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FRACTIONAL SHIFTED LEGENDRE POLYNOMIALS FOR SOLVING TIME-FRACTIONAL BIOHEAT EQUATION

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ABSTRACT

The aims of this paper: to employ the fractional shifted Legendre polynomials (FSLPs) to approximate the fractional temporal derivatives by using definition Caputo formula for fractional differential operator. The space derivatives of the order integers are approximated depending on the shifted Legendre polynomials. The numerical tests suggest for two examples that the proposed approach are efficient and robust in its solutions for the time-fractional bioheat equation. Furthermore, some the preliminaries and definitions of the fractional derivatives, theorems and lemmas related to the fractional shifted Legendre polynomials are inserted to verify the convergence results of the algorithm under consideration. Finally, we discussed the error analysis.

Keywords: *Collocation Method, Time-Fractional Bioheat Equation, Fractional Shifted Legendre Polynomials, Accuracy.*

1. INTRODUCTION

The fractional calculus have been utilized successfully to study the physical operations particularly in the scope of biomedical engineering, control theory, mechanics of solids, viscoelasticity , heat conduction, and diffusion problems etc.[1]. It is represented a dominate and control tool to improve the modelling accuracy of many phenomena in naturalistic science [2]. Heat distribution in biological tissue, is typically expressed as bioheat equation, it involves thermic conduction, perfusion of blood, convection and metabolic temperature generation in human tissue [3]. The pioneering work of Pennes' in 1948 is the mainstay of the mathematical modeling of temperature distribution in tissues, the model known as the bioheat equation as still extensively used to even today, which describes the thermal distribution in human tissue [4]. The temperature distribution in skin tissue is very important for medical application like skin cancer, skin burns etc. [5]. Fractional bio diffusion equation is a denomination of anomalous diffusive system, which is obtained from the classical heat conduction equation by replacing the first orders time derivative to a fractional derivative of order $\alpha \in (0,1]$ (in Riemann-Liouville (RL) or Caputo (C) sense)[6].

A lot of studies in the literature exists on different types of application utilize fractional bioheat models are solved by using different analytic and numerical methods, for more information see (Singh et al. [7], Jiang and Qi [8], Karatay et al. [9], Ezzat et al. [10], Ferrás et al. [11], Qin and Wu [12], Kumar and Rai [13], Roohi et. al. [14] and Hosseininia et al. [15]).

Collocation method is a powerful mathematical technique and very useful to find numerical solutions of the differential equations of fractional or integer orders and considered one of the most prominent of the approximate methods used recently because it used less number of the grid points compared with other numerical methods as well as its effectiveness and accuracy.

Indeed, the collocation method have been used to solve the various formula of fractional differential equations by using different numerical techniques that contributes to a significant rate of the ongoing research these days. for example Pedas and Tamme [16] in (2011), used spline collocation methods depend on the Lagrange fundamental polynomials for solving linear multi-term fractional differential equations. Bhrawy and Alghamdi [17] in (2012), developed a shifted Jacobi-Gauss-Lobatto polynomials for solving the nonlinear fractional Langevin equation. Khader [18] in (2013), found the numerical solution for the fractional delay differential equations by utilizing the generalized Laguerre polynomials. Tian et al. [19] and Saeed and Rehman [20] in (2014), applied the Legendre-Gauss-Lobatto and Chebyshev-Gauss-Lobatto polynomials in the matrix form for solving the fractional advection-diffusion equation, and proposed the Hermite wavelet polynomials for solving fractional linear and nonlinear delay differential equations. Bhrawy [21] in (2015), propose a new efficient techniques for solving the nonlinear fractional sub-diffusion and reaction sub-diffusion equations based basically on the operational matrix of fractional derivatives of the shifted Jacobi polynomials. Jaleb and Adibi [22] and Alshbool and Hashim [23] in (2016), employed shifted Legendre polynomials and Bernstein polynomials to find the numerical solutions of space fractional diffusion equation, fractional Riccati differential equation and fractional-order stiff systems. Rahimkhani et al. [24] in (2017), suggested the operational matrix by using Bernoulli wavelet polynomials to find approximate solutions for fractional delay differential equations. Agarwal and El-Sayed [25] and M. Bahmanpour et al. [26] in (2018), utilized the shifted Chebyshev polynomials of the second kind for solving the fractional order diffusion equation and introduced Müntz wavelets by employing Müntz Legendre polynomials and Jacobi polynomials as a test function for solving fractional differential equations. Ali et al. [27] in (2019), applied spectral collocation formula on suggest solving the fractional-order delay-differential equations depending on Chebyshev operational form.

In this article, we will be introduced the numerical methodology for solving one dimensional time-fractional bioheat equation based on FSLPs in matrix form.

2. GOVERNING EQUATION

Damor et al. [28] proposed a fractional version of the bioheat equation, by replacing the first-order time derivative with a derivative of arbitrary positive real order $\gamma \in (0,1]$. We have

$$\rho c \frac{\partial^{\gamma} T(x,t)}{\partial t^{\gamma}} - k_* \frac{\partial^2 T(x,t)}{\partial x^2} + W_b c_b (T(x,t) - T_a) = Q_{ext} + Q_{met}, \quad t > 0,\ 0 \le x \le R, \tag{1}$$

where $\rho, c, k, T, t, x, T_a, W_b = \rho_b w_b, Q_{ext}$ and Q_{met} symbolizes density, specific heat, thermal conductivity, temperature, time, distance, artillery temperature, blood perfusion rate, metabolic heat generation in skin tissue and external heat exporter in skin tissue respectively. The units and values of the symbols expressed in this equation mentioned in Table1 [29].

The initial and boundary conditions

$$T(x,0) = T_c, \tag{2}$$

$$-k_* \frac{\partial T}{\partial x}\Big|_{x=0} = q_0, \tag{3}$$

$$-k_* \frac{\partial T}{\partial x}\Big|_{x=R} = 0, \tag{4}$$

where, q_0 is the heat flux on the skin surface.

3. PRELIMINARIES AND NOTATIONS

In this section, we recall the essentials of the fractional calculus theory that will be benefit in this article.

