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Double-Stage Shrinkage Estimation of Reliability Function for Burr XII Distribution

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ABSTRACT: This study is concerned with the problem of estimating the reliability function of the parameters of the two-parameter Burr XII distribution when the data are complete. Concepts such as the likelihood estimator, bias, bias ratio, and mean square error are defined. The purpose of this research is investigate the properties of the double-stage shrinkage estimators (DSSEs) of the reliability function of Burr XII distribution and obtain the equations of bias, mean square error, and bias ratio. The relative efficiency of the Burr XII distribution of the proposed estimators is also investigated by deriving their equations. Numerical results show performance of our estimators to be better than those of the classical pooled estimators based on the relative efficiency criteria. In particular, our proposed DSSEs have better performance than the classical estimator and single-stage shrinkage estimators.

Keywords: Burr XII distribution, Shrinkage estimator, Reliability function, Bias, Mean-square error, Bias ratio, Relative efficiency, Shrinkage factor.

1. INTRODUCTION

Many researchers have attempted to find estimators with higher relative efficiency compared with those of the classical estimators, hence their proposal to use shrinkage estimators. Thompson (1968) was the first researcher who proposed the shrinkage estimator. Let θ_0 be a guess value of θ. Typically, researchers are unsure if the guess value θ_0 is close to the true parameter value θ. Thus, the shrinkage estimator needs to be obtained. The shrinkage estimator represents a linear combination between the classical estimator $\hat{\theta}$ and a prior information θ_0, and it uses a weighted shrinkage factor K that can be either a constant or a function of $\hat{\theta}$. Here, $\tilde{\theta}_{sh} = K\hat{\theta} + (1-K)\theta_0, 0 < K < 1$.

At times, the units of a sample are expensive to determine or very rarely found; therefore, inexpensive estimators are preferably used. Double-stage shrinkage estimators (DSSEs) are suitable when some prior information about θ is available in the form of initial value θ_0, which can be defined according to the following steps: Obtain a first-stage random sample of size $(n_1 < n)$, from $f(x;\theta)$. On the basis of these n_1 observations, compute an estimator $\hat{\theta}_1$ of θ. Using θ_0 and an appropriate criterion, construct a region C in the space of parameter θ. If $\hat{\theta}_1 \in C$, then shrink $\hat{\theta}_1$ towards θ_0 by using the shrinkage factor k, where $0 < k < 1$, to yield the estimator for θ_0, such that $k\theta_1 + (1-\text{k})\ \theta_0$. Otherwise take a second independent random sample of $n_2 = n + n_1$ observations and compute the estimator $\hat{\theta}_2$ of θ based on the complete sample of size $n = n_1 + n_2$. The estimators $\hat{\theta}_1$ and $\hat{\theta}_2$ are the same statistic computed for samples of sizes n_1 and n_2, respectively. The DSSE, denoted by $\tilde{\theta}$ of θ, is given by

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$$\tilde{\theta} = \begin{cases} k\hat{\theta}_1 + (1-k)\theta_0 & \text{if } H_0 \text{ accepted} \\ \frac{n_1\hat{\theta}_1 + n_2\hat{\theta}_2}{n_1+n_2} & \text{otherwise} \end{cases}, \tag{1}$$

Here, the region C defines a test of our belief in the prior information θ_0. Katti (1962) discussed the problem of estimating the mean θ of a normal population with known variance by using a shrinkage estimator of the from (1.5.4) with $k = 1$ and the region C for minimizing the MSE$\left(\tilde{\theta}/\theta_0, C\right)$, which is the mean-square error of $\widetilde{\theta}$ at θ_0. In this procedure, the prior information θ_0 is used only to construct the region C, assuming the θ_0 is the true natural origin. Shah (1964) applied the method of Katti (1962) to address the variance of a normal distribution with an unknown arithmetic mean, while Arnold and Al-Bayyati (1970) combined both methods. Thompson (1968) and Katti (1962) proposed DSSE by performing the following steps:

i. Calculate $\bar{x}_1$ based on the first sample size n_1.

ii. If $\bar{x}_1$ belongs to C, then the estimator is $k(\bar{x}_1 - \mu_0) + \mu_0$.

iii. If $\bar{x}_1$ does not belong to C, then take the remainder of the components' sample available size $n_2\,(n_2 = n - n_1)$ and calculate the pooled $\bar{x}_n$. The estimator can then be expressed as

$$\tilde{\theta} = \begin{cases} k\bar{x}_1 + (1-k)\mu_0 & \bar{x}_1 \in C \\ \frac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1+n_2} & \text{otherwise} \end{cases} \tag{2}$$

The two-parameter Burr XII distribution (hereafter referred to as $B(\gamma, \theta)$) is one of the important life distributions. In particular, the Burr type XII distribution is one of the 12 types of continuous probability distributions used for nonnegative random variables in the Burr system. The Burr distribution, which is designated as a ratio, was initially proposed by Professor Irving W. Burr in 1942. The Burr distribution is also known as the Singh Maddala distribution or the generalized log-logistic distribution. The probability density function is given by

$$f(x;\theta,\gamma) = \theta\gamma \frac{x^{\gamma-1}}{(1+x^{\gamma})^{\theta+1}} \quad ; x > 0, \theta, \gamma > 0 \tag{3}$$

while the cumulative distribution function (c.d.f) is given by

$$F(x) = 1 - (1+x^{\gamma})^{-\theta} \quad ; x > 0, \theta, \gamma > 0 \tag{4}$$

where $\theta > 0$ and $\gamma > 0$ are the scale and shape parameters, respectively. This approach has received more attention from researchers owing to its broad application in different fields. Furthermore, it gained popularity in the last decade owing to its potentiality for use it practical situations, including in fields requiring the reliability function (Johnson and Katz, 1970), such as failure-time modeling and acceptance of sampling plans, and the clinical, life-testing, survival analysis, economics, biological or experimental fields, and so on. Moreover, it is similar to log-normal distribution, which is a popular model used in reliability and life testing because it entails a non-monotone failure rate.

The reliability function is the probability in which a device or system will operate up to determined time without failure. It is defined mathematically as

$$\mathrm{R(t) = P(T > t) = 1 - F(t).} \tag{5}$$

The reliability function of the Burr(γ, θ) distribution is given by

$$R(t) = (1+t^{\gamma})^{-\theta} \quad t > 0. \tag{6}$$

The aforementioned estimators perform better than customary estimators in terms of the mean-square error value when guessing value that is close to the true value.

2. PROPOSED ESTIMATORS

Sometimes, no prior information about the unknown parameter θ is available. A classical estimation method is maximum likelihood estimation (MLE).

