Solving Non-linear Volterra-Hammerstein Integral Equations of the Second Kind Using Modified Decomposition Method

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Abstract

In this paper, non-linear Volterra-Hammerstein integral equations of the second kind (*NVHIEK2*) considered. The modified Adomian decomposition method is been used to solve the *NVHIEK2*. Some illustrative examples are prepared to show the efficiency and simplicity of the method.

الخلاصة

و استخدمة طريقة ادوميان التحليلية المتطورة ,اعتبر في هذه الدراسة، معادلة هامرستن فولتيرا اللاخطية التكاملية من النوع الثاني مع بعض امثلة ايضاحية المعدة لبيان كفاءة و سهولة الطريقة.,لحل هذه المعادلات

Keywords: Modified decomposition method, non-linear Volterra-Hammerstein integral equation.

1. Introduction:

Recently a great deal of interest has been focused on the applications of the Adomian method to solve a wide variety of stochastic and deterministic problems (Babolian and Vahidi, 2004). The solution is the sum of an infinite series which converges rapidly to the accurate solutions. A non-linear Volterra-Hammerstein integral equation (Bownds and Wood, 1976) can be written as the following:

$$u(x,t) = f(x,t) + \int_a^x k(x,t)\big[R(u(x,t)) + N(u(x,t))\big]dt \qquad , \qquad a \geq 0 \qquad (1)$$

where $u(x, t)$ is an unknown function that will be determined, $k(x, t)$ is the kernel of the integral equation, $f(x, t)$ is an analytic function, and $R(u)$ and $N(u)$ are the linear and non-linear function of u, respectively.

The basic theory of existence and uniqueness for this equation has been more or less fully developed and most time we get unique solution of this equation. An equation of this sort can arise in a number of application and the most common are probably related to problems of heat transfer, nuclear reactor dynamics, population models and renewal theory. We consider NVH rather than a more general Volterra equation because we note that most NVIE which appear in the application are in fact of the form of VHE.

Several numerical methods used to fined approximate solution of NVHIE such as Single-Term Walsh Series (Sepehrian and Razzaghi, 2005), Exact controllability of generalized (Chalishajar and George, 2006) and A Class of Volterra – Hammerstein Integral Equations With Noncompact Kernels (Hadizadeh and Mohamdsohi, 2005).

The concepts of integral equation have attracted much interest in recent years for analytical and numerical treatments. Due to the increased interest in non-linear integral equations, abroad classes of analytical and numerical solution methods have been used to handle these problems (WazWaz, 2005). However, this analytical solution method is not easy to use and require tedious work and knowledge.

To overcome the tedious work involved in the existing strategies and to minimize the proliferation of terms in the Adomian scheme, the modified Adomian decomposition method is studying.

The aim of this paper is to obtain analytical solutions for non-linear Volterra-Hammerstein integral equation of the second kind. Although the modified forms introduce a slight change in the formulation of the Adomian recursive relation, it provides a qualitative improvement over standard Adomian method. In addition, the modified technique may give the exact solution for non-linear equation without any need for the Adomian polynomials. Although this slight variation is quite simple, it demonstrates the reliability and the power of the modification.

2. Adomian Decomposition Method:

Adomian decomposition method was used to handle a variety of linear and non-linear problem among other works. The Adomian decomposition method gives a reasonable basis for studying system of equations. The method has been modified to accelerate the convergence of the solution. Both the standard decomposition method and the modification form transform the system of equations into a set of recursive relations (Wazwaz, 2003). Adomian decomposition method has been acclaimed as a significant powerful method for solving system of nonlinear equations as nonlinear reaction diffusion system of Lokta-Volterra type (Alabdullatif, Abdusalam and Fahmy, 2007) and a Class of Volterra – Hammerstein integral equations with noncompact kernels (Hadizadeh and Mohamadsohi, 2005).

3. The Modified Decomposition Method for NVHIEK2:

Adomian asserts that the decomposition method provides an efficient and computationally convenient method for generating approximate series solutions to a wide class of equations. The method is applied as follows:

The standard Adomian method defines the solution $u(x)$ by the se series

$$u(x,t)=\sum_{i=0}^{\infty}u_i(x,t) \quad . \tag{2}$$

Substituting the series decomposition (2) into both sides of equation (1) and assuming that the function f can be expressed as the sum of two parts, f_0 and f_1. Therefore we set

$$f = f_0 + f_1 \, . \tag{3}$$

In view of this assumption, we propose a slight variation is that only the part f_0 is assigned to the zeroth component u_0 and the remaining part f_1 is combined with the other terms (integral part) to define u_1, based on the suggestion, we formulate the following modified decomposition method:

$$\sum_{i=0}^{\infty}u_i(x,t)=f_0(x,t)+f_1(x,t)+\int_a^x k(x,t)\left[R(\sum_{i=0}^{\infty}u_i(x,t))+N(\sum_{i=0}^{\infty}u_i(x,t))\right]dt. \tag{4}$$

