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n-absorbing I-primary ideals in commutative rings

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ABSTRACT

We define a new generalization of n-absorbing ideals in commutative rings called n-absorbing I-primary ideals. We investigate some characterizations and properties of such new generalization. If P is an n-absorbing I-primary ideal of R and $\sqrt{IP}=I\sqrt{P}$, then $\sqrt{P}$ is a n-absorbing I-primary ideal of R. And if $\sqrt{P}$ is an (n-1)-absorbing ideal of R such that $\sqrt{(I\sqrt{P})}\subseteq IP$, then P is an n-absorbing I-primary ideal of R.

مثالية بدائية مختزلة n من نوع I

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الملخص

عرفنا تعميما جديدا للمثاليه المختزلة n في الحلقات المتبادلة وتسميها المثالية الاجتدائية المختزلة n من نوع I . بينا بعض المميزات والخواص لهذا التعميم الجديد . إذا كانت P هي نموذج بدائية مختزلة n من نوع I لـ R و $\sqrt{IP}=I\sqrt{P}$ ، فإن $\sqrt{P}$ هو نموذج بدائية مختزلة n من نوع I . وإذا كانت $\sqrt{P}$ نموذجًا ممتصًا (n−1) من R مع $\sqrt{(I\sqrt{P})}\subseteq IP$ ، ثم P بدائية مختزلة n من نوع I لـ R .

Introduction

In our article all rings are commutative ring with non-zero identity. In the recent years many generalizations of prime ideals were defined. Here state some of them. The notion of a weakly prime ideal was introduced by Anderson and Smith, where a proper ideal P of a commutative ring R is a weakly prime if $x,y\in R$, and $0\neq xy\in P$ then $x\in P$ or $y\in P$ in [5] . Ebrahimi Atani and Farzalipour defined the nation of weakly primary ideals [8] . The authors in [6] and [3] introduced the notions 2-

absorbing and n-absorbing ideals in commutative rings. A proper ideal P is said to be 2-absorbing (or n-absorbing) ideal if whenever the product of three (or n+1) elements of R in P, the product of two (or n) of these elements is in P.

In [1] and [2], the author Akray introduced the notions I-prime ideal, I-primary ideal and n-absorbing I-ideal in commutative rings as a generalization of prime ideals. For fixed proper ideal I of a commutative ring R with identity, a proper ideal P of R is an I-primary if for $c,d\in R$ with $cd\in P-IP$, then $c\in P$ or $d\in\sqrt{P}$. Throughout the paper the notation $b_1\cdots\hat{b_i}\cdots b_n$ means that b_i is excluded from the product $b_1\cdots b_n$. A proper ideal P of R is an n-absorbing I-primary ideal if for $b_1,\cdots,b_{n+1}\in R$ such that $b_1\cdots b_{n+1}\in P-IP$, then $b_1\cdots b_n\in P$ or $b_1\cdots b_{i-1}\ b_{i+1}\cdots b_{n+1}\in\sqrt{P}$ for some $i\in\{1,2,\cdots,n\}$ where $\sqrt{P}$ is the radical of the ideal P.

Assume that R is an integral domain with quotient field F. The authors in [7] introduced a "proper ideal P of R is a strongly primary if, whenever $cd\in P$ with $c,d\in F$, we have $c\in P$ or $d\in\sqrt{P}$. In [9], a proper ideal P of R is a strongly I-primary ideal if $cd\in P-IP$ with $c,d\in F$, then $c\in P$ or $d\in\sqrt{P}$. It is said that a proper ideal P of R is quotient n-absorbing I-primary" if $b_1\ b_2\cdots b_{n+1}\in P$ with $b_1,b_2,\ldots,b_{n+1}\in F$, then $b_1\ b_2\cdots b_n\in P$ or $b_1\cdots\hat{b_i}\cdots b_{n+1}\in\sqrt{P}$ for some $1\le i\le n$. Set P is an ideal of a ring R, let P be an n-absorbing I-primary ideal of R and $b_1,\ldots,b_{n+1}\in R$. The statement is that $(b_1,\ldots,b_{n+1})$ is an I-(n+1)-tuple of P if $b_1\cdots b_{n+1}\in IP$, $b_1\ b_2\cdots b_n\notin P$ and for any $1\le i\le n$, $b_1\cdots\hat{b_i}\cdots b_{n+1}\notin\sqrt{P}$.