Let $X_1, X_2, \ldots, X_n$ be a random sample from the Burr XII distribution with a size n as defined in Eq. (3). In the MLE of estimator $\hat{\theta}$, suppose the parameter γ is known, then

$$\hat{\theta} = \frac{n}{\sum_{i=1}^{n} \ln\left(1 + x_i^{\gamma}\right)}. \tag{7}$$

By referring to the invariance property of MLE and the equation with parameter $\hat{\theta}$, the MLE for the reliability function of Burr XII is

$$\hat{R}(x) = (1 + x^{\gamma})^{-\hat{\theta}} \quad , x > 0, \gamma > 0. \tag{8}$$

If X is a random variable with Burr XII distribution, then $y = \ln(1 + x^{\gamma})$, and $g(y) = \theta e^{-\theta x}$ has an exponential distribution with a mean parameter of θ. Thus, $y_i = \sum_{i=1}^{n} \ln\left(1 + x_i^{\gamma}\right)$ has a Gamma distribution denoted by $G(n,\theta)$, i.e., $\frac{\sum_{i=1}^{n} \ln\left(1+x_i^{\gamma}\right)}{n}$.

Consequently, the distribution of $\hat{\theta}$ is inverted to a Gamma distribution with the following p.d.f [21]:

$$f(\hat{\theta}) = \frac{(n\theta)^n \left(\frac{1}{\hat{\theta}}\right)^{n+1} e^{\frac{-n\theta}{\hat{\theta}}}}{\Gamma n} \quad ,\hat{\theta} > 0, \theta > 0. \tag{9}$$

Thus, we can easily show that $\frac{2n\theta_0}{\hat{\theta}} \sim X^2_{2n}$ is a Chi-square distribution with a degree-of-freedom of 2n.

3. DSSE

Supposing that the initial value θ_0 is available, then we suggest the following two single-stage shrinkage estimators to derive the equations for the bias and mean square error for calculating the bias ratio and relative efficiency for each of the suggested estimators.

The first suggested single-stage shrinkage estimator $\left(\hat{R}_{s_1}\right)$ is defined as

$$\hat{R}_{D_1} = \begin{cases} k_1\hat{R}_1 + (1-k_1)R_0 & \text{if } H_0 \text{ accepted} \\ \frac{n_1\hat{R}_1 + n_2\hat{R}_2}{n_1+n_2} & \text{otherwise} \end{cases}, \tag{10}$$

where k_1 the shrinkage factor with a constant value, i.e., $(0 < k_1 < 1)$, and $\hat{R}_i = (1 + x^{\gamma})^{-\widehat{\theta_i}}\, i = 1,2$ for the MLE of the reliability function. Consequently, $n_1 < n, n_2 = n - n_1$. C is the acceptance region for testing the hypothesis $H_0 : \theta = \theta_0$ against $H_1 : \theta \neq \theta_0$. The level of significance is set to $1 - \alpha$.

The second proposed shrinkage estimator $\left(\widehat{\mathrm{R}}_{\mathrm{D2}}\right)$ is defined as

$$\tilde{R}_{D_2} = \begin{cases} K_2\hat{R}_1 + (1-K_2)R_0 & \text{if } H_0 \text{ accepted} \\ \frac{n_1\hat{R}_1 + n_2\hat{R}_2}{n_1+n_2} & \text{otherwise ,} \end{cases} \tag{11}$$

where K_2 the shrinkage factor, which is suggested based on the acceptance of the hypotheses $\mathrm{H}_0 : \theta = \theta_0$ against hypotheses is $\mathrm{H}_1 : \theta \neq \theta_0$. Assume r_1 and r_2 are the lower and upper $100(1 - \alpha)$ percentile values, respectively, of the Chi-square distribution with a degree-of-freedom of 2n. Thus,

$$r_1 < \frac{2n\theta_0}{\hat{\theta}} < r_2$$

$$\frac{2n\theta_0}{r_2} < \hat{\theta} < \frac{2n\theta_0}{r_1}$$

when

$$C = \left\{\hat{\theta}; \frac{2n\theta_0}{r_2} < \hat{\theta} < \frac{2n\theta_0}{r_1}\right\} \tag{12}$$

is the acceptance region.

so

$$a^{\frac{-2n\theta_0}{r_1}} < a^{-\widehat{\theta}} < a^{\frac{-2n\theta_0}{r_2}} \quad ,a = (1 - x^{\gamma}), \gamma, x > 0$$

then

$$a^{\frac{-2n\theta_0}{r_1}} < \hat{R} < a^{\frac{-2n\theta_0}{r_2}}$$

$$0 < \hat{R} - a^{\frac{-2n\theta_0}{r_1}} < a^{\frac{-2n\theta_0}{r_2}} - a^{\frac{-2n\theta_0}{r_1}} < a^{\frac{-2n\theta_0}{r_2}} < a^{\frac{-2n\theta_0}{r_2}} + a^{\frac{-2n\theta_0}{r_1}}$$

therfore $$0 < \frac{\hat{R} - a^{\frac{-2n\theta_0}{r_1}}}{a^{\frac{-2n\theta_0}{r_2}} + a^{\frac{-2n\theta_0}{r_1}}} < 1$$

then $$K_2 = \frac{\hat{R} - a^{\frac{-2n\theta_0}{r_1}}}{a^{\frac{-2n\theta_0}{r_2}} + a^{\frac{-2n\theta_0}{r_1}}} \tag{13}$$

4. BIAS RATIO AND MEAN-SQUARE ERROR

The suggested bias estimator $\tilde{R}_{D_1}(t)$ is defined as

$$\text{Bias}\left(\tilde{R}_{D_1}\right) = E\left(\hat{R}_{D_1} - R\right) = E\left(\hat{R}_{D_1}\right) - R$$
$$= \int_0^{\infty}\int_c \left[k_1\hat{R}_1 + (1-k_1)R_0\right] f_1\left(\hat{\theta}_1\right) f_2\left(\hat{\theta}_2\right) d\hat{\theta}_1 d\hat{\theta}_2 + \int_0^{\infty}\int_{\bar{c}}\left[\frac{n_1\hat{R}_1 + n_2\hat{R}_2}{n_1+n_2}\right] f_1\left(\hat{\theta}_1\right) f_2\left(\hat{\theta}_2\right) d\hat{\theta}_1 d\hat{\theta}_2 - R,$$

where C is the acceptance region, i.e., $C = \left\{\hat{\theta}; \frac{2n\theta_0}{r_2} < \hat{\theta} < \frac{2n\theta_0}{r_1}\right\}$, with $\bar{C}$ as the complement region. This scheme allows for the use of the transformation

$$,z = \frac{n_2\theta}{\hat{\theta}_2} \Rightarrow \hat{\theta}_2 y = n_2\theta \Rightarrow \hat{\theta}_2 = \frac{n_2\theta}{y} \Rightarrow d\hat{\theta}_2 = \frac{-n_2\theta dy}{y^2} \quad ,x^{\gamma} = a$$

such that C^* is equal to C after the transformation, i.e. $C^* = \left(\frac{r_1}{2\lambda}, \frac{r_2}{2\lambda}\right)$, where $\lambda = \frac{\theta_0}{\theta}$