3.1 Definition

The Riemann-Liouville fractional integral operator of order $\alpha > 0$ defined as [30]:

$$I^{\alpha}T\ (t) = \frac{1}{\Gamma\ (\alpha)} \int_0^t (t-s)^{\alpha-1}\ T\ (s)ds,\ \ \alpha\ >\ 0, \quad (5)$$
$$I^0\ T\ (t) =\ T(t).$$

3.2 Definition

The Riemann-Liouville definition of fractional differential operator given as follows [31]:

$$D^{\alpha}T\ (t) = \begin{cases} \frac{1}{\Gamma\ (n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{T\ (s)}{(t-s)^{\alpha-n+1}} ds, \alpha > 0, n-1 \le \alpha < n, \\ \frac{d^n T\ (t)}{dt^n}, \qquad \alpha\ =\ n. \end{cases} \quad (6)$$

3.3 Definition

The Caputo definition of fractional differential operator defined as [8]:

$$D^{\alpha}T\ (t) = \begin{cases} \frac{1}{\Gamma\ (n-\alpha)} \int_0^t \frac{T^{(n)}\ (s)}{(t-s)^{\alpha-n+1}}\ ds,\ n-1 \le\ \alpha < n, \\ \frac{d^n T\ (t)}{dt^n}, \qquad \alpha\ =\ n. \end{cases} \quad (7)$$

The relation between the Riemann-Liouville operator and Caputo operator is given by the following expressions [7]:

$$D^{\alpha}\ I^{\alpha}T\ (t) = T\ (t),$$
$$I^{\alpha}D^{\alpha}\ T\ (t) = T(t) - \sum_{k=0}^{n-1} T^{(k)}(0^+)\frac{t^k}{k!}. \quad (8)$$

For $\alpha \ge 0,\ \ v \ge -1,$ and constant $C,$ Caputo fractional derivative has some basic properties which are needed here as follows [32]:

$$i)\ \ D^{\alpha}C\ =\ 0.$$
$$ii)\ D^{\alpha}t^{v} = \begin{cases} 0, & \text{for } v \in \mathrm{N}_0 \text{ and } v < \lceil\alpha\rceil \\ \frac{\Gamma(v+1)}{\Gamma(v+1-\alpha)} t^{v-\alpha}, & \text{for } v \in \mathrm{N}_0, v \ge \lceil\alpha\rceil \text{ or } v \notin \mathrm{N}_0, v \ge \lfloor\alpha\rfloor. \end{cases} \quad (9)$$
$$iii)\ D^{\alpha}\left(\sum_{i=0}^{n} c_i T_i(t)\right) = \sum_{i=0}^{n} c_i D^{\alpha}\ T_i(t), where\ \{c_i\}_{i=0}^{n}\ are\ constant.$$

3.4 Definition

(generalized Taylor's formula). Suppose that $D^{i\alpha}T(t) \in\ C[0,1]$ for $i\ =\ 0,1,\ldots,n-1,$ then one has

$$T(t) = \sum_{i=0}^{n-1} \frac{t^{i\alpha}}{\Gamma\ (i\alpha\ +\ 1)} D^{i\alpha}T(0^+) + \frac{t^{n\alpha}}{\Gamma(n\alpha\ +\ 1)} D^{n\alpha}T(\xi), \quad (10)$$

where $0\ <\ \xi\ \le\ \mathrm{t}, \forall t \in [0,1]$. Also, one has

$$\left| T(t) - \sum_{i=0}^{n-1} \frac{t^{i\alpha}}{\Gamma\ (i\alpha\ +\ 1)} D^{i\alpha}T(0^+) \right| \le M_{\alpha}\ \frac{t^{n\alpha}}{\Gamma(n\alpha\ +\ 1)}, \quad (11)$$

where $M_{\alpha} \ge |D^{n\alpha}T(\xi)|$.

In the case of $\alpha = 1,$ the generalized Taylor's formula (10) is the classical Taylors formula [33].

4. FRACTIONAL SHIFTED LEGENDRE POLYNOMIALS

The FSLPs are defined by the transformation $t = t^{\alpha}$ and $\alpha > 0$ on shifted Legendre polynomials. The fractional shifted Legendre polynomials $L_M(t^{\alpha})$ is denoted by $Fl_j^{\alpha}(t)$. The FSLPs are a particular solution of the normalized eigenfunctions of the singular Sturm-Liouville problem [34].

$$\left((t-\ t^{1+\alpha})(Fl_j^{\alpha}(t))'\right)' + \alpha^2\ j(j+1)t^{\alpha-1}Fl_j^{\alpha}(t) = 0, \quad t \in [0,1]. \quad (12)$$

Then $Fl_j^{\alpha}(t)$ can be obtained as follows:

$$Fl_{j+1}^{\alpha}(t) = \frac{(2j+1)(2t^{\alpha}-1)}{j+1}\ Fl_j^{\alpha}(t) - \frac{j}{j+1}Fl_{j-1}^{\alpha}(t), j = 1,2,\ldots. \quad (13)$$

The analytic formula of $Fl_j^{\alpha}(t)$ of degree $j\alpha$ has been deriving as follows:

$$Fl_j^{\alpha}(t) = \sum_{s=0}^{j} b_{sj}\ t^{s\alpha}, j = 1,2,\ldots, \quad (14)$$

where $b_{sj} = (-1)^{j+s}(j+\ s)!/(j-\ s)!\,(s!)^2$ and $Fl_{\mathrm{j}}^{\alpha}(0) = (-1)^j, Fl_j^{\alpha}(1) = 1.$

4.1 Theorem [35]

The FSLPs are orthogonal with the weight function $\omega_l^{\alpha}(t) = t^{\alpha-1}$ on the interval $[0,1]$, then be orthogonally condition is

$$\int_0^1 Fl_N^{\alpha}(t)Fl_M^{\alpha}(t)\omega_l^{\alpha}(t)\,dt = \frac{1}{(2M+1)\alpha}\delta_{NM}. \quad (15)$$

Proof. With $\int_0^1 L_N(t)\ L_M(t)\,\omega_l(t)dt\ = \frac{1}{(2M+1)}\delta_{NM},$ where δ_{NM} is the Kronecker function and the weight function $\omega_l(t) = 1,$ let $t = t^{\alpha}$, and then we have

$$\int_0^1 L_N(t)\, L_M(t)\, \omega_l(t)\, dt$$

$$= \int_0^1 L_N(t^\alpha)\, L_M(t^\alpha)\, \alpha t^{\alpha-1}\, dt,$$

$$= \frac{1}{(2M+1)} \delta_{NM},$$

$$\int_0^1 L_N(t^\alpha)\, L_M(t^\alpha)\, \alpha t^{\alpha-1}\, dt$$

$$= \int_0^1 Fl_N^\alpha(t)\, Fl_M^\alpha(t)\ \alpha t^{\alpha-1}\, dt = \frac{1}{(2M+1)} \delta_{NM}, \quad (16)$$

$$\int_0^1 Fl_N^\alpha(t)\, Fl_M^\alpha(t)\ t^{\alpha-1}\, dt = \frac{1}{(2M+1)\alpha} \delta_{NM}.$$

