$. If $sx \in H + Soc(M)$ thus $(U + \mathcal{F} - Soc(X))(sx) = 1$ this implies $s_q x_k = (sx)_t \subseteq (sx)_1 \subseteq U + \mathcal{F} - Soc(X)$.Similarly in the case, if $rx \in H + Soc(M)$ we get $r_b x_k \subseteq U + \mathcal{F} - Soc(X)$. If $sr \in (H + Soc(M) : M)$, let $m_p \subseteq X$ with $p \in [0,1]$, then $s_q r_b m_p = (srm)_\daleth$ where $\daleth = min\{q, b, p\}$ but $srm \in H + Soc(M)$. That is $(srm)_\daleth \subseteq (srx)_1 \subseteq U + \mathcal{F} - Soc(X)$, therefore $s_q r_b x_p \subseteq U + \mathcal{F} - Soc(X)$ that's mean $s_q r_b \subseteq [U + \mathcal{F} - Soc(X) : X]$.

2) If $srx \notin H$ then $U(srx) = c$ with $sx \notin H$ thus $U(sx) = c$. Since $s_q r_b x_k \subseteq U$ then $(srx)_\daleth \subseteq U$ where $\daleth = min\{q, b, k\}$, that is $U(srx) \geq \daleth$ thus $c \geq \daleth$ hus is . Now, if $\daleth = k$ this implies $(sx)_k \subseteq (sx)_c \subseteq U \subseteq U + \mathcal{F} - Soc(X)$. If $\daleth = b$, $U(n) \geq k$ for any $n \in M$, and:

$$(s_q r_b X_M)(n) = \begin{cases} b & if\ n = sra \\ 0 & otherwise \end{cases} \quad for\ some\ a \in M$$

Then we get $(s_q r_b X_M)(n) \leq U(n)$, hence $s_q r_b X_M \subseteq U \subseteq U + \mathcal{F} - Soc(X)$ that's mean $s_q r_b \subseteq [U + \mathcal{F} - Soc(X) : X]$.

Therefore U is a $\mathcal{F}$-Soc-T-ABSO sub-module of X.

Conversely

Suppose U is a $\mathcal{F}$-Soc-T-ABSO of X. Let $zvy \in U_t$, with $z, v \in \mathcal{R}$ and $y \in X_t$ it follows that $(zvx)_t \subseteq U$, that is $z_t v_t x_t \subseteq U$. But U is a $\mathcal{F}$-Soc-T-ABSO of X, then either $z_t x_t \subseteq U + \mathcal{F} - Soc(X)$ or $v_t x_t \subseteq U + \mathcal{F} - Soc(X)$ or $z_t v_t \subseteq [U + \mathcal{F} - Soc(X) :_R X]$. Thus we get either $(U + \mathcal{F} - Soc(X))(zx) \geq t$ or $(U + \mathcal{F} - Soc(X))(vx) \geq t$ or $(U + \mathcal{F} - Soc(X) :_R X)(zv) \geq t$, hence by (Lemma 1.12) and (Lemma 2.2) either $zx \in (U + \mathcal{F} - Soc(X))_t = U_t + (\mathcal{F} - Soc(X))_t = U_t + Soc(X_t)$ or$vx \in (U + \mathcal{F} - Soc(X))_t = U_t + Soc(X_t)$ or $zv \in (U + \mathcal{F} - Soc(X) :_R X)_t \subseteq ((U + \mathcal{F} - Soc(X))_t :_R X_t) = (U_t + (\mathcal{F} - Soc(X))_t :_R X_t) = (U_t + Soc(X_t) :_R X_t)$. Therefor we have either $zx \in U_t + Soc(X_t)$ or $zx \in U_t + Soc(X_t)$ or $zv \in (U_t + Soc(X_t) :_R X_t)$. That's mean U_t is T-ABSO sub-module of X_t .

For every $t > 0$ with $U_t \neq M$, hence $U_1 = H$ is T-ABSO sub-module of M.

Remark 3.4. ***(1)*** *The following example show that sum of T-ABSO $\mathcal{F}$-sub-module is need not to be T-ABSO $\mathcal{F}$-sub-module*

Let X:M→$[0,1]$ *such that with X(m)=1 for each $m \in M = Z$ It is obvious that X be F. M. of Z-M. Z.*

Let U: Z→$[0,1]$ *such that*

$$U(m) = \begin{cases} 1 & if \quad m \in 2Z \\ k & if \quad m \notin 2Z \end{cases} \quad with\ 0 < k < 1$$

Let V: Z→$[0,1]$ such that

$$V(m) = \begin{cases} 1 & if \quad m \in 3Z \\ h & if \quad m \notin 3Z \end{cases} \quad with\ 0 < h < 1$$

That's clear U and V are a F- sub-module of X .

Now, $U_1 = 2Z$, $V_1 = 3Z$ where U_1 and $V_1$are Soc-T-ABSO submodules of Z, but $U_1 + V_1 = Z = X_1$is not Soc-T-ABSO subm. hence A+B=X is not $\mathcal{F}$-Soc-T-ABSO submodule

***(2)** The intersection of two T-ABSO F- sub-module need not be T-ABSO F- sub-module, for example:*

Let X:M→$[0,1]$ such that with X(m)=1 for each $m \in M = Z$ It is obvious that X be $\mathcal{F}$-module of an Z-module Z

Let U: Z→$[0,1]$ such that

$$U(m) = \begin{cases} 1 & if \quad m \in 12Z \\ k & if \quad m \notin 12Z \end{cases} \quad with\ 0 < k < 1$$

Let V: Z→$[0,1]$ such that

$$V(m) = \begin{cases} 1 & if \quad m \in 10Z \\ h & if \quad m \notin 10Z \end{cases} \quad with\ 0 < h < 1$$

That's clear U and V are a F- sub-module of X .