$$\text{ias}\left(\tilde{R}_{D_1}\right) = \left(k_1 - \frac{n_1}{n_1+n_2}\right)\frac{1}{\Gamma n_1}\int_{C^*}(1+a)^{\frac{-n_1\theta}{y}} e^{-y} y^{n_1-1} dy + \frac{(1-k_1)(1+a)^{-\theta_0}}{\Gamma n_1}\int_{C^*} e^{-y} y^{n_1-1} dy$$
$$+ \frac{n_1}{\Gamma n_1 (n_1+n_2)}\int_0^{\infty}(1+a)^{\frac{-n_1\theta}{y}} e^{-y} y^{n_1-1} dy + \frac{n_2}{\Gamma n_2(n_1+n_2)}\int_0^{\infty}(1+a)^{\frac{-n_2\theta}{z}} e^{-z} z^{n_2-1} dz - (1+a)^{-\theta}$$
$$- \frac{n_2}{(n_1+n_2))\Gamma n_1 \Gamma n_2}\left(\int_{C^*} e^{-y} y^{n_1-1} dy\right) \times \left(\int_0^{\infty}(1+a)^{\frac{-n_2\theta}{z}} e^{-z} z^{n_2-1} dz\right).$$

After performing the streamlines,

$$\begin{aligned}\text{Bias}\left(\tilde{R}_{D2}\right) &= (1+a)^{-\theta}\Bigg[(1+a)^{\theta}\Bigg[\left(k_1 - \frac{n_1}{n_1+n_2}\right)\int_{C^*}(1+a)^{\frac{-2n_1\theta}{y}} e^{-y} y^{n_1-1} dy \\ &+ \frac{n_1}{(n_1+n_2)\Gamma n_1}\left(\frac{1}{n_1\theta\ln(1-a)}\right)^{\frac{1-n_1}{2}} K_{-n}\left(2\sqrt{n_1\theta\ln(1-a)}\right. \\ &+ \frac{n_2}{(n_1+n_2)\Gamma n_2}\left(\frac{1}{n_2\theta\ln(1-a)}\right)^{\frac{1-n_2}{2}} K_{-n}\left(2\sqrt{n_2\theta\ln(1-a)} - \frac{2n_2}{(n_1+n_2)\Gamma n_2}\left(I\left(\frac{r_2}{2\lambda}, n_1\right) - I\left(\frac{r_1}{2\lambda}, n_1\right)\right)\right. \\ &\times\left(\left(\frac{1}{n_2\theta\ln(1-a)}\right)^{\frac{1-n_2}{2}} K_{-n}\left(2\sqrt{n_2\theta\ln(1-a)}\right)\right] + (1-K_1)(1+a)^{(1-\lambda)\theta}\left(I\left(\frac{r_2}{2\lambda}, n_1\right) - I\left(\frac{r_1}{2\lambda}, n_1\right)\right) - 1\Bigg]\end{aligned} \tag{14}$$

where $I(b,n) = \frac{\int_0^b x^{n-1}e^{-x}dx}{n}$ is an incomplete Gamma function, and $K_n = \frac{(-1)^i t^{n+2i}}{2^{n+1}i!(n+i)!}$ is the second modified Bessel function. The variables r_1 and r_2 are the lower and upper percentile values, respectively, of the Chi-square dis. with a degree-of-freedom of $2n$.

As for bias ratio in relation to shape, we have

$$BR\left(\tilde{R}_{D_1}, R\right) = \frac{B(\tilde{R}(T), R)}{R},$$

$$BR\left(\tilde{R}_{D_1}, R\right) = \left(k_1 - \frac{n_1}{n_1+n_2}\right)\int_{C^*}(1+a)^{\frac{-2n_1\theta}{y}} e^{-y} y^{n_1-1} dy + \tag{15}$$

$$\frac{n_1}{(n_1+n_2)\Gamma n_1}\left(\frac{1}{n_1\theta\ln(1-a)}\right)^{\frac{-n_1}{2}} K_{-n}\left(2\sqrt{n_1\theta\ln(1-a)} + \frac{n_2}{(n_1+n_2)\Gamma n_2}\left(\frac{1}{n_2\theta\ln(1-a)}\right)^{\frac{1-n_2}{2}} K_{-n}\left(2\sqrt{n_2\theta\ln(1-a)} - \right.\right.$$

$$\frac{2n_2}{(n_1+n_2)\Gamma n_2}\left(I\left(\frac{r_2}{2\lambda}, n_1\right) - I\left(\frac{r_1}{2\lambda}, n_1\right)\right)\left(\left(\frac{1}{n_2\theta\ln(1-a)}\right)^{\frac{1-n_2}{2}} K_{-n}\left(2\sqrt{n_2\theta\ln(1-a)}\right) + (1-K_1)(1+\right.$$
$$a)^{(1-\lambda)\theta}\left(I\left(\frac{r_2}{2\lambda}, n_1\right) - I\left(\frac{r_1}{2\lambda}, n_1\right)\right) - 1. \tag{16}$$

The mean square error with the estimated $\widetilde{\mathrm{R}}_{s_1}(\mathrm{t})$ is defined as

$$\mathrm{MSE}\left(\tilde{R}_{D_1}\right)=E(\tilde{R}-R)^2$$
$$=\int_0^\infty\int_c\left[k_1\hat{R}+(1-k_1)R_0-R\right]^2 f_1\hat{\theta}_1 f_2\hat{\theta}_2 d\hat{\theta}_1 d\hat{\theta}_2+\int_0^\infty\int_{\bar{c}}\left[\frac{n_1\hat{R}_1+n_2\hat{R}_2}{n_1+n_2}-\right.$$
$$\left.R\right]f_1\hat{\theta}_1 f_2\hat{\theta}_2 d\hat{\theta}_1 d\hat{\theta}_2$$
$$=\int_c\left[k_1\left(\hat{R}-R_0\right)-(R-R_0)\right]^2 f_1\left(\hat{\theta}_1\right)f_2\left(\hat{\theta}_2\right)d\hat{\theta}_1 d\hat{\theta}_2+\int_0^\infty\int_0^\infty\left[\frac{n_1\left(\hat{R}_1-R\right)+n_2\left(\hat{R}_2-R\right)}{n_1+n_2}\right]^2 f_1\left(\hat{\theta}_1\right)f_2\left(\hat{\theta}_2\right)d\hat{\theta}_1 d\hat{\theta}_2-$$
$$\int_0^\infty\int_c\left[\frac{n_1\left(\hat{R}_1-R\right)+n_2\left(\hat{R}_2-R\right)}{n_1+n_2}\right]^2 f_1\left(\hat{\theta}_1\right)f_2\left(\hat{\theta}_2\right)d\hat{\theta}_1 d\hat{\theta}_2$$
$$=\int_c\left[k_1^2\left(\hat{R}_1^2-2R_0\hat{R}_1+R_0^2\right)-2k_1(R-R_0)\left(\hat{R}_1-R_0\right)+(R-R_0)^2\right]f_1\hat{\theta}_1 d\hat{\theta}_1+\int_0^\infty\int_0^\infty\left[\frac{n_1^2\left(\hat{R}_1^2-2\hat{R}_1R+R^2\right)}{(n_1+n_2)^2}+\right.$$
$$\left.\frac{2n_1n_2\left(\left(\hat{R}_1-R\right)\left(\hat{R}_2-R\right)\right.}{(n_1+n_2)^2}+\frac{n_2^2\left(\hat{R}_2^2-2\hat{R}_2R+R^2\right)}{(n_1+n_2)^2}\right]f_1\left(\hat{\theta}_1\right)f_2\left(\hat{\theta}_2\right)d\hat{\theta}_1 d\hat{\theta}_2-\int_0^\infty\int_c\left[\frac{n_1^2\left(R_1^2-2\hat{R}_1R+R^2\right)}{(n_1+n_2)^2}+\frac{2n_1n_2(R_1-R)(R_2-R)}{(n_1+n_2)^2}+\right.$$
$$\left.\frac{n_2^2\left(R_2^2-2R_2R+R^2\right)}{(n_1+n_2)^2}\right]f_1\left(\hat{\theta}_1\right)f_2\left(\hat{\theta}_2\right)d\hat{\theta}_1 d\hat{\theta}_2$$