2 n-absorbing I-primary ideals

In this section, we start with to define the definition of an n-absorbing I-primary ideal of a ring R.

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Definition A proper ideal P of R is an I-primary if for $c,d\in R$ with $cd\in P-IP$, then $c\in P$ or $d\in\sqrt{P}$.

Definition A proper ideal P of R is an n-absorbing I-primary ideal if for $b_1,\cdots,b_{n+1}\in R$ such that $b_1\cdots b_{n+1}\in P-IP$, then $b_1\cdots b_n\in P$ or $b_1\cdots b_{i-1}\ b_{i+1}\cdots b_{n+1}\in\sqrt{P}$ for some $i\in\{1,2,\cdots,n\}$.

Example 2.1 Consider the ring $A=k[t_1,t_2,\ldots,t_{n+2}]$, where k is a field and suppose that $P=\langle t_1\ t_2\cdots t_{n+1}, t_1^2\ t_2\cdots t_n, t_1^2\ t_{n+2}\rangle$, $I=\langle t_1\ t_2\cdots t_n, t_1\ t_2\cdots t_{n+1}\rangle$. Then $P-IP=\langle t_1\cdots t_{n+1}, t_1^2\ t_2\cdots t_n, t_1^2\ t_{n+2}\rangle-\langle t_1\ t_2\cdots t_{n+1}, t_1^2\cdots t_n, t_1^2\cdots t_{n+1}, t_1^2\ t_2\cdots t_{n+2}\rangle$. Hence P is an n-absorbing I-primary ideal but P is not n-absorbing.

Proposition 2.2 We set that R is a ring. Based on this, the following statements can be considered equivalent:

(i) P is an n-absorbing I-primary ideal of R;

(ii) For any elements $\alpha_1,\ldots,\alpha_n\in R$ with $\alpha_1\cdots\alpha_n$ not in $\sqrt{P}$, $(P:_R \alpha_1\cdots\alpha_n)\subseteq[\cup_{i=1}^{n-1}(\sqrt{P}:_R \alpha_1\cdots\hat{\alpha}_i\cdots\alpha_n)]\cup(P:_R \alpha_1\cdots\alpha_{n-1})\cup(IP:_R \alpha_1\cdots\alpha_n)$.

Proof. (i) $\Rightarrow$ (ii) Set $\alpha_1,\ldots,\alpha_n\in R$ such that $\alpha_1\cdots\alpha_n\notin\sqrt{P}$. Let $r\in(P:_R \alpha_1\cdots\alpha_n)$. So $r\alpha_1\cdots\alpha_n\in P$. If $r\alpha_1\cdots\alpha_n\in IP$, then $r\in(IP:_R \alpha_1\cdots\alpha_n)$. Let $r\alpha_1\cdots\alpha_n\notin IP$. Since $\alpha_1\cdots\alpha_n\notin\sqrt{P}$, either $r\alpha_1\cdots\alpha_{n-1}\in P$, that is, $r\in(P:_R \alpha_1\cdots\alpha_{n-1})$ or for some $1\le i\le n-1$ we have $r\alpha_1\cdots\hat{\alpha_i}\cdots\alpha_n\in\sqrt{P}$, that is, $r\in(\sqrt{P}:_R \alpha_1\cdots\hat{\alpha_i}\cdots\alpha_n)$ or $\alpha_1\ \alpha_2\cdots\alpha_n\in\sqrt{P}$. By assumption the last case is not hold. Consequently

$$((P:_R \alpha_1\cdots\alpha_n)\subseteq[\cup_{i=1}^{n-1}(\sqrt{P}:_R \alpha_1\cdots\hat{\alpha}_i\cdots\alpha_n)]\cup(P:_R \alpha_1\cdots\alpha_{n-1})\cup(IP:_R \alpha_1\cdots\alpha_n).)$$