Now, $U_1 = 12Z$, $V_1 = 10Z$ where U_1 and $V_1$are Soc-T-ABSO sub-modules of Z, but $U_1 \cap V_1 = 120Z$is not Soc-T-ABSO sub-module since 2.6.10∈120Z, but 2.10∉120Z + Soc(Z) =120Z, 6.10∉120Z + Soc(Z) =120Z and 2.6∉120Z + Soc(Z) =120Z.

Hence U and V are two $\mathcal{F}$-Soc-T-ABSO $\mathcal{F}$-sub-module, but A∩B is not $\mathcal{F}$-T-ABSO F- sub-module

Proposition 3.5. *Let U is $\mathcal{F}$-Soc-T-ABSO $\mathcal{F}$-sub-module of F. M. X of an R-M with $\mathcal{F} - Soc(X) = \mathcal{F} - Soc(V)$., then for each V⊆X, either V⊆U or V∩U is $\mathcal{F}$-Soc-T-ABSO $\mathcal{F}$-sub-module of V.*

Proof. suppose that V⊈U then V∩U⊊V . Let $s_q r_b m_t \subseteq V \cap U$ where r_b, s_q are $\mathcal{F}$-singletons of $\mathcal{R}$ and m_t ⊆V, implies $s_q r_b m_t \subseteq U$. But U is $\mathcal{F}$-Soc-T-ABSO F- sub-module, hence either $s_q r_b \subseteq [U + \mathcal{F} - Soc(X) :_{\mathcal{R}} X]$ or $r_b m_t \subseteq U + \mathcal{F} - Soc(X)$ or $s_q m_t \subseteq U + \mathcal{F} - Soc(X)$. Then either $s_q r_b \subseteq [(U + \mathcal{F} - Soc(V)) \cap (V + \mathcal{F} - Soc(V)) :_{\mathcal{R}} V]$ or $r_b m_t \subseteq (U + \mathcal{F} - Soc(V)) \cap (V + \mathcal{F} - Soc(V))$ or $s_q m_t \subseteq (U + \mathcal{F} - Soc(V)) \cap (V + \mathcal{F} - Soc(V))$. But $\mathcal{F} - Soc(V)$ ⊆V, so we have either $s_q r_b \subseteq [(V \cap U) + \mathcal{F} - Soc(V) :_{\mathcal{R}} V]$ or $r_b m_t \subseteq (V \cap U) + \mathcal{F} - Soc(V)$ or $s_q m_t \subseteq (V \cap U) + \mathcal{F} - Soc(V)$ Thus V∩U is T-ABSO F- sub-module of V.

Proposition 3.6. *Let U and V are $\mathcal{F}$-sub-modules of a $\mathcal{F}$-module X of an $\mathcal{R}$-module M with V is $\mathcal{F}$-T-ABSO prime sub-module of X. Then $[U :_{\mathcal{R}} V]$ is $\mathcal{F}$-Soc-T-ABSO ideal of $\mathcal{R}$.*

Proof. Suppose that $s_q r_b a_k \subseteq [U :_{\mathcal{R}} V]$, for $s_q, r_b, a_k \subseteq \mathcal{R}$, where $q, k, b \in [0, 1]$. ,thus $s_q r_b a_k V \subseteq U$.So we have, $s_q r_b(a_k V) \subseteq U$, but V is $\mathcal{F}$-T-ABSO sub-module of X. thus either $s_q(a_k V) \subseteq U$ or $r_b(a_k V) \subseteq U$ or $s_q r_b X \subseteq U$, therefore either $s_q a_k \subseteq [U :_{\mathcal{R}} V] \subseteq [U :_{\mathcal{R}} V] + \mathcal{F} - Soc(\mathcal{R})$ or $r_b a_k \subseteq [U :_{\mathcal{R}} V] + \mathcal{F} - Soc(\mathcal{R})$ or $s_q r_b \subseteq [U :_{\mathcal{R}} V] + \mathcal{F} - Soc(\mathcal{R})$. That's mean $[U :_{\mathcal{R}} V]$ is a $\mathcal{F}$-Soc-T-ABSO ideal of $\mathcal{R}$.

Remark 3.7. *Every $\mathcal{F}$-Soc-prime submodule of a $\mathcal{F}$-module X of an $\mathcal{R}$-module M is $\mathcal{F}$-Soc-T-ABSO submodule.*

Proof. Let U be a $\mathcal{F}$-Soc-prime submodule of X with $s_q r_b m_t \subseteq U$ for r_b, s_q are $\mathcal{F}$-singletons of $\mathcal{R}$ and m_t ⊆X. that is $(sr)_k m_t = (s_q r_b) m_t \subseteq U$ with k = min{q,b}, but U is a $\mathcal{F}$-Soc-prime submodule of X then either $(sr)_k \subseteq [U + \mathcal{F} - Soc(X) :_{\mathcal{R}} X]$ or $m_t \subseteq U + \mathcal{F} - Soc(X)$, that's mean either $s_q r_b \subseteq [U + \mathcal{F} - Soc(X) :_{\mathcal{R}} X]$ or $r_b m_t \subseteq U + \mathcal{F} - Soc(X)$ and $s_q m_t \subseteq U + \mathcal{F} - Soc(X)$.

Remark 3.8. *Every $\mathcal{F}$-T-ABSO sub-module is $\mathcal{F}$-Soc-T-ABSO sub-module , but the converse is not true .*

Proof. Suppose U be a $\mathcal{F}$-T-ABSO sub-module of a $\mathcal{F}$-module X of an $\mathcal{R}$-module M and $s_q r_b x_k \subseteq U$, for $x_k \subseteq X$ and $s_q, r_b \subseteq \mathcal{R}$, where $q, k, b \in [0, 1]$. Since U is a $\mathcal{F}$-T-ABSO sub-module, then we get either $r_b x_k \subseteq U \subseteq U + \mathcal{F} - Soc(X)$ or $s_q x_k \subseteq U + \mathcal{F} - Soc(X)$ or $s_q r_b X \subseteq U + \mathcal{F} - Soc(X)$. Therefore U is a $\mathcal{F}$-Soc-T-ABSO sub-module.