□

A temperature function $T(t)$ square integrable in $0 \le t \le 1$, may be expressed in terms of the FSLPs as:

$$T(t) = \sum_{j=0}^{\infty} c_j\, Fl_j^\alpha(t), \quad (17)$$

where the coefficients c_j are obtained by

$$c_j = \alpha(2j+1) \int_0^1 Fl_j^\alpha(t) T(t)\, \omega_l^\alpha\ (t) dt,$$

$$j = 0,1,2, \ldots. \quad (18)$$

If we consider truncated series when $(M+1)$-term FSLPs in (17), we obtain

$$T(t) \approx T_M(t) = \sum_{j=0}^{M} c_j\, Fl_j^\alpha(t)$$

$$= C'\emptyset(t), \quad (19)$$

where the fractional shifted Legendre coefficient vector C and $\emptyset(t)$ are given by

$$C' = [c_0, c_1, \ldots, c_M],\ \emptyset(t) = [Fl_0^\alpha(t), Fl_1^\alpha(t), \ldots, Fl_M^\alpha(t)]'.$$

4.2 Theorem

Suppose that $D^{j\alpha} T(t) \in C[0,1]$ for $j = 0,1, \ldots, M-1$. $(2M+1)\alpha \ge 1$ and $\boldsymbol{P}_M^\alpha = span\{Fl_0^\alpha(t), Fl_1^\alpha(t), \ldots, Fl_M^\alpha(t)\}$.If $T_M(t) = C'\emptyset(t)$ is the best approximation for $T(t)$ from $\boldsymbol{P}_M^\alpha$ then the error bound presented as follows:

$$\|T(t) - T_M(t)\|_\omega \le \frac{M_\alpha}{\Gamma(\alpha M+1)} \sqrt{\frac{1}{(2M+1)\alpha}}, \quad (20)$$

where $M_\alpha \ge |D^{\alpha M} T(t)|,\ t \in [0,1]$.

Proof. This theorem can be proved in the same approach in [33]

For arbitrary function $T(x,t) \in L^2([0,1] \times [0,1])$, they can be expanded as the following formula:

$$T(x,t) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} c_{ij}\, L_i(x) Fl_j^\alpha(t), \quad (21)$$

where

$$c_{ij} = (2i+1)(2j+1)\alpha \int_0^1 \int_0^1 T(x,t)\, L_i(x) Fl_j^\alpha(t) \omega_l^\alpha(t) dx dt,$$

$$i,j = 0,1, \ldots. \quad (22)$$

4.3 Theorem

If the series $\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} c_{ij}\, L_i(x) Fl_j^\alpha(t)$ converges uniformly to $T(x,t)$ on the square $[0,1] \times [0,1]$, that give the coefficients c_{ij} in equation (22).

Proof. By multiplying both sides of equation (21) by $\omega_l^\alpha(t) L_N(x) Fl_M^\alpha(t)$ where i and j are fixed then the resulting equation respect to x and t on $[0,1] \times [0,1]$, then

$$\int_0^1 \int_0^1 T(x,t)\, L_N(x) Fl_M^\alpha(t) \omega_l^\alpha(t) dx dt$$

$$= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} c_{ij} \int_0^1 \int_0^1 L_i(x) Fl_j^\alpha(t) L_N(x) Fl_M^\alpha(t) \omega_l^\alpha(t) dx dt$$

$$= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} c_{ij} \int_0^1 L_i(x)\, L_N(x) dx \int_0^1 \omega_l^\alpha(t)\, Fl_j^\alpha(t) Fl_M^\alpha(t) dt$$

$$= c_{ij} \int_0^1 [L_i(x)]^2\, dx \int_0^1 \omega_l^\alpha(t) \left[Fl_j^\alpha(t)\right]^2 dt, \quad (23)$$

$$= c_{ij} \frac{1}{(2i+1)} \frac{1}{(2j+1)\alpha}.$$

□

4.4 Theorem

If the function $T(x,t)$ is a continuous function on $[0,1] \times [0,1]$ and the series $\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} c_{ij}\, L_i(x) Fl_j^\alpha(t)$ converges uniformly to $T(x,t)$, then $\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} c_{ij}\, L_i(x) Fl_j^\alpha(t)$ is the FSLPs expansion of $T(x,t)$.

Proof. Using contradiction, let

$$\left.\begin{aligned} T(x,t) &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} c_{ij}\, L_i(x) Fl_j^{\alpha}(t), \\ T(x,t) &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} g_{ij}\, L_i(x) Fl_j^{\alpha}(t). \end{aligned}\right\} \quad (24)$$

Then there is at least one coefficient such that $c_{NM} \neq g_{NM}$ however

$$\begin{aligned} & c_{NM} \\ &= (2N+1)(2M \\ &+ 1)\alpha \int_0^1 \int_0^1 T(x,t)\, L_N(x) Fl_M^{\alpha}(t)\omega_l^{\alpha}(t)dxdt \\ &= g_{NM}. \end{aligned} \quad (25)$$

□

4.5 Theorem

If two continuous functions defined on $[0,1]\times[0,1]$ have the identical FSLPs expansions, then these two function are identical.

Proof. Suppose that $T(x,t)$ and $f(x,t)$ can be expended by FSLPs as follows:

$$T(x,t) \approx \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} c_{ij}\, L_i(x) Fl_j^{\alpha}(t), \quad (26)$$

$$f(x,t) \approx \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} c_{ij}\, L_i(x) Fl_j^{\alpha}(t). \quad (27)$$

By subtracting equation (27) from (26), we have

$$\begin{aligned} T(x,t) - f(x,t) &\approx \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \left(c_{ij} - c_{ij}\right) L_i(x) Fl_j^{\alpha}(t) \\ &= 0. \end{aligned} \quad (28)$$

□

4.6 Theorem

If the sum of the absolute value of the FSLPs coefficients of a continuous function $T(x,t)$ forms a convergent series, then the FSLPs expansion is absolutely uniformly convergent and converges to the function $T(x,t)$.