By using the same assumption and compensations for the bias, we have

$$\begin{aligned}
\mathrm{MSE}(\tilde{R})&=(1+a)^{-2\theta}\left[\left(k_1^2-\left(\frac{n_1}{n_1+n_2}\right)^2\right)\cdot\frac{(1+a)^{2\theta}}{\Gamma n_1}\times\int_{C^*}e^{-\left(y+\frac{2n_1\theta}{y}\ln((1+a)\right.}y^{n_1-1}dy-\left[2k_1^2(1+a)^{-(2+\lambda)\theta}+\right.\right.\\
&2k_1\left((1+a)^{\theta}-(1+a)^{-(2+\lambda)\theta}\right)+\left.\frac{2n_1(1+a)^{\theta}}{(n_1+n_2)}\right]\frac{1}{\Gamma n_1}\int_{C^*}(1+a)^{\frac{-n_1\theta}{y}}e^{-y}y^{n_1-1}dy+\left[k_1^2(1+a)^{-2(1+\lambda)\theta}+\right.\\
&2k_1(1+a)^{-(1+\lambda)\theta}\left(1-(1+a)^{-(1+\lambda)\theta}\right)+\left((1+a)^{3\theta}-(1+a)^{-(4+\lambda)\theta}\right)^2+1\Big]\times\frac{1}{\Gamma n_1}\int_{C^*}e^{-y}y^{n_1-1}dy-\\
&\frac{2n_1(1+a)^{\theta}}{(n_1+n_2)\Gamma n_1}\times\int_0^\infty(1+a)^{\frac{-2n_1\theta}{y}}e^{-y}y^{n_1-1}dy+\frac{2n_1n_2(1+a)^{-2\theta}}{(n_1+n_2)\Gamma n_1\Gamma n_2}\times\left(\int_0^\infty(1+a)^{\frac{-n_1\theta}{y}}e^{-y}y^{n_1-1}dy\right)\left(\int_0^\infty(1+\right.\\
&\left.a)^{\frac{-n_2\theta}{y}}e^{-y}y^{n_2-1}dy\right)\quad-\frac{2n_2(1+a)^{\theta}}{(n_1+n_2)\Gamma n_2}\int_0^\infty e^{-\left(z+\frac{n_2\theta}{z}\ln(1+a)\right.}z^{n_2-1}dz+\\
&\left(\frac{n_2}{n_1+n_2}\right)^2\frac{(1+a)^{2\theta}}{\Gamma n_2}\int_0^\infty e^{-\left(z+\frac{2n_2\theta}{z}\ln(1+a)\right.}z^{n_2-1}dz+\frac{2n_1n_2(1+a)^{2\theta}}{(n_1+n_2)^2\Gamma n_1\Gamma n_2}\left(\int_0^\infty e^{-\left(z+\frac{n_2\theta}{z}\ln(1+a)\right.}z^{n_2-1}dz\right)\times\\
&\left(\int_{C^*}e^{-\left(y+\frac{n_1\theta}{y}\ln((1+a)\right.}y^{n_1-1}dy\right)-\left(\frac{n_2(1+a)^{\theta}}{n_1+n_2}\right)^2\frac{1}{\Gamma n_1\Gamma n_2}\left(\int_0^\infty e^{-\left(z+\frac{2n_2\theta}{z}\ln(1+a)\right.}z^{n_2-1}dz\right)\times\left(\int_{C^*}e^{-y}y^{n_1-1}dy\right)+\\
&\frac{2n_2(1+a)^{\theta}}{(n_1+n_2)\Gamma n_1\Gamma n_2}\left(\int_0^\infty e^{-\left(z+\frac{2n_2\theta}{z}\ln(1+a)\right.}z^{n_2-1}dz\right)\times\left(\int_{C^*}e^{-y}y^{n_1-1}dy\right)+1\Big].
\end{aligned}\tag{17}$$

The bias of estimator $\widetilde{\mathrm{R}}_{\mathrm{D}_2}$ is derived as follows:

$$\mathrm{Bias}\left(\tilde{R}_{D2}\right)=E\left(\tilde{R}_{D2}-R\right)=E\left(\tilde{R}_{D2}\right)-R$$
$$=\int_0^\infty\int_c\left(k_2\hat{R}_1+(1-k_2)R_0\right)f_1\left(\hat{\theta}_1\right)f_2\left(\hat{\theta}_2\right)d\hat{\theta}_1 d\hat{\theta}_2+$$
$$\int_0^\infty\int_{\bar{c}}\left[\frac{n_1\hat{R}_1+n_2\hat{R}_2}{n_1+n_2}\right]f_1\left(\hat{\theta}_1\right)f_2\left(\hat{\theta}_2\right)d\hat{\theta}_1 d\hat{\theta}_2-R.$$

Similarly, by using same the transformation $y = \frac{n_1\theta}{\hat{\theta}_1}, z = \frac{n_2\theta}{\hat{\theta}_2}, x^\gamma = a$, we can obtain