(ii) ⇒ (i) Let $\beta_1\beta_2\cdots\beta_{n+1}\in P-IP$ for some $\beta_1,\beta_2,\ldots,\beta_{n+1}\in R$ such that $\beta_1\beta_2\cdots\beta_n\notin P$. Then $\beta_1\in(P:_R \beta_2\cdots\beta_{n+1})$. If $\beta_2\cdots\beta_{n+1}\in\sqrt{P}$, then we are done. Hence we may set $\beta_2\cdots\beta_{n+1}\notin\sqrt{P}$ and so by part (ii), $(P:_R \beta_2\cdots\beta_{n+1})\subseteq[\cup_{i=2}^{n}(\sqrt{P}:_R \beta_2\cdots\hat{\beta}_i\cdots\beta_{n+1})]\cup(P:_R \beta_2\cdots\beta_n)\cup(IP:_R \beta_2\cdots\beta_{n+1})$. Since $\beta_1\beta_2\cdots\beta_{n+1}\notin IP$ and $\beta_1\beta_2\cdots\beta_n\notin P$, the only possibility is that $\beta_1\in\cup_{i=2}^{n}(\sqrt{P}:_R \beta_2\cdots\hat{\beta}_i\cdots\beta_{n+1})$. Then $\beta_1\beta_2\cdots\hat{\beta}_i\cdots\beta_{n+1}\in\sqrt{P}$ for some $2\le i\le n$. Hence P is an n-absorbing I-primary ideal of R.

Proposition 2.3 If V be a valuation domain with the quotient field F. Then all n-absorbing I-primary ideal of V is a quotient n-absorbing I-primary ideal of R.

Proof. We can certainly assume that P is n-absorbing I-primary ideal of V, and $a_1a_2\cdots a_{n+1}\in P$ for some $a_1,a_2,\ldots,a_{n+1}\in V$ such that $a_1a_2\cdots a_n\notin P$. If $a_{n+1}\notin V$, then $a_{n+1}^{-1}\in V$. So $a_1\cdots a_n a_{n+1}a_{n+1}^{-1}=a_1\cdots a_n\in P$, which is a contradiction. So $a_{n+1}\in V$. If $a_i\in V$ for all $1\le i\le n$, then there is nothing to prove. If $a_i\notin V$ for some $1\le i\le n$, then $a_1\cdots\hat{a}_i\cdots a_{n+1}\in P\subseteq\sqrt{P}$. Consequently, P is a quotient n-absorbing I-primary.

Proposition 2.4 Set P be an n-absorbing I-primary ideal of R such that $\sqrt{IP}=I\sqrt{P}$, then $\sqrt{P}$ is a n-absorbing I-primary ideal of\

Proof. Let us assume $a_1a_2\cdots a_{n+1}\in\sqrt{P}-I\sqrt{P}$ for some $a_1,a_2,\ldots,a_{n+1}\in R$ such that $a_1\cdots\hat{a}_i\cdots a_{n+1}\notin\sqrt{P}$ for every $1\le i\le n$. Thus we have $n\in N$ such that $a_1^n a_2^n\cdots a_{n+1}^n\in P$. If $a_1^n a_2^n\cdots a_{n+1}^n\in IP$, then $a_1a_2\cdots a_{n+1}\in\sqrt{IP}=I\sqrt{P}$, which is a contradiction. Since P is an n-absorbing I-primary, our hypothesis implies $a_1^n a_2^n\cdots a_n^n\in P$. So $a_1a_2\cdots a_n\in\sqrt{P}$ and $\sqrt{P}$ is an n-absorbing I-primary ideal of R.

Theorem 2.5 Assume that "for any $1\le i\le k$, I_i is an n_i-absorbing I-primary ideal of R such that $\sqrt{P_i}=q_i$ is an n_i-absorbing I-primary ideal of R, respectively. Let $n=n_1+n_2+\cdots+n_k$. The following statements does hold:

(1) $P_1\cap P_2\cap\cdots\cap P_k$ is an n-absorbing I-primary ideal of R.

(2) $P_1P_2\cdots P_k$ is an n-absorbing I-primary ideal of R".