The following example show that the converse is not true

Example 3.9. *Consider* $M = Z_{18}$ *as a Z-module and* $X : M \to [0,1]$, $U : M \to [0,1]$ *defined by:*

$$X(m) = 1 \qquad if m \in Z_{18}$$

$$U(m) = \begin{cases} 1 & if \;\; m \in \langle \overline{0} \rangle \\ 1/2 & if \;\; m \notin \langle \overline{0} \rangle \end{cases}$$

And a $\mathcal{F}$*-socle of X defined by* $\mathcal{F} - Soc(X) : M \to [0,1]$ *such that :*

$$\mathcal{F} - Soc(X)(m) = \begin{cases} 1 & if \;\; m \in \langle \overline{3} \rangle \\ 1/4 & if \;\; m \notin \langle \overline{3} \rangle \end{cases}$$

Where $Soc(M) = \langle \overline{3} \rangle$*. That's clear X is a* $\mathcal{F}$*-module and U be a* $\mathcal{F}$*-sub-module of X.*
$\langle \overline{0} \rangle$ *is Soc-T-ABSO sub-module of M, so by (Lemma 2.3) we get U is a* $\mathcal{F}$*-Soc-semi-prime sub-module of X.*
Now, $\left[\langle \overline{0} \rangle :_{\mathcal{R}} Z_{18} \right] = 18Z$ *. So we define* $[U :_{\mathcal{R}} X]$

$$[U :_{\mathcal{R}} X](m) = \begin{cases} 1 & if \;\; m \in 18Z \\ 1/3 & if \;\; m \notin 18Z \end{cases}$$

But U is not a $\mathcal{F}$*-T-ABSO sub-module of X, since for a* $\mathcal{F}$*-singleton* $3_{\frac{1}{4}} \subseteq X$ *and a* $\mathcal{F}$*-singletons* $2_{\frac{1}{5}}, 2_{\frac{1}{7}}$ *of* $\mathcal{R}$ *such that* $2_{\frac{1}{5}} 2_{\frac{1}{7}} 3_{\frac{1}{4}} = 0_{\frac{1}{7}}$ *where* $0_{\frac{1}{7}} \subseteq U$ *since* $U(0) = 1 > \frac{1}{7}$ *. but* $2_{\frac{1}{5}} 3_{\frac{1}{4}} = 6_{\frac{1}{5}} \nsubseteq U$ *since* $U(6) = \frac{1}{5} \ngtr \frac{1}{2}$ *and*$2_{\frac{1}{7}} 3_{\frac{1}{4}} = 6_{\frac{1}{7}} \nsubseteq U$ *since* $U(6) = \frac{1}{7} \ngtr \frac{1}{2}$ *and* $2_{\frac{1}{5}} 2_{\frac{1}{7}} = 4_{\frac{1}{7}} \nsubseteq [U :_{\mathcal{R}} X]$ *since* $[U :_{\mathcal{R}} X](4) = \frac{1}{7} \ngtr \frac{1}{3}$
Hence, U is not a $\mathcal{F}$*-T-ABSO sub-module of X.*

Remark 3.10. *Let U be a* $\mathcal{F}$*- sub-module of X, if (* $U + \mathcal{F} - Soc(X)$ *) is a* $\mathcal{F}$*-T-ABSO sub-module of X. Then U is a* $\mathcal{F}$*-Soc-T-ABSO sub-module of X.*

Proof. Assume $s_q r_b x_k \subseteq U$, for $x_k \subseteq X$ and $s_q, r_b \subseteq \mathcal{R}$. hence $s_q r_b x_k \subseteq U + \mathcal{F} - Soc(X)$, but $(U + \mathcal{F} - Soc(X))$ is a $\mathcal{F}$-T-ABSO sub-module of X. So we have either $r_b x_k \subseteq U + \mathcal{F} - Soc(X) = [U + \mathcal{F} - Soc(X)] + \mathcal{F} - Soc(X)$ or $s_q x_k \subseteq U + \mathcal{F} - Soc(X) = [U + \mathcal{F} - Soc(X)] + \mathcal{F} - Soc(X)$ or $s_q r_b X \subseteq U + \mathcal{F} - Soc(X) = [U + \mathcal{F} - Soc(X)] + \mathcal{F} - Soc(X)$. That's mean U is a $\mathcal{F}$-Soc-T-ABSO sub-module of X.

Remark 3.11. *Every completely* $\mathcal{F}$*-sub-module of a* $\mathcal{F}$*-module X of an* $\mathcal{R}$*-module M is* $\mathcal{F}$*-Soc-T-ABSO sub-module of X, but the converse is not true .*

Proof. We take U as a completely $\mathcal{F}$-sub-module of X where $s_q r_b m_t \subseteq U$, for $r_b, s_q \subseteq \mathcal{R}$,$m_t \subseteq X$,where $q, t, b \in [0,1]$.Now, if $r_b = 0_1$ then $r_b m_t = 0_t \subseteq 0_1 \subseteq U + \mathcal{F} - Soc(X)$.we get U is $\mathcal{F}$-Soc-T-ABSO sub-module of X. If $r_b \neq 0_1$,thus $s_q(r_b m_t) \subseteq U$, we get $s_q(rm)_d \subseteq U$ where $d = min\{b, t\}$.Now, since U is completely $\mathcal{F}$-sub-module of X, then we have $(rm)_d \subseteq U \subseteq U + \mathcal{F} - Soc(X)$, thus $r_b m_t \subseteq U + \mathcal{F} - Soc(X)$. Therefore U is a $\mathcal{F}$-Soc-T-ABSO sub-module.
The following example show that the converse is not true