$$= \left(k_1^2 - \left(\frac{n_1}{n_1+n_2}\right)^2\right)\frac{1}{\Gamma n_1}\int_{C^*}(1+a)^{\frac{-2n_1\theta}{y}}e^{-y}y^{n_1-1}d - \left[2k_1^2(1+a)^{-\theta_0} + 2k_1\left((1+a)^{-\theta} - (1+a)^{-\theta_0}\right) - \frac{2n_1(1+a)^{-\theta}}{n_1+n_2}\right]\frac{1}{\Gamma n_1}\int_{C^*}(1+a)^{\frac{-n_1\theta}{y}}e^{-y}y^{n_1-1}d + \left[k_1^2(1+a)^{-2\theta_0} + 2k_1(1+a)^{-\theta_0}\left((1+a)^{-\theta} - (1+a)^{-\theta_0}\right) + \left((1+a)^{-\theta} - (1+a)^{-\theta_0}\right)^2 - (1+a)^{-2\theta}\right] \times \frac{1}{\Gamma n_1}\int_{C^*}e^{-y}y^{n_1-1}dy - \frac{2n_1(1+a)^{-\theta}}{(n_1+n_2)\Gamma n_1}\times\int_0^\infty(1+a)^{\frac{-n_1\theta}{y}}e^{-x}y^{n_1-1}dy + (1+a)^{-2\theta} + \frac{2n_1n_2}{(n_1+n_2)\Gamma n_1\Gamma n_2}\times\left(\int_0^\infty(1+a)^{\frac{-n_1\theta}{y}}e^{-x}y^{n_1-1}dy\right)\times\left(\int_0^\infty(1+a)^{\frac{-2n_2\theta}{z}}e^{-z}z^{n_2-1}dz\right) - \frac{2n_2(1+a)^{-\theta}}{(n_1+n_2)\Gamma n_2}\int_0^\infty(1+a)^{\frac{-2n_2\theta}{z}}e^{-z}z^{n_2-1}dz + \left(\frac{n_2}{n_1+n_2}\right)^2\frac{1}{\Gamma n_2}\int_0^\infty(1+a)^{\frac{-2n_2\theta}{y}}e^{-y}y^{n_2-1}d - \frac{2n_1n_2}{(n_1+n_2)^2\Gamma n_1\Gamma n_2}\left(\int_0^\infty(1+a)^{\frac{-2n_2\theta}{z}}e^{-z}z^{n_2-1}dz\right)\times\left(\int_{C^*}(1+a)^{\frac{-n_1\theta}{y}}e^{-y}y^{n_1-1}dy\right) - \left(\frac{n_2}{n_1+n_2}\right)^2\frac{1}{\Gamma n_1\Gamma n_2}\left(\int_0^\infty(1+a)^{\frac{-n_2\theta}{z}}e^{-z}z^{n_2-1}dz\right)\times\left(\int_{C^*}e^{-y}y^{n_1-1}dy\right) + \frac{2n_2(1+a)^{-\theta}}{(n_1+n_2)\Gamma n_1\Gamma n_2}\times\left(\int_0^\infty(1+a)^{\frac{-2n_2\theta}{z}}e^{-z}z^{n_2-1}dz\right)\times\left(\int_{C^*}e^{-y}y^{n_1-1}dy\right)$$

As for the bias ratio of estimator $\widetilde{R}_{s_2}$ in relation to shape, it is expressed as

$$BR\left(\tilde{R}_{D_2}, R\right) = \frac{B\left(\tilde{R}_{D_2}(T), R\right)}{R}$$

$$BR\left(\tilde{R}_{D_2}, R\right) = \frac{1}{\left((1+a)^{-\left(1+\frac{2n\lambda}{r_2}\right)\theta} + (1+a)^{-\left(1+\frac{2n\lambda}{r_1}\right)\theta}\right)\Gamma n_1}\int_{C^*}y^{-(n_1-1)}e^{-\left(y+\frac{2nn_1\theta ln(1-a)}{y}\right)}d - \left(\frac{1}{\left((1+a)^{-\theta} - (1+a)^{-\left(1+2n\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\theta\right)}\right)} + \left(I\left(\frac{r_2}{2\lambda}, n_1\right) - I\left(\frac{r_1}{2\lambda}, n_1\right)\right) + (1+a)^{\theta}\left(\frac{2n_1}{(n_1+n_2)\Gamma n_1}\right)\times\left(\frac{1}{n_1\theta\ln(1-a)}\right)^{\frac{1-n_1}{2}}K_{-n}\left(2\sqrt{n_1\theta\ln(1-a)} + \frac{n_2}{(n_1+n_2)\Gamma n_1}\left(\frac{1}{n_2\theta\ln(1-a)}\right)^{\frac{1-n_2}{2}}K_{-n}\left(2\sqrt{n_2\theta\ln(1-a)} - \frac{2n_2}{(n_1+n_2)\Gamma n_2}\left(I\left(\frac{r_2}{2\lambda}, n_1\right) - I\left(\frac{r_1}{2\lambda}, n_1\right)\right)\times\left(\left(\frac{1}{n_2\theta\ln(1-a)}\right)^{\frac{1-n_2}{2}}K_{-n}\left(2\sqrt{n_2\theta\ln(1-a)} + 1\right]. \tag{18}$$

The mean square error of the estimator $\widetilde{R}_{D_2}$ is derived as follows:

$$\mathrm{MSE}\left(\tilde{R}_{D_2}\right) = E(\hat{R} - R)^2 = \int_0^\infty\int_c\left[K_2\hat{R} + (1-K_2)R_0 - R\right]^2 f_1\left(\hat{\theta}_1\right)f_2\left(\hat{\theta}_2\right)d\hat{\theta}_1 d\hat{\theta}_2 + \int_0^\infty\int_{\bar{c}}\left[\frac{n_1\hat{R}_1 + n_2\hat{R}_2}{n_1+n_2} - R\right]^2 f_1\left(\hat{\theta}_1\right)f_2\left(\hat{\theta}_2\right)d\hat{\theta}_1 d\hat{\theta}_2.$$

By using same the transformation, we have $y = \frac{n_1\theta}{\hat{\theta}_1}, z = \frac{n_2\theta}{\hat{\theta}_2}, x^\gamma = a$.

After calculating the simplifications and integrals, we can obtain

$$\mathrm{MSE}\left(\tilde{R}_{D_2}\right)=(1+a)^{-2\theta}\left[\frac{1}{\left((1+a)-\left(1+\frac{2n\lambda}{r_2}\right)_{+}+(1+a)-\left(1+\frac{2n\lambda}{r_1}\right)\theta\right)^2\Gamma n_1}\int_{C^*}y^{n_1-1}(1+a)^{-\frac{4n_1\theta}{y}}e^{-y}dy-\right.$$

$$\frac{2\left((1+a)\left(1-\frac{2n\lambda}{r_1}\right)\theta+(1+a)^{(1-\lambda)\theta}\right.}{\left((1+a)-\left(\frac{1}{2}+\frac{2n\lambda}{r_2}\right)\theta+(1+a)-\left(\frac{1}{2}+\frac{2n\lambda}{r_1}\right)\theta\right)^2_{\Gamma n_1}}\int_{C^*}y^{n_1-1}(1+a)^{-\frac{3n_1\theta}{y}}e^{-y}dy+\left(\left(\frac{(1+a)^{(1-2\lambda)\theta}+4(1+a)^{\left(1-\left(1-\frac{2n}{r_1}\right)\lambda\right)\theta}+h_1{}^2}{\left((1+a)^{-\left(\frac{1}{2}+\frac{2n\lambda}{r_2}\right)\theta}+(1+a)\right.}-\right.\right.$$