Proof. Consider

$$\begin{aligned} & \left|\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} c_{ij}\, L_i(x) Fl_j^{\alpha}(t)\right| \\ &\leq \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \left|c_{ij}\right| \left|L_i(x)\right| \left|Fl_j^{\alpha}(t)\right| \\ &\leq \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \left|c_{ij}\right|. \end{aligned} \quad (29)$$

□

If we consider truncated series in (21), we satisfy

$$\begin{aligned} T(x,t) &\approx \sum_{i=0}^{N}\sum_{j=0}^{M} c_{ij}\, L_i(x) Fl_j^{\alpha}(t) \\ &= \emptyset'(x) C \emptyset(t), \end{aligned} \quad (30)$$

where

$$\emptyset(x) = [L_0(x), L_1(x), \dots, L_N(x)]',$$

$$\emptyset(t) = [Fl_0^{\alpha}(t), Fl_1^{\alpha}(t), \dots, Fl_M^{\alpha}(t)]',$$

$$C = \{c_{ij}\}_{i,j=0}^{N,M}.$$

5. TWO DIMENSIONAL FRACTIONAL SHIFTED LEGENDRE OPERATIONAL MATRIX OF FRACTIONAL DIFFERENTIATION

The derivative of the $\emptyset(t)$ can be approximated as follows

$$\emptyset^{(\gamma)}(t) \approx D_t^{\gamma}\emptyset(t), \quad (31)$$

where D_t^{γ} is called the FSLPs operational matrix of time derivative.

5.1 Theorem [30, 33,35]

Suppose D_t^{γ} is $(M+1)\times(M+1)$ operational matrix of Caputo fractional derivatives of order $\gamma > 0$, $\alpha > \frac{\gamma}{2}$, when $\alpha \notin \mathbb{N}$; then the elements of D_t^{γ} are obtained as

$$\begin{aligned} & \{d_{ij}\}_{i,j=0}^{M,M} \\ &= (2i+1)\alpha \sum_{s=0}^{j}\sum_{r=0}^{i} b_{ri}\, b'_{sj} \frac{\Gamma(s\alpha+1)}{\Gamma(s\alpha-\gamma+1)} \frac{1}{(s+r+1)\alpha-\gamma}, \end{aligned} \quad (32)$$

where

$$\begin{aligned} & b'_{sj} \\ &= \begin{cases} 0, s\alpha \in N_0, s\alpha < \gamma, \\ b'_{sj} = b_{sj}, s\alpha \notin N_0, s\alpha \geq \lceil\gamma\rceil \text{ or } s\alpha \in N_0, s\alpha \geq \gamma. \end{cases} \end{aligned} \quad (33)$$

Proof. One can be proved this theorem in the same manner in [30, 33, 35].

6. METHOD FOR SOLUTION

We will structure the numerical solution methodology of equation (1), under given conditions, as the following series form

$$T(x,t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} t_{ij}\, L_i(x) Fl_j^{\alpha}(t), \tag{34}$$

which equivalent the matrix form

$$T(x,t) = \emptyset(x)' T \emptyset(t), \tag{35}$$

where

$T = \{t_{ij}\}_{i,j=0}^{N,M}$,
$\emptyset(x) = [L_0(x), L_1(x), \dots, L_N(x)]'$,
$\emptyset(t) = [Fl_0^{\alpha}(t), Fl_1^{\alpha}(t), \dots, Fl_M^{\alpha}(t)]'$.
The approximate of the first and second spatial derivatives as

$$\frac{\partial T(x,t)}{\partial x} = \emptyset'(x) D_x{}' T \emptyset(t), \tag{36}$$

$$\frac{\partial^2 T(x,t)}{\partial x^2} = \emptyset'(x)(D_x{}')^2 T \emptyset(t), \tag{37}$$

and the fractional temporal derivative as

$$\frac{\partial^{\gamma} T(x,t)}{\partial t^{\gamma}} = \emptyset'(x) T D_t^{\gamma} \emptyset(t), \tag{38}$$

for equation (1), we have

$$\begin{aligned} \rho c\, \emptyset'(x) T D_t^{\gamma} \emptyset(t) - k_* \emptyset'(x)(D_x{}')^2 T \emptyset(t) \\ + W_b c_b \emptyset'(x) T \emptyset(t) \\ = \emptyset'(x) Q_{ext} \emptyset(t) + \emptyset'(x) Q_{met} \emptyset(t) \\ + \emptyset'(x) W_b c_b T_a I \emptyset(t), \end{aligned} \tag{39}$$

where

$g(x,t) = Q_{ext} + Q_{met} + W_b c_b T_a$,
$\emptyset'(x) G\, \emptyset(t) = \emptyset'(x) Q_1 \emptyset(t) + \emptyset'(x) Q_2 \emptyset(t) + \emptyset'(x) Q_3 \emptyset(t)$ and $G = \{g_{ij}\}_{i,j=0}^{N,M}$,

$$g_{ij} = \alpha(2i+1)(2j+1) \int_0^1 \int_0^1 g(x,t)\, L_i(x) Fl_j^{\alpha}(t) \omega_l^{\alpha}(t) dx dt. \tag{40}$$

Then we generate $NM - M$ algebraic equations by multiplying $\omega_l^{\alpha}(t) L_i(x) Fl_j^{\alpha}(t)$ for $i = 1,2,\dots,N-1; j = 1,2,\dots,M$, integrating from 0 to 1 and using the orthogonal property; we have

$$T(\rho c\, D_t^{\gamma} - \omega_b \rho_b c_b I) - K(D_x{}')^2 T = G, \tag{41}$$

with the initial condition from equation (2) in matrix form

$$T \emptyset(0) \approx F, \tag{42}$$

where $F = [f_0, f_1, \dots, f_M]'$,

$$f_j = (2j+1) \int_0^1 T(x,0)\, L_j(x) dx, \tag{43}$$

and boundary conditions respectively from equations (3) and (4) in matrix form

$$-k_* \emptyset'(0) D_x{}' T \approx K', \tag{44}$$

$$-k_* \emptyset'(R) D_x{}' T \approx H', \tag{45}$$

where $K = [k_0, k_1, \dots, k_N]'$ and $H = [h_0, h_1, \dots, h_N]'$ and

$$k_i = \alpha(2i+1) \int_0^1 T_x(0,t)\, Fl_i^{\alpha}(t) \omega_l^{\alpha}(t) dt, \tag{46}$$

$$h_i = \alpha(2i+1) \int_0^1 T_x(R,t)\, Fl_i^{\alpha}(t) \omega_l^{\alpha}(t) dt. \tag{47}$$

We generate $NM + N + M + 1$ linear algebraic equations by equation (41) together with equations (42), (44) and (45). These unknown coefficients T can be solved by solving Sylvester system.