Example 3.12. *Consider* $M = Z$ *as a Z-module and* $X : M \to [0,1]$, $U : M \to [0,1]$ *defined by:*

$$X(m) = 1 \qquad if \;\; m \in Z$$

$$U(m) = \begin{cases} 1 & if \;\; m \in 2Z \\ 1/4 & if \;\; m \notin 2Z \end{cases}$$

And a $\mathcal{F}$*-socle of X defined by* $\mathcal{F} - Soc(X) : M \to [0,1]$ *such that :*

$$\mathcal{F} - Soc(X)(m) = \begin{cases} 1 & if \;\; m \in \{0\} \\ 1/3 & if \;\; m \notin \{0\} \end{cases}$$

$$(U + \mathcal{F} - Soc(X))(m) = \begin{cases} 1 & if \;\; m \in 2Z \\ 1/3 & if \;\; m \notin 2Z \end{cases}$$

Where $Soc(M) = \{0\}$*. That's clear X is a* $\mathcal{F}$*-module and U be a* $\mathcal{F}$*-sub-module of X.*

2Z is T-ABSO sub-module of M, so by (Lemma 2.3) we get U is a $\mathcal{F}$-Soc- T-ABSO sub-module of X.

But U is not completely $\mathcal{F}$-sub-module of X, since for a $\mathcal{F}$-singleton $5_{\frac{1}{3}} \subseteq X$ and a $\mathcal{F}$-singleton $2_{\frac{1}{2}}$ of $\mathcal{R}$ such that $2_{\frac{1}{2}}5_{\frac{1}{3}} = 10_{\frac{1}{3}}$ where $10_{\frac{1}{3}} \subseteq U$ since $U(0) = 1 > \frac{1}{3}$. but $5_{\frac{1}{3}} \nsubseteq U$ since $U(5) = \frac{1}{4} \ngtr \frac{1}{3}$.

Hence, U is not completely $\mathcal{F}$-sub-module of X.

Proposition 3.13. *Let U be a $\mathcal{F}$-sub-module of a $\mathcal{F}$-module X of an $\mathcal{R}$-module M, Then U is a $\mathcal{F}$-Soc-T-ABSO sub-module of X if and only if $\forall$ $\mathcal{F}$-sub-module D of X and every $\mathcal{F}$-singletons s_q, r_b of $\mathcal{R}$ with $s_q r_b S \subseteq U$, implies that either $s_q D \subseteq U + F - Soc(X)$ or $r_b D \subseteq U + F - Soc(X)$ or $s_q r_b X \subseteq U + F - Soc(X)$*

Proof. ($\Rightarrow$) Let $s_q r_b S \subseteq$U , for S is a $\mathcal{F}$-sub-module of X and s_q, r_b are $\mathcal{F}$-singletons of $\mathcal{R}$, So for every $y_h \subseteq S$, we have $s_q r_b y_h \subseteq U$ get ve ine .But U is $\mathcal{F}$-Soc-T-ABSO sub-module of X, implies that either $r_b y_h \subseteq U + \mathcal{F} - Soc(X)$ or $c_h y_h \subseteq U + \mathcal{F} - Soc(X)$ or $c_h r_b X \subseteq U + \mathcal{F} - Soc(X)$. That is, either s_qS $\subseteq U + \mathcal{F} - Soc(X)$ or $r_b S \subseteq U + F - Soc(X)$ or $s_q r_b X \subseteq U + F - Soc(X)$.

($\Leftarrow$) Let $c_h r_b x_t \subseteq U$ for $c_h, r_b \subseteq \mathcal{R}$ and $x_t \subseteq X$ then $c_h r_b \langle x_t \rangle \subseteq U$, that is by hypothesis we get either $c_h \langle x_t \rangle \subseteq U + \mathcal{F} - Soc(X)$ or $r_b \langle x_t \rangle \subseteq U + \mathcal{F} - Soc(X)$ or $c_h r_b X \subseteq U + \mathcal{F} - Soc(X)$. Then, either $r_b x_t \subseteq U + \mathcal{F} - Soc(X)$ or $c_h x_t \subseteq U + \mathcal{F} - Soc(X)$ or $c_h r_b X \subseteq U + \mathcal{F} - Soc(X)$. That's mean U is a $\mathcal{F}$-Soc-T-ABSO sub-module of X.

Proposition 3.14. *Let U be a $\mathcal{F}$-Soc-T-ABSO sub-module of a $\mathcal{F}$-module X of an $\mathcal{R}$-module M, Then U is a $\mathcal{F}$-Soc-T-ABSO sub-module of X if and only if $\forall$ $\mathcal{F}$-sub-module S of X and a $\mathcal{F}$-ideal J of $\mathcal{R}$ with JKS$\subseteq$U, implies either $JS \subseteq U + F - Soc(X)$ or $KS \subseteq U + F - Soc(X)$ or $JKX \subseteq U + F - Soc(X)$*

Proof. ($\Rightarrow$) Assume that $JKS \subseteq$U , for S is a $\mathcal{F}$-sub-module of X and J, K are $\mathcal{F}$-ideals of $\mathcal{R}$, So for every $c_h \subseteq J$ and $r_b \subseteq K$ we get $c_h r_b S \subseteq U$ get ve ine .But U is $\mathcal{F}$-Soc-T-ABSO sub-module of X, so by proposition(2.11) implies that either $r_b S \subseteq U + \mathcal{F} - Soc(X)$ or $c_h S \subseteq U + \mathcal{F} - Soc(X)$ or $c_h r_b X \subseteq U + \mathcal{F} - Soc(X)$. Therefore, either $JS \subseteq U + \mathcal{F} - Soc(X)$ or $KS \subseteq U + F - Soc(X)$ or $JKX \subseteq U + F - Soc(X)$.