$$\left.\frac{2\left(1-(1+a)^{(1-\lambda)\theta}\right)}{\left((1+a)^{-\left(1+\frac{2n\lambda}{r_2}\right)\theta}+(1+a)-\left(1+\frac{2n\lambda}{r_1}\right)\theta\right)}-\frac{n_1^2(1+a)^{2\theta}}{(n_1+n_2)^2}\right)\frac{1}{\Gamma n_1}\int_{C^*}y^{n_1-1}(1+a)^{-\frac{2n_1\theta}{y}}e^{-y}dy-$$

$$\frac{2((1+a))^{\left(1-\left(2-\frac{2n}{r_1}\right)\lambda\right)\theta}+(1+a)^{\left(1-\left(1-\frac{4n}{r_1}\right)\lambda\right)\theta}\Big)}{\left((1+a)^{-\left(\frac{1}{2}+\frac{2n\lambda}{r_2}\right)\theta}+(1+a)^{-\left(\frac{1}{2}+\frac{2n\lambda}{r_1}\right)\theta}\right)^2}-\frac{2\left((1+a)^{-\frac{1}{2}\theta}-(1+a)^{\left(\frac{1}{2}-\lambda\right)\theta}\right)\left((1+a)^{\left(\frac{1}{2}-\lambda\right)\theta}+(1+a)^{\left(\frac{1}{2}-\frac{2n\lambda}{r_1}\right)\theta}\right)}{\left((1+a)^{-\left(1+\frac{2n\lambda}{r_2}\right)\theta}+(1+a)^{-\left(1+\frac{2n\lambda}{r_1}\right)\theta}\right)}-$$

$$\left.\frac{2n_1\left(n_1(1+a)^{2\theta}+n_2(1+a)^{\theta}\right)}{(n_1+n_2)^2}\right)\frac{1}{\Gamma n_1}\int_{C^*}y^{n_1-1}(1+a)^{-\frac{n_1\theta}{y}}e^{-y}d+\left(\frac{(1+a)\left(1-2\left(1-\frac{2n}{r_1}\right)\lambda\right)^{\theta}}{(h_2+h_1)^2}-\right.$$

$$\left.\frac{2(1+a)^{\left(\frac{1}{2}-\left(1-\frac{2n}{r_1}\right)\lambda\right)\theta}\left((1+a)^{-\frac{1}{2}\theta}-(1+a)^{\left(\frac{1}{2}-\lambda\right)\theta}\right)}{(h_2+h_1)}-2(1+a)^{(1-\lambda)\theta}+(1+a)^{(1-2\lambda)\theta}\right)\left\{I\left(\frac{r_2}{2\lambda},n_1\right)-I\left(\frac{r_1}{2\lambda},n_1\right)\right\}+$$

$$\frac{2n_1^2(1+a)^{2\theta}}{(n_1+n_2)^2\Gamma n_1}\left(\frac{1}{2n_1\theta\ln(1-a)}\right)^{\frac{1-n_1}{2}}K_{-n_1}\left(2\sqrt{2n_1\theta\ln(1-a)}\right.+$$

$$\frac{2n_2^2(1+a)^{2\theta}}{(n_1+n_2)^2\Gamma n_2}\left(\frac{1}{n_2\theta\ln(1-a)}\right)^{\frac{1-n_2}{2}}K_{-n_2}\left(\sqrt{2n_2\theta\ln(1-a)}-\frac{4n_1(1+a)\theta}{(n_1+n_2)\Gamma n_1}\left(\frac{1}{n_1\theta\ln(1-a)}\right)^{\frac{1-n_1}{2}}K_{-n_1}\left(2\sqrt{n_1\theta\ln(1-a)}\right.\right.-$$

$$\frac{2n_2(1+a)^{\theta}}{(n_1+n_2)\Gamma n_2}\left(\frac{1}{n_2\theta\ln(1-a)}\right)^{\frac{1-n_2}{2}}K_{-n_2}\left(\sqrt{2n_2\theta\ln(1-a)}+1+\right.$$

$$\frac{8n_1n_2(1+a)^{2\theta}}{(n_1+n_2)^2\Gamma n_1\Gamma n_2}\left(\left(\frac{1}{n_1\theta\ln(1-a)}\right)^{\frac{1-n_1}{2}}K_{-n_1}\left(2\sqrt{n_1\theta\ln(1-a)}\right)\times\left(\left(\frac{1}{n_2\theta\ln(1-a)}\right)^{\frac{1-n_2}{2}}K_{-n_2}\left(\sqrt{2n_2\theta\ln(1-a)}\right)-\right.\right. \tag{19}$$

$$\frac{8n_1n_2(1+a)^{2\theta}}{(n_1+n_2)^2\Gamma n_1\Gamma n_2}\left(\left(\frac{1}{n_1\theta\ln(1-a)}\right)^{\frac{1-n_1}{2}}K_{-n_1}\left(\sqrt{2n_1\theta\ln(1-a)}\right)\times\right.$$

$$\left(\left(\frac{1}{n_2\theta\ln(1-a)}\right)^{\frac{1-n_2}{2}}K_{-n_2}\left(\sqrt{2n_2\theta\ln(1-a)}\right)-\frac{2n_2^2(1+a)^{2\theta}}{(n_1+n_2)^2\Gamma n_2}\left(\left\{I\left(\frac{r_2}{2\lambda},n_1\right)-I\left(\frac{r_1}{2\lambda},n_1\right)\right\}\right)\times\right.$$

$$\left(\left(\frac{1}{n_2\theta\ln(1-a)}\right)^{\frac{1-n_2}{2}}K_{-n_2}\left(\sqrt{2n_2\theta\ln(1-a)}\right)+\frac{2n_2(1+a)^{\theta}(n_2+2n_1)}{(n_1+n_2)^2\Gamma n_2}\left(\left\{I\left(\frac{r_2}{2\lambda},n_1\right)-I\left(\frac{r_1}{2\lambda},n_1\right)\right\}\right)\times\right.$$

$$\left.\left(\left(\frac{1}{n_2\theta\ln(1-a)}\right)^{\frac{1-n_2}{2}}K_{-n_2}\left(\sqrt{2n_2\theta\ln(1-a)}\right)\right].$$

5. EXPECTED SAMPLE SIZE

The sample size depends on test results of the hypothesis $H_0 : \theta = \theta_0$ against the hypothesis $H_1 : \theta \neq \theta_0$ at the significance level (α). Here, we replace the true n by its expected sample size value, especially since the true sample size is random.

$$\begin{aligned} E\left(n\backslash\hat{R}_D\right) &= n_1 p\left(\hat{R}_1 \in C\right) + np\left(\hat{R}_1 \notin C\right) \\ &= n_1 p\left(\hat{R}_1 \in C\right) + n\left(1 - p\left(\hat{R}_1 \in C\right)\right. \\ &= n - n_2\left[\left\{I\left(\frac{r_2}{2\lambda}, n_1\right) - I\left(\frac{r_1}{2\lambda}, n_1\right)\right\}\right] \end{aligned} \tag{20}$$

Qusay (2019) used the same approach to calculate the expected value sample (Appendix).