7. ERROR ANALYSIS

We consider $e(x,t) = T(x,t) - T_{NM}(x,t)$ as the error function where $T_{NM}(x,t)$ approximate solution and $T(x,t)$ is the exact solution of equation (1).

Therefore, $T_{NM}(x,t)$ satisfies the following problem

$$\rho c \frac{\partial^{\gamma} T_{NM}(x,t)}{\partial t^{\gamma}} - k_* \frac{\partial^2 T_{NM}(x,t)}{\partial x^2} + W_b c_b T_{NM}(x,t) + R_{NM}(x,t) = g(x,t), \tag{48}$$

where $R_{NM}(x,t)$ is the residual function,

$$R_{NM}(x,t) = \rho c \frac{\partial^{\gamma} T_{NM}(x,t)}{\partial t^{\gamma}} - k_* \frac{\partial^2 T_{NM}(x,t)}{\partial x^2} + W_b c_b T_{NM}(x,t) - g(x,t). \tag{49}$$

We can find an approximation $\tilde{e}_{NM}(x,t)$ to the error function $e_{NM}(x,t)$ in the same previous

procedure, so for the solution of the problem, the error function satisfies the problem

$$\rho c \frac{\partial^{\gamma} e_{NM}(x,t)}{\partial t^{\gamma}} - k_* \frac{\partial^2 e_{NM}(x,t)}{\partial x^2} + W_b c_b e_{NM}(x,t) = R_{NM}(x,t). \quad (50)$$

Note that in order to construct the approximate $\tilde{e}_{NM}(x,t)$ to the error function $e_{NM}(x,t)$, only equation (50) needs to be recalculated in the same way as doing in previous for the solution of equation (1).

8. NUMERICAL EXPERIMENTS

In this section, we apply the algorithm that presented in section 6 for solving the time-fractional bioheat equation in two cases based on FSLPs. In order to show the ability and effective of the suggested method for achieving the high accuracy, the solution obtained from this technique comparing with the exact solution. In these cases, we take $\gamma = \alpha$, $R = 1$ and use equally spaced grid points.

Utilize the method presented in Section 6, the time-fractional bioheat equation transformed into linear algebraic equations $(41-42), (44)$ and (45) respectively.

The results shown that the numerical solutions have a high reliability and good affinity when increasing $N = M$, and $x \in [0,R]$ for different values of order of numerical solutions, we noted that the effect increasing t at the accuracy

8.1 Case

The fractional bioheat equation (1) considered where the parameters $\rho, c, k_*, T, t, x, T_a, W_b = \rho_b w_b$ and Q_{met} are obtained from Table 1 and by choosing Q_{ext} so that the exact solution[29]:

$$T(x,t) = \mathrm{e}^{-t}\cos(\pi x) + \tanh(x) + 37, \quad (51)$$

with the initial condition

$$T(x,0) = \cos(\pi x) + \tanh(x) + 37, \quad x \in [0,R] \quad (52)$$

and boundary conditions

$$-k_* \frac{\partial T(0,t)}{\partial x} = 1 \ , t > 0 \quad (53)$$

$$-k_* \frac{\partial T(R,t)}{\partial x} = -\pi e^{-t}\sin(R\pi) + \sec^2(R), t > 0. \quad (54)$$

Tables 2-3 shown the errors obtained from solving the time-fractional bioheat equation by using FSLPs at t = 0.1 and 1. Figs. 1-4 clarified a comparison between the exact solution and numerical solutions of case1.

8.2 Case

The fractional bioheat equation (1) considered where the parameters $\rho, c, k_*, T, t, x, T_a, W_b = \rho_b w_b$ and Q_{met} are obtained from Table 1 and by choosing Q_{ext} so that the exact solution [11]:

$$T(x,t) = \mathrm{t}^{\frac{3}{2}} x^2 \left(\frac{3}{2} - x\right) + 37, \quad (55)$$

with the initial condition

$$T(x,0) = 37, \quad x \in [0,R], \quad (56)$$

and boundary conditions

$$-k_* \frac{\partial T(0,t)}{\partial x} = 0 \ , t > 0, \quad (57)$$

$$-k_* \frac{\partial T(R,t)}{\partial x} = 3R(1-R)\mathrm{t}^{\frac{3}{2}}, t > 0. \quad (58)$$

Tables 4-5 show the errors obtained from solving the time-fractional bioheat equation by using FSLPs at $t = 0.1$, 1 and 5 and $x \in [0,R]$ for different values of order of numerical solutions. Figs. 5-10 clarify a comparison between the exact solution and numerical solutions of case 2.

9. CONCLUSIONS

In this article, we employed fractional shifted Legendre polynomials in the matrix form to solve time-fractional bioheat equation successfully. The approximate results show that the present technique has higher accuracy, good convergence and reasonable stability as well as a less computation workload by using few grid points.

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Table 1: *The Unit and Value of the Symbols Expressed in the Fractional Bioheat Equation.*

Symbol	T_a	ρ and ρ_b	c and c_b	k_*	ω_b	Q_{met}
Unit	℃	kg/m^3	J/kg℃	W/m℃	m^3/s/ m^3	W/m^3
value	37	1000	4000	0.5	0.0005	420