Proof. The proof of the two parts are similar, so we prove just the first. Let $H=P_1\cap P_2\cap\cdots\cap P_k$. Then $\sqrt{H}=P_1\cap P_2\cap\cdots\cap P_k$. Let $a_1a_2\cdots a_{n+1}\in H-IH$ for some $a_1,a_2,\ldots,a_{n+1}\in R$ and $a_1\cdots\hat{a}_i\cdots a_{n+1}\notin\sqrt{H}$ for any $1\le i\le n$. By, $\sqrt{H}=P_1\cap P_2\cap\cdots\cap P_k$ is an n-absorbing I-primary, then $a_1a_2\cdots a_n\in P_1\cap P_2\cap\cdots\cap P_k$. We prove that $a_1a_2\cdots a_n\in H$. For all $1\le i\le k$, P_i is an n_i-absorbing I-primary and $a_1a_2\cdots a_n\in P_i-IP_i$, then we have $1\le\beta_1^i,\beta_2^i,\ldots,\beta_{n_i}^i\le n$ such that $a_{\beta_1^i}a_{\beta_2^i}\cdots a_{\beta_{n_i}^i}\in P_i$. If $\beta_r^l=\beta_s^m$ it is for two couples l,r and m,s, then

$$\begin{matrix} a_{\beta_1^1}a_{\beta_2^1}\cdots a_{\beta_{n_1}^1}\cdots a_{\beta_1^l}a_{\beta_2^l}\cdots a_{\beta_r^l}\cdots a_{\beta_{n_l}^t}\cdots \\ a_{\beta_1^m}a_{\beta_2^m}\cdots\hat{a}_{\beta_m^m}\cdots a_{\beta_{m_m}^m}\cdots a_{\beta_1^k}a_{\beta_2^k}\cdots a_{\beta_{n_k}^k}\in\sqrt{H}\end{matrix}$$

Therefore $a_1\cdots\hat{a}_{\beta_n^m}\cdots a_n a_{n+1}\in\sqrt{H}$, which is a contradiction. So β_j^i 's are distinct. Hence $\{a_{\beta_1^1},a_{\beta_2^1},\ldots,a_{\beta_{n_1}^1},a_{\beta_1^2},a_{\beta_2^2},\ldots,a_{\beta_{n_2}^2},\ldots,a_{\beta_1^k},a_{\beta_2^k},\ldots,a_{\beta_{n_k}^k}\}=\{a_1,a_2,\ldots,a_n\}$. If $a_{\beta_1^i}a_{\beta_2^i}\cdots a_{\beta_{n_i}^i}\in P_i$ for any $1\le i\le k$, then

$$a_1a_2\cdots a_n=a_{\beta_1^1}a_{\beta_2^1}\cdots a_{\beta_{n_1}^1}a_{\beta_1^2}a_{\beta_2^2}\cdots a_{\beta_{n_2}^2}\cdots a_{\beta_1^k}a_{\beta_2^k}\cdots a_{\beta_{n_k}^k}\in H,$$

thus we are done. Therefore we may assume that $a_{\beta_1^1} a_{\beta_2^1}\cdots a_{\beta_{n_1}^1}\notin P_1$. Since P_1 is $I=n_1$-absorbing I-primary and

$$a_{\beta_1^1} a_{\beta_2^1}\cdots a_{\beta_{n_1}} a_{\beta_1^2} a_{\beta_2^2}\cdots a_{\beta_{n_2}^2}\cdots a_{\beta_1^k} a_{\beta_2^k}\cdots a_{\beta_{n_k}^k} a_{n+1}=a_1\cdots a_{n+1}\in P_1-IP_1,$$

then we have $a_{\beta_1^2} a_{\beta_2^2}\cdots a_{\beta_{n_2}^2}\cdots a_{\beta_1^k} a_{\beta_2^k}\cdots a_{\beta_{n_k}^k} a_{n+1}\in P_1$. On the other hand $a_{\beta_1^2} a_{\beta_2^2}\cdots a_{\beta_{n_2}^2}\cdots a_{\beta_1^k} a_{\beta_2^k}\cdots a_{\beta_{n_k}^k} a_{n+1}\in P_2\cap\cdots\cap P_k$. Consequently $a_{\beta_1^2} a_{\beta_2^2}\cdots a_{\beta_{n_2}^2}\cdots a_{\beta_1^k} a_{\beta_2^k}\cdots a_{\beta_{n_k}^k} a_{n+1}\in\sqrt{H}$, which is a contradiction. Similarly $a_{\beta_1^i} a_{\beta_2^i}\cdots a_{\beta_{n_i}^i}\in P_i$ for every $2\le i\le k$. Then $a_1 a_2\cdots a_n\in H$.