5.1 RELATIVE EFFICIENCY

To determine the properties of estimators $\hat{R}_{D_1}$ and $\hat{R}_{D_2}$, we compare their relative efficiencies with respect to the pooled estimator $\hat{R}_D$. For this purpose, we obtain the mean square error of the pooled estimator $\hat{R}_D$ as follows:

$$\begin{aligned} \mathrm{MSE}\left(\hat{R}_D\right) &= E\left(\hat{R}_D - R\right)^2 \\ &= \int_0^\infty \int_0^\infty \left(\hat{R}_D - R\right)^2 f_1\left(\hat{\theta}_1\right) f_2\left(\hat{\theta}_2\right) d\hat{\theta}_1 d\hat{\theta}_2. \end{aligned}$$

By simplifying the integration cited above and using the same transformation of $y = \frac{n_1\theta}{\hat{\theta}_1}, z = \frac{n_2\theta}{\hat{\theta}_2}, x^\gamma = a$, we have

$$\begin{aligned} \mathrm{MSE}\left(\hat{R}_D\right) &= \frac{2n_1^2}{(n_1+n_2)^2\Gamma n_1}\left(\frac{1}{2n_1\theta\ln(1-a)}\right)^{\frac{1-n_1}{2}} K_{-n_1}\left(2\sqrt{2n_1\theta\ln(1-a)}\right. \\ &- \frac{4n_1(1+x^\gamma)^{-\widehat{\theta}}}{(n_1+n_2)\Gamma n_1}\left(\frac{1}{n_1\theta\ln(1-a)}\right)^{\frac{1-n_1}{2}} K_{-n_1}\left(2\sqrt{n_1\theta\ln(1-a)}\right. \\ &+ \frac{8n_1n_2}{(n_1+n_2)^2\Gamma n_1\Gamma n_2}\left(\left(\frac{1}{n_1\theta\ln(1-a)}\right)^{\frac{1-n_1}{2}} K_{-n_1}\left(2\sqrt{n_1\theta\ln(1-a)}\right)\right. \\ &\times\left(\left(\frac{1}{n_2\theta\ln(1-a)}\right)^{\frac{1-n_2}{2}} K_{-n_2}\left(2\sqrt{n_2\theta\ln(1-a)}\right)\right. \\ &- \frac{4n_2(1+x^\gamma)^{-\widehat{\theta}}}{(n_1+n_2)\Gamma n_2}\left(\frac{1}{n_2\theta\ln(1-a)}\right)^{\frac{1-n_2}{2}} K_{-n_2}\left(2\sqrt{n_2\theta\ln(1-a)}\right. \\ &+ \frac{2n_2^2}{(n_1+n_2)^2\Gamma n_2}\left(\frac{1}{2n_2\theta\ln(1-a)}\right)^{\frac{1-n_2}{2}} K_{-n_2}\left(2\sqrt{2n_2\theta\ln(1-a)}\right. \end{aligned} \tag{21}$$

Then, we define the relative efficiency of estimator $\tilde{R}_{S_1}$ and compare it with the classical estimator $\tilde{R}_s$ as follows:

$$RE\left(\tilde{R}_{D_1}\right) = \frac{\mathrm{MSE}\left(\tilde{R}_D\right)}{\mathrm{MSE}\left(\tilde{R}_{D_1}\right)}. \tag{22}$$

Similarly, we determine the relative efficiency of estimator $\tilde{R}_{D_2}$ with respect to $\tilde{R}_D$ as follows:

$$RE\left(\tilde{R}_{D_2}\right) = \frac{\mathrm{MSE}\left(\tilde{R}_D\right)}{\mathrm{MSE}\left(\tilde{R}_{D_2}\right)}. \tag{23}$$

The bias ratio and relative efficiency were computed for the shrunken estimators of the reliability functions considered for $n_1, n_2, \lambda, \alpha$, and a to take the following value assumptions:

$$\begin{aligned} &n_1 = 4, 8, 12, \quad n_2 = 4, 8, 12, \quad a = 0.5, 1, 1.5, \quad K = 0.1, 0.2, \\ &\alpha = 0.01, 0.05, \quad \lambda = 0.25, 0.50, 0.75, 1, 1.25, 1, 05, 0.75. \end{aligned}$$

The R language is used for statistical programming. Some of the results are presented in Figures (3.1)-(3.24) and Tables (3.1)- (3.12).

The following conclusions can be derived from the computations:

6. NUMERICAL RESULTS

The following conclusions can be derived from the computations:

1. The suggested DSSEs of the reliability functions $\tilde{R}_{D_1}$ and $\tilde{R}_{D_2}$ give a higher relative efficiency compared with the pooled classical estimator $\hat{R}_S$ in neighborhood $\theta = \theta_0$, $\lambda \approx 1$ and when the farther values decrease from $\lambda \approx 1$.

2. Figures (1)-(9) indicate that the absolute values for the bias ratios of DSSEs of the reliability functions $\tilde{R}_{D_1}$ and $\tilde{R}_{D_2}$ are increasing functions when ($\lambda > 0.75$).However, when ($0.5 < \lambda < 0.75$), they only somewhat decrease.

3. The absolute values for bias ratios of DSSEs of the reliability functions $\tilde{R}_{D_1}$ and $\tilde{R}_{D_2}$ are increasing functions of α when ($1 < \lambda < 1.75$), as shown in Figures (2) and (10).

4. The absolute values for bias ratios of DSSEs of the reliability functions $\tilde{R}_{D_1}$ and $\tilde{R}_{D_2}$ are increasing functions of α when ($1 < \lambda < 1.75$), as shown in Figures (2) and (10). 4. The absolute value for the bias ratio of DSSEs with respect to the reliability function n_1 is shown in Figures (11) and (12). $\tilde{R}_{D1}$ decreases more than $\tilde{R}_{D2}$ for n_1 when ($\lambda > 0.57$). However, when ($\lambda < 0.\ 75$), no conclusion can be given for this period.

5. As shown in Figure (3.4), the absolute value for the bias ratio of DSSEs with respect to the reliability function $\tilde{R}_{D1}$ of n_2 is similarly a decreasing function when ($0.75 < \lambda < 1.75$). Meanwhile, as shown in Figure (12), the absolute value for the bias ratio of DSSEs with respect to the reliability functions $\tilde{R}_{D2}$ of n_2 is a less-decreasing function. Therefore $\tilde{R}_{D1}$ is better $\tilde{R}_{D2}$ for n_2.

6. The absolute value for the bias ratio of DSSEs of the reliability functions of $\tilde{R}_{D1}$ under $\alpha = 0.01, 0.05$ is a better decreasing function of $\tilde{R}_{D2}$ when ($0.75 < \lambda < 1.75$). However, increasing functions are observed in ($0.5 < \lambda < 1.75$). No conclusions about k can be made when ($0.25 < \lambda < 0.5$), as shown in Figure (12).