($\Leftarrow$) Let $c_h r_b x_t \subseteq U$ for $c_h, r_b \subseteq \mathcal{R}$ and $x_t \subseteq X$ then $\langle c_h \rangle \langle r_b \rangle \langle x_t \rangle \subseteq U$, that is by hypothesis we get either $\langle c_h \rangle \langle x_t \rangle \subseteq U + \mathcal{F} - Soc(X)$ or $\langle r_b \rangle \langle x_t \rangle \subseteq U + \mathcal{F} - Soc(X)$ or $\langle c_h \rangle \langle r_b \rangle X \subseteq U + \mathcal{F} - Soc(X)$. Thus, either $r_b x_t \subseteq U + \mathcal{F} - Soc(X)$ or $c_h x_t \subseteq U + \mathcal{F} - Soc(X)$ or $c_h r_b X \subseteq U + \mathcal{F} - Soc(X)$. That's mean U is a $\mathcal{F}$-Soc-T-ABSO sub-module of X.

Corollary 3.15. *(1) If L is a $\mathcal{F}$-ideal of $\mathcal{R}$. Then L is a $\mathcal{F}$-Soc-T-ABSO ideal of $\mathcal{R}$ if and only if $\forall$ $\mathcal{F}$-ideal K of $\mathcal{R}$ and every $\mathcal{F}$-singletons s_q, r_b of $\mathcal{R}$ with $s_q r_b K \subseteq L$, implies that either $s_q K \subseteq L + F - Soc(\mathcal{R})$ or $r_b K \subseteq L + F - Soc(\mathcal{R})$ or $s_q r_b \subseteq K + F - Soc(\mathcal{R})$*

(2) L is a $\mathcal{F}$-Soc-T-ABSO ideal of $\mathcal{R}$ if and only if $\forall$ $\mathcal{F}$-ideals K,P and J of $\mathcal{R}$ with PJK$\subseteq$L , implies that either $PK \subseteq L + F - Soc(\mathcal{R})$ or $JK \subseteq L + F - Soc(\mathcal{R})$ or $PJ \subseteq K + F - Soc(\mathcal{R})$

Proof. Similarly as proof of Propositions 2.13 and 2.14.

Remark 3.16. *If U is a $\mathcal{F}$-Soc-T-ABSO sub-module of $\mathcal{F}$-module X, with $\mathcal{F} - Soc(X) \subseteq U$. Then U is $\mathcal{F}$- T-ABSO sub-module.*

Proof. Assume that U is a $\mathcal{F}$-Soc- T-ABSO sub-module of a $\mathcal{F}$-module X of an $\mathcal{R}$-module M. Let $s_q r_b x_k \subseteq U$, for $s_q, r_b \subseteq \mathcal{R}, x_k \subseteq X$,where $q, k, b \in [0, 1]$. Since U is a $\mathcal{F}$-Soc-T-ABSO sub-module, thus either $s_q x_k \subseteq U + F - Soc(X)$ or $r_b x_k \subseteq U + F - Soc(X)$ or $s_q r_b X \subseteq U + F - Soc(X)$ but $\mathcal{F} - Soc(X) \subseteq U$. Hence U is a $\mathcal{F}$-T-ABSO sub-module.

Corollary 3.17. *If U is a $\mathcal{F}$-Soc-T-ABSO sub-module of $\mathcal{F}$-module X, with U be an $\mathcal{F}$-essential sub-module of X. Then U is a $\mathcal{F}$-T-ABSO sub-module.*

Proof. Since U be an $\mathcal{F}$-essential sub-module of X, then by definition of $\mathcal{F}$-socle we have $\mathcal{F} - Soc(X) \subseteq U$ and by (Remark 2.16) that's complete the proof.

Corollary 3.18. *If U is a $\mathcal{F}$-sub-module of $\mathcal{F}$-module X, with $\mathcal{F} - Soc(X) \subseteq U$. Then U is a $\mathcal{F}$-T-ABSO sub-module of X if and only if U is a $\mathcal{F}$-T-ABSO sub-module of X.*

Proof. Consequently from (Remark 2.8) and (Remark 2.16).

Remark 3.19. *Let U and V are $\mathcal{F}$-sub-modules of $\mathcal{F}$-module X. If U+V is a $\mathcal{F}$-T-ABSO sub-module of X with $V \subseteq \mathcal{F} - Soc(X)$, then U is a $\mathcal{F}$-Soc-T-ABSO sub-module of X.*

Proof. By (definition 1.18) then U+V is a $\mathcal{F}$-sub-module of X. Now, let $s_q r_b x_k \subseteq U$, for $s_q, r_b \subseteq \mathcal{R}, x_k \subseteq X$,where $q,k,b \in [0,1]$. this implies $s_q r_b x_k \subseteq U+V$. But U+V is a $\mathcal{F}$-T-ABSO sub-module of X, therefore either $s_q x_t \subseteq U+V \subseteq U+F-Soc(X)$ or $r_b x_t \subseteq U+V \subseteq U+F-Soc(X)$ or $s_q r_b X \subseteq U+V \subseteq U+F-Soc(X)$. That is U is a $\mathcal{F}$-Soc-T-ABSO sub-module of X.

Theorem 3.20. *Any $\mathcal{F}$-sub-module of semi-simple $\mathcal{F}$-module X is a $\mathcal{F}$-Soc-T-ABSO sub-module of X.*

Proof. If U is a $\mathcal{F}$-sub-module of a $\mathcal{F}$-module X of an $\mathcal{R}$-module M. Let $s_q r_b m_t \subseteq U$, for $r_b, s_q \subseteq \mathcal{R}, m_t \subseteq X$,where $q,t,b \in [0,1]$. But X is semi-simple $\mathcal{F}$-module, thus $X = \mathcal{F} - Soc(X)$. We have $m_t \subseteq X = \mathcal{F} - Soc(X) \subseteq U + \mathcal{F} - Soc(X)$, this implies $r_b m_t \subseteq r_b X = r_b \mathcal{F} - Soc(X) \subseteq r_b(U + \mathcal{F} - Soc(X)) \subseteq U + \mathcal{F} - Soc(X)$ that's mean U is a $\mathcal{F}$-Soc-T-ABSO sub-module of X.