Proposition 2.6 Assume that P " is an ideal of a ring R with $\sqrt{I\sqrt{P}}\subseteq IP$. If $\sqrt{P}$ is an $(n-1)$-absorbing ideal of R, then P is an n-absorbing I-primary ideal of R."

Proof. Let $\sqrt{P}$ be an $(n-1)$-absorbing, and consider $b_1 b_2\cdots b_{n+1}\in P-IP$ for some $b_1,b_2,\ldots,b_{n+1}\in R$ and $b_1 b_2\cdots b_n\notin P$. Since

$$(b_1 b_{n+1})(b_2 b_{n+1})\cdots(b_n b_{n+1})=(b_1 b_2\cdots b_n) b_{n+1}^n\in P\subseteq\sqrt{P}-I\sqrt{P}.$$

Then for some $1\le i\le n$,

$$(b_1 b_{n+1})\cdots\widehat{(x_i b_{n+1})}\cdots(b_n b_{n+1})=(b_1\cdots\widehat{b_i}\cdots b_n) b_{n+1}^{n-1}\in\sqrt{P},$$

and so $b_1\cdots\widehat{b_i}\cdots b_n b_{n+1}\in\sqrt{P}$. Consequently P is an n-absorbing I-primary ideal of R.

We recall that a proper ideal Q of R is an n-absorbing primary if $a_1, a_2,\cdots, a_{n+1}\in R$ and $a_1 a_2\cdots a_{n+1}\in Q$, then $a_1 a_2\cdots a_n\in Q$ or the product of a_{n+1} with $(n-1)$ of $a_1, a_2,\cdots, a_n$ is in $\sqrt{Q}$. It is clearly every n-absorbing primary is an n-absorbing I-primary.

Proposition 2.7 Suppose that R is a ring and $r\in R$, a nonunit and $m\ge 2$ is not negative integer. Let $(0:_R r)\subseteq\langle a\rangle$, then $\langle r\rangle$ is an n-absorbing I-primary, for some I with $IP\subseteq I^m$ if and only if $\langle a\rangle$ is n-absorbing primary.

Proof. Let $\langle r\rangle$ be n-absorbing I^m-primary, and $a_1 a_2\cdots a_{n+1}\in\langle r\rangle$ for some $a_1,a_2,\ldots,a_{n+1}\in R$. If $a_1 a_2\cdots a_{n+1}\notin\langle r^m\rangle$, then $a_1 a_2\cdots a_n\in\langle r\rangle$ or $a_1\cdots\widehat{a_i}\cdots a_{n+1}\in\sqrt{\langle r\rangle}$ for some $1\le i\le n$. Based on this assumption, $a_1 a_2\cdots a_{n+1}\in\langle r^m\rangle$. Hence $a_1 a_2\cdots a_n (a_{n+1}+r)\in\langle r\rangle$. If $a_1 a_2\cdots a_n (a_{n+1}+r)\notin\langle r^m\rangle$, then $a_1 a_2\cdots a_n\in\langle r\rangle$ or $a_1\cdots\widehat{a_i}\cdots a_n (a_{n+1}+r)\in\sqrt{\langle r\rangle}$ for some $1\le i\le n$. So $a_1 a_2\cdots a_n\in\langle r\rangle$ or $a_1\cdots\widehat{a_i}\cdots a_{n+1}\in\sqrt{\langle r\rangle}$ for some $1\le i\le n$. Hence, suppose that $a_1 a_2\cdots a_n (a_{n+1}+r)\in\langle r^m\rangle$. Thus $a_1 a_2\cdots a_{n+1}\in\langle r^m\rangle$ implies that $a_1 a_2\cdots a_n r\in\langle r^m\rangle$. Therefore, there exists $s\in R$ such that $a_1 a_2\cdots a_n-sr^{m-1}\in(0:_R r)\subseteq\langle r\rangle$. Consequently $a_1 a_2\cdots a_n\in\langle r\rangle$.