Table2. *Absolut Errors Obtained for Case 1 with* $R = 1$ *and* $\alpha = 0.5$

(x,t)	Absolute error $N\times M=4\times4$ CPU= 223.975082s	Absolute error $N\times M=6\times6$ CPU=432.679099s	Absolute error $N\times M=8\times8$ CPU=890.914626s	Absolute error $N\times M=10\times10$ CPU=1377.631638s
(0,0)	9.0495e-03	1.1476e-04	2.4495e-06	1.5132e-08
(0.1,0.1)	8.3466e-03	1.0507e-04	2.8832e-06	3.4709e-08
(0.2,0.2)	7.7205e-03	6.8539e-05	4.0683e-06	3.8744e-08
(0.3,0.3)	6.9239e-03	1.3309e-05	5.4753e-06	3.9579e-08
(0.4,0.4)	5.9756e-03	5.3205e-05	6.8183e-06	3.8019e-08
(0.5,0.5)	4.8975e-03	1.2531e-04	7.9152e-06	3.4270e-08
(0.6,0.6)	3.7098e-03	1.9844e-04	8.6545e-06	2.8687e-08
(0.7,0.7)	2.4307e-03	2.6887e-04	8.9789e-06	2.1863e-08
(0.8,0.8)	1.0768e-03	3.3357e-04	8.8740e-06	1.4550e-08
(0.9,0.9)	3.3637e-04	3.9016e-04	8.3581e-06	7.5408e-09
(1,1)	1.7945e-03	4.3679e-04	7.4745e-06	1.5693e-09

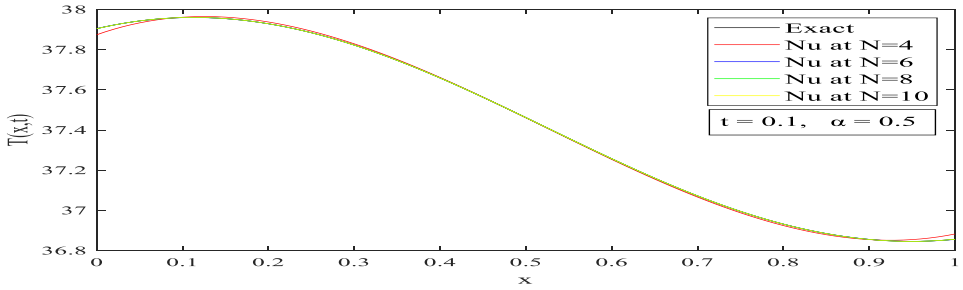

Figure1: *Comparison Between Exact and Numerical Solutions for Case 1 at* $\alpha = 0.5, t = 0.1, R = 1, N = 4,6,8$ *and* 10.

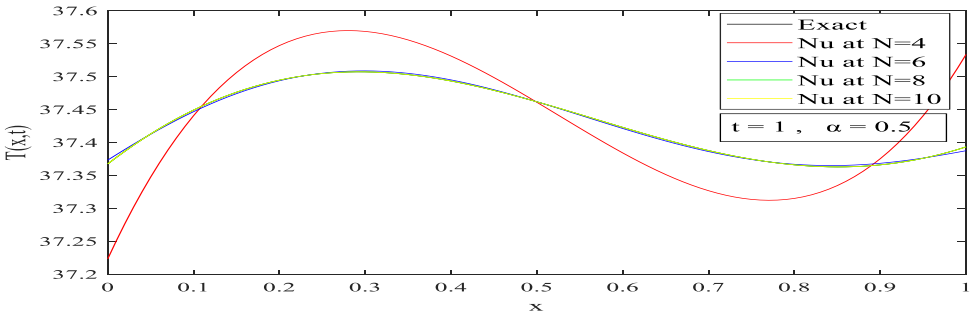

Figure2: *Comparison Between Exact and Numerical Solutions for Case 1 at* $\alpha = 0.5\ t = 1, R = 1, N = 4,6,8$ *and* 10.

Table3. *Absolut Errors Obtained for Case 1 with* $R = 1$ *and* $\alpha = 0.999$

(x,t)	Absolute error $N\times M=4\times4$ CPU= 195.254485s	Absolute error $N\times M=6\times6$ CPU=467.916514s	Absolute error $N\times M=8\times8$ CPU=882.050002s	Absolute error $N\times M=10\times10$ CPU=1778.943196s
(0,0)	9.0494e-03	2.8182e-04	4.2854e-05	2.8063e-05
(0.1,0.1)	8.4586e-03	2.5341e-04	3.7446e-05	2.4976e-05
(0.2,0.2)	7.7204e-03	2.1324e-04	4.1169e-05	2.7983e-05
(0.3,0.3)	6.8563e-03	1.5928e-04	4.5697e-05	3.0136e-05
(0.4,0.4)	5.8792e-03	9.5516e-05	4.9575e-05	3.0996e-05
(0.5,0.5)	4.8010e-03	2.6148e-05	5.2336e-05	3.0806e-05
(0.6,0.6)	3.6334e-03	4.4946e-05	5.3869e-05	2.9926e-05
(0.7,0.7)	2.3876e-03	1.1436e-04	5.4229e-05	2.8685e-05
(0.8,0.8)	1.0749e-03	1.7919e-04	5.3553e-05	2.7343e-05
(0.9,0.9)	2.9328e-04	2.3703e-04	5.2016e-05	2.6080e-05
(1,1)	1.7059e-03	2.8596e-04	4.9807e-05	2.5010e-05

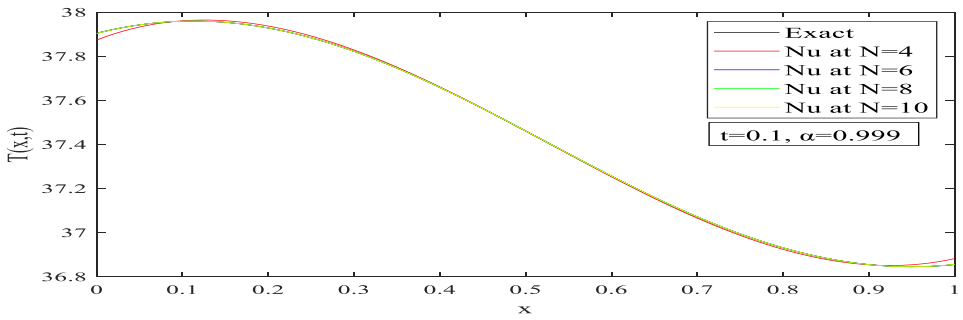

Figure3: *Comparison Between Exact and Numerical Solutions for Case 1 at* $\alpha = 0.999\ t = 0.1\ , R = 1, N = 4, 6, 8$ *and* 10.