Lemma 3.21. $\left(A \bigoplus B\right) + F - Soc\left(X \bigoplus Y\right) = (A + F - Soc(X)) \bigoplus (B + F - Soc(Y))$ *for every $\mathcal{F}$-sub-modules A and B of $\mathcal{F}$-modules X and Y respectively.*

Proposition 3.22. *If U and V are $\mathcal{F}$-sub-modules of $\mathcal{F}$-modules X and Y respectively, then*
1) U is a $\mathcal{F}$-Soc-T-ABSO sub-module of X if and only if $U \bigoplus Y$ is a $\mathcal{F}$-Soc-T-ABSO sub-module of $X \bigoplus Y$.
2) V is a $\mathcal{F}$-Soc-T-ABSO sub-module of X if and only if $X \bigoplus V$ is a $\mathcal{F}$-Soc-T-ABSO sub-module of $X \bigoplus Y$.

Proof. 1) Assume that U is a $\mathcal{F}$-Soc-T-ABSO sub-module of X , s_q, r_b are $\mathcal{F}$-singletons of $\mathcal{R}$ and $(x_t, y_p) \subseteq X \bigoplus Y$ where $q,b,t,p \in [0,1]$such that $s_q r_b(x_t, y_p) \subseteq U \bigoplus Y$. This implies $s_q r_b x_t \subseteq U$, but U is a $\mathcal{F}$-Soc-T-ABSO sub-module of X, then either $s_q x_t \subseteq U + F - Soc(X)$ or $r_b x_t \subseteq U + F - Soc(X)$ or $s_q r_b X \subseteq U + F - Soc(X)$. So, either $(s_q x_t, y_p) \subseteq (U + F - Soc(X)) \bigoplus Y$ or $(r_b x_t, y_p) \subseteq (U + F - Soc(X)) \bigoplus Y$ or $(s_q r_b X \bigoplus r_b Y) \subseteq (U + F - Soc(X)) \bigoplus Y$. But $(U + F - Soc(X)) \bigoplus Y = (U + F - Soc(X)) \bigoplus (Y + F - Soc(Y))$ since $F - Soc(Y) \subseteq Y$, thus we have either $(s_q x_t, y_p) \subseteq (U + F - Soc(X)) \bigoplus (Y + F - Soc(Y))$ or $(r_b x_t, y_p) \subseteq (U + F - Soc(X)) \bigoplus (Y + F - Soc(Y))$ or $(s_q r_b X \bigoplus r_b Y) \subseteq (U + F - Soc(X)) \bigoplus (Y + F - Soc(Y))$. Now, by (Lemma 2.21) we get either $s_q(x_t, y_p) \subseteq \left(U \bigoplus Y\right) + F - Soc\left(X \bigoplus Y\right)$ or $r_b(x_t, y_p) \subseteq \left(U \bigoplus Y\right) + F - Soc\left(X \bigoplus Y\right)$ or $s_q r_b\left(X \bigoplus Y\right) = (s_q r_b X \bigoplus s_q r_b Y) \subseteq \left(U \bigoplus Y\right) + F - Soc\left(X \bigoplus Y\right)$. Hence $U \bigoplus Y$ is $\mathcal{F}$-Soc-T-ABSO sub-module of $X \bigoplus Y$.

Conversely

Suppose that $U \bigoplus Y$ is $\mathcal{F}$-Soc-T-ABSO sub-module of $X \bigoplus Y$ with r_b, s_q are $\mathcal{F}$-singletons of $\mathcal{R}$ and $x_t \subseteq X$ such that $s_q r_b x_t \subseteq U$. Then $s_q r_b(x_t, y_p) = (s_q r_b x_t, s_q r_b y_p) \subseteq U \bigoplus Y$, for any $\mathcal{F}$-singleton $y_p \subseteq Y$, but $U \bigoplus Y$ is $\mathcal{F}$-Soc-T-ABSO sub-module of $X \bigoplus Y$. Thus either $s_q(x_t, y_p) \subseteq \left(U \bigoplus Y\right) + F - Soc\left(X \bigoplus Y\right)$ or $r_b(x_t, y_p) \subseteq \left(U \bigoplus Y\right) + F - Soc\left(X \bigoplus Y\right)$ or $s_q r_b\left(X \bigoplus Y\right) = (s_q r_b X \bigoplus s_q r_b Y) \subseteq \left(U \bigoplus Y\right) + F - Soc\left(X \bigoplus Y\right)$, by (Lemma 2.21) we get either $(s_q x_t, s_q y_p) \subseteq (U + F - Soc(X)) \bigoplus (Y + F - Soc(Y))$ or $(r_b x_t, r_b y_p) \subseteq (U + F - Soc(X)) \bigoplus (Y + F - Soc(Y))$ or $(s_q r_b X \bigoplus s_q r_b Y) \subseteq (U + F - Soc(X)) \bigoplus (Y + F - Soc(Y))$. That either $s_q x_t \subseteq U + F - Soc(X)$ or $r_b x_t \subseteq U + F - Soc(X)$ or $s_q r_b X \subseteq U + F - Soc(X)$, therefore U is $\mathcal{F}$-Soc-T-ABSO sub-module of X.

2) Similarly as the idea in (1).

Lemma 3.23. *If X is a $\mathcal{F}$-module of an $\mathcal{R}$-module M, and M be a faithful multiplication $\mathcal{R}$-module, then:*
$\mathcal{F} - Soc(X) = X\mathcal{F} - Soc(\mathcal{R})$.