Proposition 2.8 Assume V is a valuation domain and $n\in N$. Let P be an ideal of V such that P^{n+1} is not principal. Then P is an n-absorbing I^{n+1}-primary if and only if it is an n-absorbing primary.

Proof. $(\Rightarrow)$ Let P be an n-absorbing I^n-primary that is n't n-absorbing primary. Therefore there are $a_1,\ldots,a_{n+1}\in R$ such that $a_1\cdots a_{n+1}\in P$, but neither $a_1\cdots a_n\in P$ nor $a_1\cdots\widehat{a_i}\cdots a_{n+1}\in\sqrt{P}$ for any $1\le i\le n$. Hence $\langle a_i\rangle\nsubseteq P$ for any $1\le i\le n+1$. And so V is a valuation domain, thus $P\subset\langle a_i\rangle$ for any $1\le i\le n+1$, and so $P^{n+1}\subseteq\langle a_1\cdots a_{n+1}\rangle$. Therefore P^{n+1} is not principal, then $a_1\cdots a_{n+1}\in P-P^{n+1}$. Therefore P is an n-absorbing I^{n+1}-primary implies that either $a_1\cdots a_n\in P$ or $a_1\cdots\widehat{a_i}\cdots a_{n+1}\in\sqrt{P}$ for some $1\le i\le n$, which is a contradiction. Hence P is an n-absorbing primary ideal of R.

$(\Leftarrow)$ Is trivial.

Theorem 2.9 We consider that $J\subseteq P$ are a proper ideals of a ring R.

(1) Let P is an n-absorbing I-primary ideal of R, then P/J is a n-absorbing I-primary ideal of R/J.

(2) Let $J\subseteq IP$ and P/J be an n-absorbing I-primary ideal of R/J, then P is an n-absorbing I-primary ideal of R.

(3) Let $IP\subseteq J$ and P be an n-absorbing I-primary ideal of R, then P/J is a weakly n-absorbing primary ideal of R/J.

(4) Let $JP\subseteq IP$, J be an n-absorbing I-primary ideal of R and P/J be a weakly n-absorbing primary ideal of R/J, then P is an n-absorbing I-primary ideal of R.

Proof. (1) Set $b_1,b_2,\dots,b_{n+1}\in R$ such that $(b_1+J)(b_2+J)\cdots(b_{n+1}+J)\in (P/J)-I(P/J)=(P/J)-(I(P)+J)/J$. Then $b_1 b_2\cdots b_{n+1}\in P-IP$ and from being P is an n-absorbing I-primary, we obtain $b_1\cdots b_n\in P$ or $b_1\cdots\hat{b_i}\cdots b_{n+1}\in\sqrt{P}$ for some " $1\leq i\leq n$. And so $(b_1+J)\cdots(b_n+J)\in P/J$ or $(b_1+J)\cdots\widehat{(b_i+J)}\cdots(b_{n+1}+J)\in\sqrt{P}/J=\sqrt{(P/J)}$ for some $1\leq i\leq n$. Hence we prove that P/J is n-absorbing I-primary ideal of R/J."

(2) Set $b_1 b_2\cdots b_{n+1}\in P-IP$ for some $b_1,b_2,\dots,b_{n+1}\in R$. Then $(b_1+J)(b_2+J)\cdots(b_{n+1}+J)\in(P/J)-(I(P)/J)=(P/J)-I(P/J)$. From being P/J is an n-absorbing I-primary, we obtain that $(b_1+J)\cdots(b_n+J)\in P/J$ or $(b_1+J)\cdots\widehat{(b_i+J)}\cdots(b_{n+1}+J)\in\sqrt{(P/J)}=\sqrt{P}/J$ for some $1\leq i\leq n$. Therefore $b_1\cdots b_n\in P$ or $b_1\cdots\hat{b_i}\cdots b_{n+1}\in\sqrt{P}$ for some $1\leq i\leq n$, hence P is an n-absorbing I-primary ideal of R.

(3) Resulted directly from part (1).