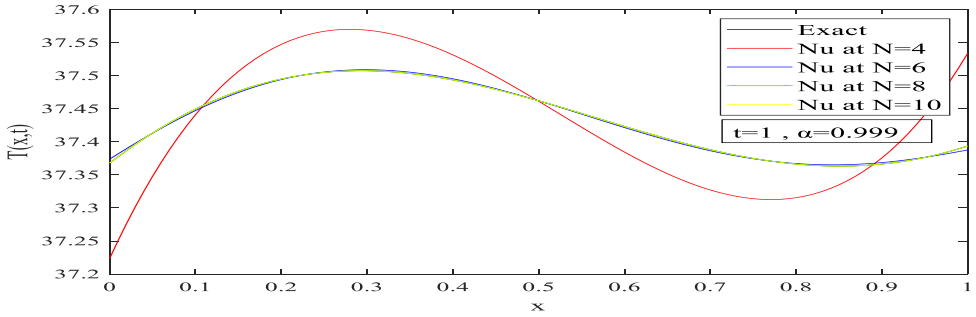

Figure4: *Comparison Between Exact and Numerical Solutions for Case 1 at* $\alpha = 0.999, t = 1\ , R = 1, N = 4, 6, 8$ *and* 10.

Table 4. *Absolut Errors Obtained for Case 2 with* $R = 1$ *and* $\alpha = 0.5$.

(x,t)	**Absolute error** $N \times M = 6 \times 6$ **CPU=168.841006s**	**Absolute error** $N \times M = 8 \times 8$ **CPU=310.074466s**	**Absolute error** $N \times M = 10 \times 10$ **CPU=620.646941s**	**Absolute error** $N \times M = 12 \times 12$ **CPU= 1145.971471s**
(0,0)	7.2235e-08	2.0132e-07	6.3733e-08	1.1620e-08
(0.1,0.1)	7.2204e-08	2.0127e-07	6.3738e-08	1.1494e-08
(0.2,0.2)	7.2151e-08	2.0123e-07	6.3779e-08	1.1158e-08
(0.3,0.3)	7.2076e-08	2.0119e-07	6.3842e-08	1.0662e-08
(0.4,0.4)	7.1986e-08	2.0113e-07	6.3921e-08	1.0060e-08
(0.5,0.5)	7.1889e-08	2.0108e-07	6.4007e-08	9.4057e-09
(0.6,0.6)	7.1793e-08	2.0102e-07	6.4093e-08	8.7512e-09
(0.7,0.7)	7.1704e-08	2.0097e-07	6.4172e-08	8.1499e-09
(0.8,0.8)	7.1630e-08	2.0093e-07	6.4237e-08	7.6546e-09
(0.9,0.9)	7.1580e-08	2.0090e-07	6.4281e-08	7.3186e-09
(1,1)	7.1560e-08	2.0089e-07	6.4295e-08	7.1946e-09

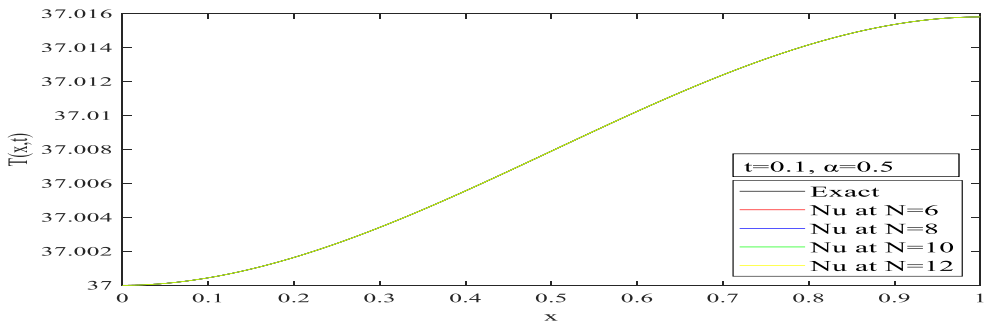

Figure 5: *Comparison Between Exact and Numerical Solutions for Case 2 at* $\alpha = 0.5, t = 0.1\ , R = 1, N = 6, 8, 10$ *and* 12.

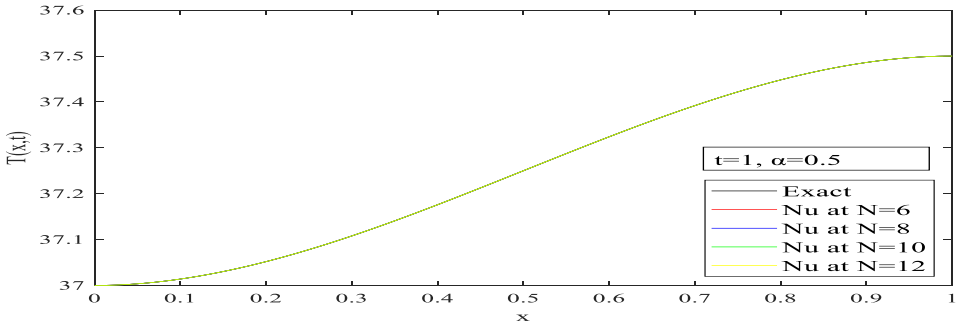

Figure 6: *Comparison Between Exact and Numerical Solutions for Case 2 at* $\alpha = 0.5$ $t = 1$,$R = 1, N = 6,8,10$ *and* 12.

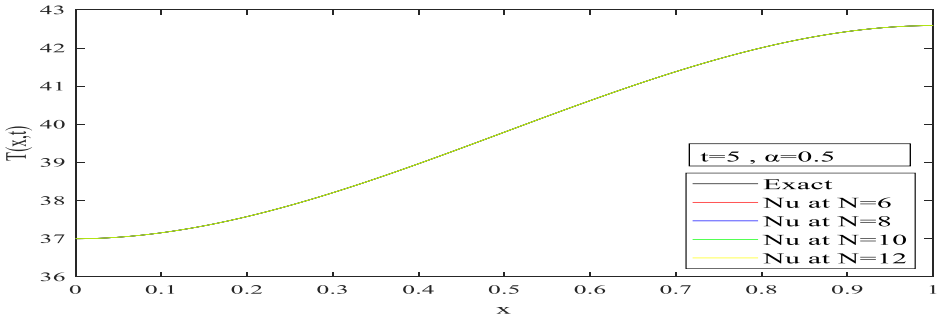

Figure7: *Comparison Between Exact and Numerical Solutions for Case 2 at* $\alpha = 0.5, t = 5$,$R = 1, N = 6,8,10$ *and* 12.

Table 5. *Absolut Errors Obtained for Case 2 with* $R = 1$ *and* $\alpha = 0.75$.