Proposition 3.24. *Let X be a finitely generated multiplication and faithful $\mathcal{F}$-module of an $\mathcal{R}$-module M, if J is a $\mathcal{F}$-Soc-T-ABSO ideal of $\mathcal{R}$ then JX is a $\mathcal{F}$-Soc-T-ABSO sub-module of X.*

Proof. Let r_b, s_q are $\mathcal{F}$-singletons of $\mathcal{R}$ and $m_t \subseteq X$ where $q,b,t \in [0,1]$, such that $s_q r_b m_t \subseteq JX$, where $q,t,b \in [0,1]$.that is $s_q r_b \langle x_t \rangle \subseteq JX$. But X is a multiplication $\mathcal{F}$-module, thus there exist an $\mathcal{F}$-ideal L of $\mathcal{R}$ with $\langle x_t \rangle = LX$. Then we get $s_q r_b LX \subseteq JX$, so $s_q r_b L \subseteq J + \mathcal{F} - ann(X) = J$ since X is a faithful $\mathcal{F}$- module. But J is a $\mathcal{F}$-Soc-T-ABSO ideal of $\mathcal{R}$, then by (Corollary 2.15 (1)) implies that either $s_q r_b \subseteq J + \mathcal{F} - Soc(\mathcal{R})$ or $r_b L \subseteq J + \mathcal{F} - Soc(\mathcal{R})$ or $s_q L \subseteq J + \mathcal{F} - Soc(\mathcal{R})$. Now, by multiplying both sides with X and using (Lemma 2.21) we have either $s_q r_b X \subseteq JX + \mathcal{F} - Soc(\mathcal{R})X = JX + \mathcal{F} - Soc(X)$ or $r_b LX \subseteq JX + \mathcal{F} - Soc(X)$ or $s_q LX \subseteq JX + \mathcal{F} - Soc(X)$. Therefore, JX is a $\mathcal{F}$-Soc-T-ABSO sub-module of X.

Proposition 3.25. *Suppose that U is a $\mathcal{F}$-Soc-T-ABSO sub-module of a $\mathcal{F}$-module X and V is T-ABSO $\mathcal{F}$- sub-module of X with $\mathcal{F} - Soc(X) \subseteq V$. Then the intersection of U and V is a $\mathcal{F}$-Soc-T-ABSO sub-module of X.*

Proof. Assume that r_b, s_q are $\mathcal{F}$-singletons of $\mathcal{R}$ and $m_t \subseteq X$ where $q,b,t \in [0,1]$, such that $s_q r_b m_t \subseteq U \cap V$,where. This implies $s_q r_b m_t \subseteq U$ and $s_q r_b m_t \subseteq V$, but U is a $\mathcal{F}$-Soc-T-ABSO sub-module of X. So, we get either $s_q r_b \subseteq [U + \mathcal{F} - Soc(X) :_{\mathcal{R}} X]$ or $r_b m_t \subseteq U + \mathcal{F} - Soc(X)$ or $s_q m_t \subseteq U + \mathcal{F} - Soc(X)$..Now, since V is a T-ABSO $\mathcal{F}$-sub-module of X this implies either $s_q r_b \subseteq [V :_{\mathcal{R}} X]$ or $r_b m_t \subseteq V$ or $s_q m_t \subseteq V$.Therefore either $s_q r_b X \subseteq (U + \mathcal{F} - Soc(X)) \cap V$ or $r_b m_t \subseteq (U + \mathcal{F} - Soc(X)) \cap V$ or $s_q m_t \subseteq (U + \mathcal{F} - Soc(X)) \cap V$, but $\mathcal{F} - Soc(X) \subseteq V$ then by using (Lemma 1.28) we have either $s_q r_b X \subseteq (U \cap V) + \mathcal{F} - Soc(X)$ or $r_b m_t \subseteq (U \cap V) + \mathcal{F} - Soc(X)$ or $s_q m_t \subseteq (U \cap V) + \mathcal{F} - Soc(X)$. That's mean $U \cap V$ is a $\mathcal{F}$-Soc-T-ABSO sub-module of X.

Proposition 3.26. *Let X be a faithful multiplication $\mathcal{F}$-module of an $\mathcal{R}$-module M, then a proper $\mathcal{F}$- sub-module U is $\mathcal{F}$-Soc-T-ABSO sub-module of if and only if $[U :_R X]$ is $\mathcal{F}$-Soc-T-ABSO ideal of $\mathcal{R}$.*

Proof. Let $s_q r_b c_t \subseteq [U :_R X]$,where $c_t, r_b and s_q$ are $\mathcal{F}$-singletons of $\mathcal{R}$ where $q, b, t \in [0, 1]$, implies that $s_q r_b (c_t X) = (sr)_h (c_t X) \subseteq U$, with $h = min\{q, b\}$ But U is $\mathcal{F}$-Soc-T-ABSO sub-module, so by (Corollary 2.12) then either $s_q r_b \subseteq [U + \mathcal{F} - Soc(X) :_{\mathcal{R}} X]$ or $r_b (c_t X) \subseteq U + \mathcal{F} - Soc(X)$ or $s_q (c_t X) \subseteq U + \mathcal{F} - Soc(X)$. Since X is multiplication $\mathcal{F}$-module, then by (Preposition 1.31) $U = [U :_R X] X$, and since X is faithful multiplication, so by (Lemma 2.23) $F - Soc(X) = \mathcal{F} - Soc(\mathcal{R}) X$. Thus either $s_q r_b X \subseteq [U :_R X] X + \mathcal{F} - Soc(\mathcal{R}) X$ or $r_b c_t X \subseteq [U :_R X] X + \mathcal{F} - Soc(\mathcal{R}) X$ or $s_q c_t X \subseteq [U :_R X] X + \mathcal{F} - Soc(\mathcal{R}) X$, this implies either $r_b m_t \subseteq [U :_R X] + \mathcal{F} - Soc(\mathcal{R})$ or $r_b c_t \subseteq [U :_R X] + \mathcal{F} - Soc(\mathcal{R})$ or $s_q c_t \subseteq [U :_R X] + \mathcal{F} - Soc(\mathcal{R})$. Thus $[U :_R X]$ is $\mathcal{F}$-Soc-T-ABSO ideal of $\mathcal{R}$.