(4) Set $b_1\cdots b_{n+1}\in P-IP$ where $b_1,\dots,b_{n+1}\in R$. Note that $b_1\cdots b_{n+1}\notin JP$ because $JP\subseteq IP$. If $b_1\cdots b_{n+1}\in J$, then either $b_1\cdots b_n\in J\subseteq P$ or $b_1\cdots\hat{b_i}\cdots b_{n+1}\in\sqrt{J}\subseteq\sqrt{P}$ for some $1\leq i\leq n$, since J is an n-absorbing I-primary. If $b_1\cdots b_{n+1}\notin J$, then $(b_1+J)\cdots(b_{n+1}+J)\in(P/J)-\{0\}$ and so either $(b_1+J)\cdots(b_n+J)\in P/J$ or $(b_1+J)\cdots\widehat{(b_i+J)}\cdots(b_{n+1}+J)\in\sqrt{(P/J)}=\sqrt{P}/J$ for some $1\leq i\leq n$. Therefore $b_1\cdots b_n\in P$ or $b_1\cdots\hat{b_i}\cdots b_{n+1}\in\sqrt{P}$ for some $1\leq i\leq n$. Hence P is an n-absorbing I-primary ideal of R.

Proposition 2.10 Suppose that P is an ideal of a ring R such that IP is an n-absorbing primary ideal of R. If P is an n-absorbing I-primary ideal of R, then P is an n-absorbing primary ideal of R.

Proof. Let $a_1 a_2\cdots a_{n+1}\in P$ for some elements $a_1,a_2,\dots,a_{n+1}\in R$ such that $a_1 a_2\cdots a_n\notin P$. If $a_1 a_2\cdots a_{n+1}\in IP$, then IP n-absorbing primary and $a_1 a_2\cdots a_n\notin IP$ implies that $a_1\cdots\hat{a_i}\cdots a_{n+1}\in\sqrt{IP}\subseteq\sqrt{P}$ for some $1\leq i\leq n$, and so we are done. When $a_1 a_2\cdots a_{n+1}\notin IP$ clearly the result follows.

Theorem 2.11 If P is an n-absorbing I-primary ideal of a ring R and $(a_1,\dots,a_{n+1})$ is an I-(n+1)-tuple of P for some $a_1,\dots,a_{n+1}\in R$. Then for every elements $\alpha_1,\alpha_2,\dots,\alpha_m\in\{1,2,\dots,n+1\}$ which $1\leq m\leq n$,

$$a_1\cdots\widehat{a_{\alpha_1}}\cdots\widehat{a_{\alpha_2}}\cdots\widehat{a_{\alpha_m}}\cdots a_{n+1}\, I^m\subseteq IP$$

Proof. We claim that by using induction on m. We take $m=1$ and assume $a_1\cdots\widehat{a_{\alpha_1}}\cdots a_{n+1}\,x\notin IP$ for some $x\in P$. Then $a_1\cdots\widehat{a_{\alpha_1}}\cdots a_{n+1}\,(a_{\alpha_1}+x)\notin IP$. Since P is a n-absorbing I-primary ideal of R and $a_1\cdots\widehat{a_{\alpha_1}}\cdots a_{n+1}\notin P$, we conclude that $a_1\cdots\widehat{a_{\alpha_1}}\cdots\widehat{a_{\alpha_2}}\cdots a_{n+1}\,(a_{\alpha_1}+x)\in\sqrt{P}$, for some $1\leq\alpha_2\leq n+1$ different from α_1. Hence $a_1\cdots\widehat{a_{\alpha_2}}\cdots a_{n+1}\in\sqrt{P}$, a contradiction. Thus $a_1\cdots\widehat{a_{\alpha_1}}\cdots a_{n+1}\,P\subseteq IP$. Here assume that $m>1$ and for every integers less than m the prove does hold. Let $a_1\cdots\widehat{a_{\alpha_1}}\cdots\widehat{a_{\alpha_2}}\cdots\widehat{a_{\alpha_m}}\cdots a_{n+1}\,x_1 x_2\cdots x_m\notin IP$ for some $x_1,x_2,\dots,x_m\in P$. According to the induction assumption, we conclude that there exists ζ∈IP such that

(&a_1⋯(a_(α_1))̂⋯(a_(α_2))̂⋯(a_(α_m))̂⋯a_(n+1) (a_(α_1)+x_1)(a_(α_2)+x_2)⋯(a_(α_m)+x_m)@ =&ζ+a_1⋯(a_(α_1))̂⋯(a_(α_2))̂⋯(a_(α_m))̂⋯a_(n+1) x_1 x_2⋯x_m∉IP.)