The modified decomposition method introduces the use of the recurrence relation

$$u_0(x,t) = f_0(x,t)$$

$$u_1(x,t) = f_1(x,t)+\int_a^x k(x,t)\left[R(u_0(x,t))+N(u_0(x,t))\right]dt,$$

$$u_{k+2}(x,t)=\int_a^x k(x,t)\left[R(u_{k+1}(x,t))+N(u_{k+1}(x,t))\right]dt \, , \; k \geq 0 \, . \tag{5}$$

As stated before, we may need only two iterations to derive the exact solution. If more than two iterations are needed, $N(u(x, t))$ should be represented by Adomian polynomials which can easily be generated for all types of non-linearity. The choice of $f_0(x,t)$ contain the minimal number of terms has a strong influence on facilitating the recurrence relation, and as a consequence, accelerates the convergence of the solution. Also the exact solution must be a term of $f(x, t)$ or to be a part of series of $f(x, t)$. This means that the success of this method depends mainly on the proper choice of f_0 and f_1. We have been unable to establish any criteria to judge what forms of f_0 and f_1 can be used to yield the acceleration demanded. At present f_0 and f_1 are selected by trials. Several illustrative examples are used to show the pertinent features of the modified method.

4. The Noise Terms phenomenon:

The noise terms phenomenon gives a useful tool in that, if it appears, it gives a fast convergence of the solution by using two iterations only. It is significant to note that the noise terms may appear only for inhomogeneous problems. It is important to note that these terms may appear for inhomogeneous problems, whereas homogeneous problems do not general noise terms (WazWaz, 2003).

A necessary condition for obtaining exact solution in minimum iteration (only two iterations) is that the exact solution must be a part of noise terms (or a part of series of noise terms).
To give a clear overview of the content of this work, several illustrative examples of *NVHIEK*2 have been selected to demonstrate the efficiency of the method and to confirm the necessary condition for obtaining exact solution.

5. Illustrative Examples:

The method of this paper is useful for finding the solutions of non-linear *VHIEK*2 in terms of modified decomposition method. We illustrate this by solving the following five examples (Hadizadeh and Mohamadsohi, 2005):

Example 1:

Consider a non-linear *VHIEK*2's:

$$u(x) = x + \frac{1}{x+1}(\exp(-(x+1)) - 1) + \int_0^x (\frac{1}{x}\exp(\frac{-t}{x})\exp(-u(t)))dt.$$

We decompose $f(x)$ as follows:

$$f_0 = x,$$

$$f_1 = \frac{1}{x+1}(\exp(-(x+1)) - 1).$$

The modified decomposition technique admits the use of the recursive relation given in the form

$$u_0 = x,$$

$$u_1 = \frac{1}{x+1}(\exp(-(x+1)) - 1) + \int_1^x \frac{1}{x}\exp(\frac{-t}{x})\exp(u_0(t))dt.$$

This gives

$$u_0 = x,$$

$$u_1 = \frac{1}{x+1}(\exp(-(x+1)) - 1) + \int_1^x \frac{1}{x}\exp(\frac{-t}{x})(\exp(-t))dt = 0.$$

It follows immediately that $u_k = 0,\ k \geq 2$, and the exact solution $u(x) = x$, readily obtain.

Example 2:

Solve a non-linear *VHIEK*2's:

$$u(x) = \frac{-x^4}{10} + \frac{5x^2}{6} + \frac{3}{8} + \int_0^x (\frac{1}{2x}(u(t))^2)dt.$$

We decompose $f(x)$ as follows:

$$f_0 = x^2 + \frac{1}{2},$$

$$f_1 = \frac{-x^4}{10} - \frac{x^2}{6} - \frac{1}{8}.$$

The modified decomposition technique admits the use of the recursive relation given in the form

$u_0 = x^2 + \frac{1}{2}$,

$$u_1(x) = \frac{-x^4}{10} + \frac{5x^2}{6} + \frac{3}{8} + \int_0^x \left(\frac{1}{2x}\left(t^2 + \frac{1}{2}\right)^2\right)dt.$$

This gives

$u_0 = x$,

$$u_1(x) = \frac{-x^4}{10} + \frac{5x^2}{6} + \frac{3}{8} + \int_0^x \left(\frac{1}{2x}\left(t^2 + \frac{1}{2}\right)^2\right)dt = 0.$$

It follows immediately that $u_k = 0,\ k \geq 2$, and the exact solution $u = x^2 + \frac{1}{2}$, readily obtain.

6. Conclusions:

This paper presents the use of the reliable modified Adomian decomposition method for solving non-linear Volterra-Hammerstein integral equations of the second kind. The modified Adomian decomposition method is implemented in a straight forward manner and provided significant improvement by requiring only two iterations to obtain the exact solution in most cases but the solution of the same examples if we comparison with (Hadizadeh and Mohamadsohi 2005)we see that after ten step gets an approximate solution. Accelerating the convergence of the modified Adomian method requires that the exact solution must be a part of $f(x, t)$ or series of $f(x, t)$.

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