Conversely

Let $s_q r_b E \subseteq U$ with s_q, r_b are $\mathcal{F}$-singletons of $\mathcal{R}$ and E is $\mathcal{F}$-sub-module of X. Since X is multiplication $\mathcal{F}$-module, then $E = JX$ for some $\mathcal{F}$-ideal J of $\mathcal{R}$, we get $s_q r_b JX \subseteq U$ that's mean $s_q r_b J \subseteq [U :_R X]$,but $[U :_R X]$ is $\mathcal{F}$-Soc-T-ABSO ideal of $\mathcal{R}$, so by (Corollary 2.13 (1)) we have either $r_b J \subseteq [U :_R X] + \mathcal{F} - Soc(\mathcal{R})$ or $s_q J \subseteq [U :_R X] + \mathcal{F} - Soc(\mathcal{R})$ or $s_q r_b \subseteq [U :_R X] + \mathcal{F} - Soc(\mathcal{R})$, this implies either $r_b JX \subseteq [U :_R X] X + \mathcal{F} - Soc(\mathcal{R}) X$ or $s_q J \subseteq [U :_R X] X + \mathcal{F} - Soc(\mathcal{R}) X$ or $s_q r_b \subseteq [U :_R X] X + \mathcal{F} - Soc(\mathcal{R}) X$, then by (Lemma 2.23) we have U is $\mathcal{F}$-Soc-T-ABSO sub-module of X.

Lemma 3.27. *Let $f : M \to \overline{M}$ be isomorphism mapping from an $\mathcal{R}$-module M into an $\mathcal{R}$-module $\overline{M}$ .If X and $\overline{X}$are $\mathcal{F}$-modules of an $\mathcal{R}$-modules M and $\overline{M}$ respectively. Then $f(\mathcal{F} - Soc(X)) \subseteq \mathcal{F} - Soc\left(\overline{X}\right)$.*

Proposition 3.28. *Let $f : X \to \overline{X}$ be an $\mathcal{F}$-isomorphism from $\mathcal{F}$-module X into $\mathcal{F}$-module $\overline{X}$, with U is an $\mathcal{F}$-Soc-T-ABSO submodule of X, such that $ker(f) \subseteq U$. Then $f(U)$ is an $\mathcal{F}$-Soc-T-ABSO sub-module of $\overline{X}$.*

Proof. $f(U)$ is a proper $\mathcal{F}$-sub-module of $\overline{X}$. If not, then $f(U) = \overline{X}$. Let $x_t \subseteq X$, so $f(x_t) \subseteq \overline{X} = f(U)$, that is there exist $y_s \subseteq U$ where $s, t \in [0, 1]$such that $f(x_t) = f(y_s)$, implies that $f(x_t) - f(y_s) = 0_1$ then $f(x_t - y_s) = 0_1$, thus $x_t - y_s \subseteq ker(f) \subseteq U$, it follows that $x_t \subseteq U$.Thus $U = X$ that's contradiction. Now, Let $s_q r_b m_t \subseteq f(U)$,where r_b, s_q are $\mathcal{F}$-singletons of $\mathcal{R}$ and m_t is a $\mathcal{F}$-singleton of $\overline{X}$. But f is onto$f(n_t) = m_t$ for some $n_t \subseteq X$,that is $s_q r_b m_t = s_q r_b f(n_t) = f\left(s_q r_b n_t\right) \subseteq f(U)$implies that $f\left(s_q r_b n_t\right) = f(z_h)$ for some $z_h \subseteq U$. Thus $f\left(s_q r_b n_t\right) - f(z_h) = 0_1$, then $f\left(s_q r_b n_t - z_h\right) = 0_1$ that's mean $s_q r_b n_t - z_h \subseteq ker(f) \subseteq U$,an s . So, $s_q r_b n_t \subseteq U$ but U is $\mathcal{F}$-Soc-T-ABSO sub-module of X. hence ,at is lemma we have either $s_q r_b \subseteq [U + \mathcal{F} - Soc(X) :_{\mathcal{R}} X]$ or $r_b n_t \subseteq U + \mathcal{F} - Soc(X)$ or $s_q n_t \subseteq U + \mathcal{F} - Soc(X)$.Now, we get either $f(s_q r_b X) = s_q r_b \overline{X} \subseteq f(U + \mathcal{F} - Soc(X))$ or $f(r_b n_t) = r_b m_t \subseteq f(U + \mathcal{F} - Soc(X))$ or $f(s_q n_t) = s_q m_t \subseteq f(U + \mathcal{F} - Soc(X))$. Then by (Lemma 2.27) we have either $s_q n_t \overline{X} \subseteq f(U) + f(\mathcal{F} - Soc(X)) \subseteq f(U) + \mathcal{F} - Soc\left(\overline{X}\right)$ or $r_b m_t \subseteq f(U) + \mathcal{F} - Soc\left(\overline{X}\right)$ or $s_q m_t \subseteq f(U) + \mathcal{F} - Soc\left(\overline{X}\right)$. Hence $f(U)$ is an $\mathcal{F}$-Soc-T-ABSO sub-module of $\overline{X}$.

4. CONCLUSION

The idea of fuzzy socle T-Absorbing sub-modules is dualised in this study by introducing several characteristics and properties of fuzzy T-Absorbing sub-modules. This approach has opened up new possibilities for studying the fuzzy dimension. For example, socle T-Absorbing module and completely socle T-Absorbing sub-modules can be defined utilising the concept of fuzzy socle T-Absorbing sub-modules**.**

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CONFLICTS OF INTEREST

The authors declare no conflict of interest.

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