(x,t)	**Absolute error** $N \times M = 6 \times 6$ **CPU=166.562920s**	**Absolute error** $N \times M = 8 \times 8$ **CPU=384.579042s**	**Absolute error** $N \times M = 10 \times 10$ **CPU=792.067181s**	**Absolute error** $N \times M = 12 \times 12$ **CPU= 1469.992834s**
(0,0)	3.6291e-09	4.9725e-08	1.9869e-07	1.9163e-07
(0.1,0.1)	3.6404e-09	4.9717e-08	1.9871e-07	1.9167e-07
(0.2,0.2)	3.6718e-09	4.9706e-08	1.9880e-07	1.9179e-07
(0.3,0.3)	3.7181e-09	4.9691e-08	1.9894e-07	1.9199e-07
(0.4,0.4)	3.7745e-09	4.9673e-08	1.9912e-07	1.9223e-07
(0.5,0.5)	3.8358e-09	4.9655e-08	1.9931e-07	1.9250e-07
(0.6,0.6)	3.8972e-09	4.9637e-08	1.9951e-07	1.9276e-07
(0.7,0.7)	3.9535e-09	4.9620e-08	1.9969e-07	1.9301e-07
(0.8,0.8)	3.9999e-09	4.9605e-08	1.9983e-07	1.9321e-07
(0.9,0.9)	4.0313e-09	4.9595e-08	1.9993e-07	1.9334e-07
(1,1)	4.0428e-09	4.9590e-08	1.9996e-07	1.9338e-07

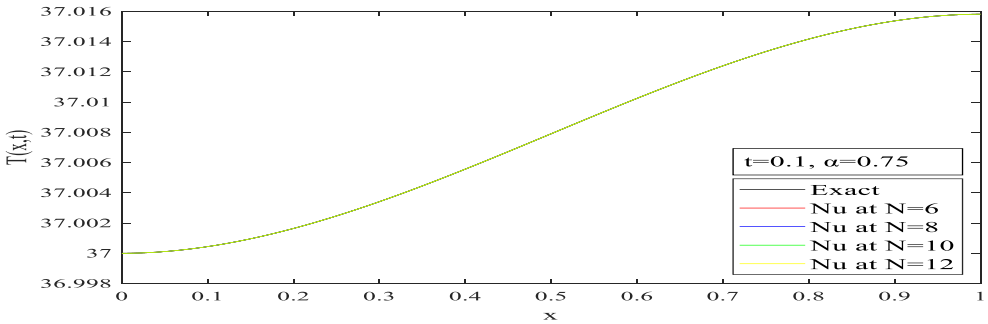

Figure8: *Comparison Between Exact and Numerical Solutions for Case 2 at* $\alpha = 0.75, t = 0.1$,$R = 1, N = 6,8,10$ *and* 12.

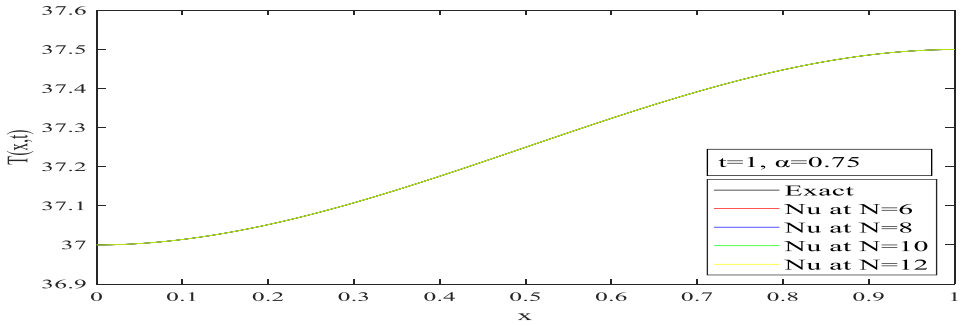

Figure9: *Comparison between exact and numerical solutions for case 2 at* $\alpha = 0.75$, $t = 1$, $R = 1, N = 6,8,10$ *and* 12.

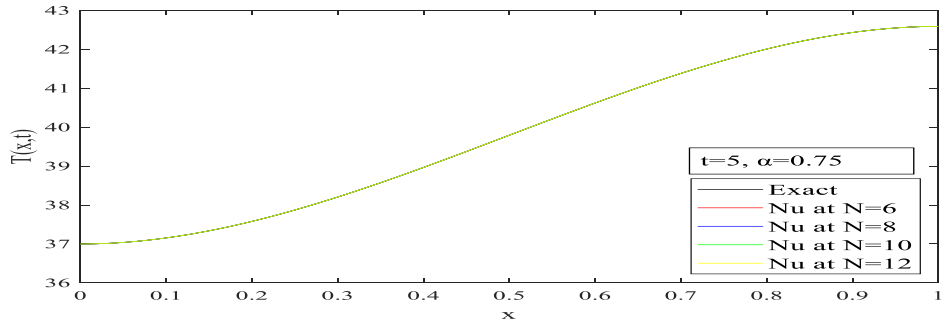

Figure10: *Comparison Between Exact and Numerical Solutions for Case 2 at* $\alpha = 0.75$, $t = 5$, $R = 1, N = 6,8,10$ *and* 12.

متعددات حدود ليجندر الكسورية لحل معادلة الحرارة الحيوية الكسورية-الزمانية

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المستخلص

يهدف هذه البحث: إلى توظيف متعددات حدود ليجندر الكسورية (FSLPs) لتقريب المشتقات الزمانية باستخدام تعريف صيغة كابوتو للمؤثر التفاضلي الكسوري. المشتقات الكسورية المكانية للرتب الصحيحة تكون مقربة بالاعتماد متعددات حدود ليجندر. تشير الاختبارات العددية لمثالين الى ان النهج المقترح يكون كفوء ومتين لحلول معادلة الحرارة الحيوية الكسورية–الزمانية. علاوة على ذلك، بعض الاوليات والتعاريف للمشتقات الكسورية، والمبرهنات والبديهيات المتعلقة بمتعددات حدود ليجندر الكسورية تكون مضافة لتحقيق نتائج التقارب للخوارزمية قيد الدراسة. في النهاية، ناقشنا تحليل الخطأ.

الكلمات المفتاحية: طريقة التجميع، معادلة الحرارة الحيوية الكسورية–الزمانية، متعددات حدود ليجندر الكسورية، الدقة.