Now, we have two cases.

Case 1. Set α_m<n+1. Since from being P is an n-absorbing I-primary, then

a_1⋯(a_(α_1))̂⋯(a_(α_2))̂⋯(a_(α_m))̂⋯a_n (a_(α_1)+x_1)(a_(α_2)+x_2)⋯(a_(α_m)+x_m)∈P,

or

(a_1⋯(a_(α_1))̂⋯(a_(α_2))̂⋯(a_(α_m))̂⋯(a_j)̂⋯a_(n+1) (a_(α_1)+x_1)(a_(α_2)+x_2)⋯(a_(α_m)+x_m)@∈√P)

for some j<n+1 distinct from α_i 's; or

a_1⋯(a_(α_1))̂⋯(a_(α_2))̂⋯(a_(α_m))̂⋯a_(n+1) (a_(α_1)+x_1)⋯((a_(α_i)+x_i)̂)⋯(a_(α_m)+x_m)∈√P

for some 1≤i≤m. Thus either a_1 a_2⋯a_n∈P or a_1⋯a ̂_j⋯a_(n+1)∈√P or a_1⋯(a_(α_i))̂⋯a_(n+1)∈√P, which each of these cases which is a contradiction.

Case 2. Set α_m=n+1. Since from being P is an n-absorbing I-primary, then a_1⋯(a_(α_1))̂⋯(a_(α_2))̂⋯(a_(α_(m-1)))̂⋯(a_(n+1))̂(a_(α_1)+x_1)(a_(α_2)+x_2)⋯(a_(α_m)+x_m)∈P, or

(a_1⋯(a_(α_1))̂⋯(a_(α_2))̂⋯(a_(α_(m-1)))̂⋯(a_j)̂⋯(a_(n+1))̂(a_(α_1)+x_1)(a_(α_2)+x_2)⋯(a_(α_m)+x_m)@∈√P,)

for some j<n+1 different from α_i 's; or

a_1⋯(a_(α_1))̂⋯(a_(α_2))̂⋯(a_(α_(m-1)))̂⋯(a_(n+1))̂(a_(α_1)+x_1)⋯((a_(α_i)+x_i)̂)⋯(a_(α_m)+x_m) ∈√P for some 1≤i≤m-1. Thus either a_1 a_2⋯a_n∈P or a_1⋯(a_j)̂⋯a_(n+1)∈√P or a_1⋯(a_(α_i))̂⋯a_(n+1)∈√P, which each of these cases which are a contradiction. Thus

a_1⋯(a_(α_1))̂⋯(a_(α_2))̂⋯(a_(α_m))̂⋯a_(n+1) I^m⊆IP

Theorem 2.12 If P isan n-absorbing I-primary ideal of R which is not an n-absorbing primary ideal. Then

(i) P^(n+1)⊆IP.

(ii) √P=√IP.

Proof. (i) Since P is assumed not to be an n-absorbing primary ideal of R, so P has an I-(n+1)-tuple zero (b_1,…,b_(n+1)) for some b_1,…,b_(n+1)∈R. Let c_1 c_2⋯c_(n+1)∉ IP for some c_1,c_2,…,c_(n+1)∈P. Therefore, according to the Theorem 2.11, there is λ∈IP such that (b_1+c_1)⋯(b_(n+1)+c_(n+1))=λ+c_1 c_2⋯c_(n+1)∉IP. Hence either (b_1+c_1)⋯(b_n+c_n)∈P or (b_1+c_1)⋯(b_i+c_i)̂⋯(b_(n+1)+c_(n+1))∈√P for some 1≤i≤n. Thus either b_1⋯b_n∈P or b_1⋯(b_i)̂⋯b_(n+1)∈√P for some 1≤i≤n, which is a contradiction. Hence P^(n+1)⊆IP.

(ii) Clearly, √IP⊆√P. As P^(n+1)⊆IP, we obtain √P⊆√IP, we are done.

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