7. The relative efficiency of the estimators of the reliability functions $\tilde{R}_{D1}$ and $\tilde{R}_{D2}$ with respect to the pooled estimator R_D when ($0.5 < \lambda < 1.25$) is an increasing function of a. Meanwhile, the relative efficiency of the abovementioned estimators when ($\lambda < 0.5$) and ($\lambda > 1.5$) is a decreasing function, as shown in Figures (15) and (20).

8. As shown in Figures (18) and (21), the relative efficiency of the estimators of reliability $\tilde{R}_{D1}$ with respect to the pooled estimator $\hat{R}_D$ when ($0.5 < \lambda < 1.75$) is an increasing function in all possibilities n_1 of n_2. However, for $\tilde{R}_{D2}$, when $\alpha = 0.01$ be decreasing function shifts to a higher trend from $\alpha = 0.05$.Therefore, the relative efficiency $\tilde{R}_{D2}$ is less than the relative efficiency $\tilde{R}_{D1}$. Meanwhile, when ($1.5 < \lambda < 1.75$) and ($0.25 < \lambda < 0.5$), a decreasing function is observed.

9. The relative efficiency of the estimator of reliability $\tilde{R}_{D2}$ with respect to classical pooled estimator $\hat{R}_D$ when ($0.75 < \lambda < 1.25$) is shown in Figures (17) and (19), and it is an increasing function. By contrast, the relative efficiency of the estimator of reliability $\tilde{R}_{D2}$ with respect to classical pooled estimator $\hat{R}_D$ when ($\lambda < 0.75, 1.25 > \lambda$) is a decreasing function.

10. Figure (24) shows a comparison between estimators $\tilde{R}_{D1}$ and $\tilde{R}_{D2}$. For the relative efficiency of the estimator of the reliability functions, when ($0.75 < \lambda < 1.25$), the $\tilde{R}_{D2}$ estimator is less than $\tilde{R}_{D1}$. Thus, $\tilde{R}_{D2}$ is better than $\tilde{R}_{D1}$ when ($1.25 < \lambda < 1.75$). However, the estimators $\tilde{R}_{D1}$ and $\tilde{R}_{D2}$ are equal when ($\lambda < 0.75$).

11. As shown in Figures (25), the expected sample size $E\left(n \mid \tilde{\theta}_D\right)$ defined in (23) is a convex function of λ with a point concave of $\lambda = 1$. Furthermore, as shown in Figure (26), the overall saving of sample size $\left(\frac{n_2}{n} P(H_0 \text{ accepted }) * 100\right)$ is a concave function of λ with a point concave of $\lambda = 1$. This finding implies a saving for the sampling units if the true values θ are in the neighborhood of θ_0.

7. CONCLUSIONS

a) In general, the DSSEs proposed in this work perform better than the pooled estimator R_D, both under $\tilde{R}_{D1}$ and $\tilde{R}_{D2}$.

b) The proposed DSSEs of the reliability function perform better than the classical estimator and the single-stage shrinkage estimator, which are both under constant and statistic conditions.

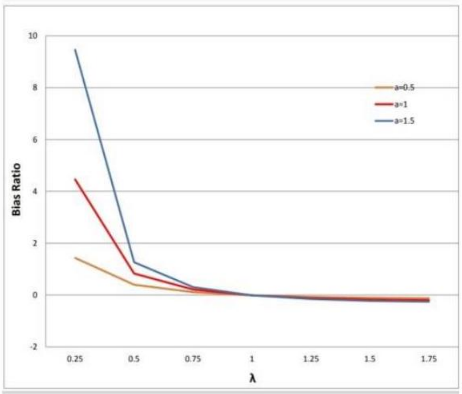

FIGURE 1. **Bias ratio $\widetilde{\mathbf{R}}_1$** a = 0.5, 1, 1.5

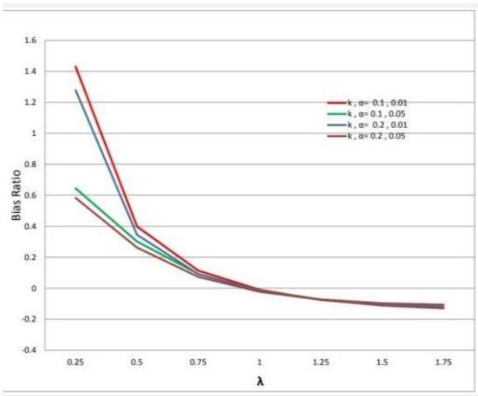

FIGURE 2. **Bias ratio k = 0.1, 0.2**

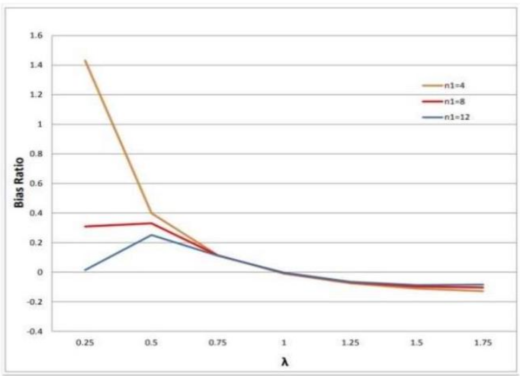

FIGURE 3. **Bias ratio n1 = 4, 8,** 12

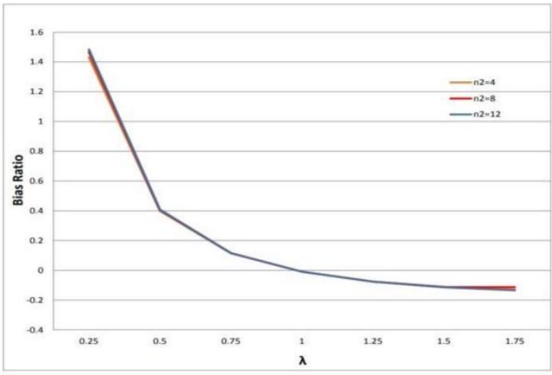

FIGURE 4. **Bias ratio** $n2 = 4, 8, 12$

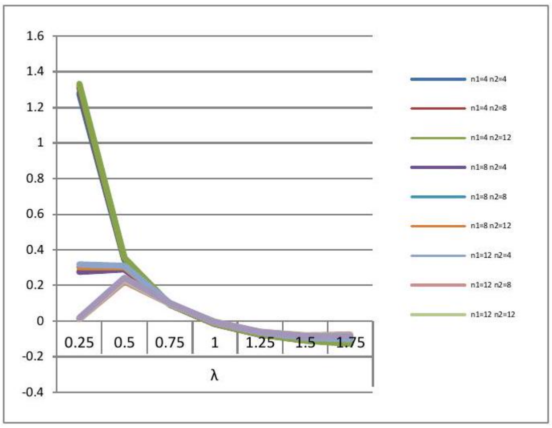

FIGURE 5. **Bias ratio** $\widetilde{\mathrm{R}}_{\mathrm{D1}}$ $\mathbf{k = 0.2}, \boldsymbol{\alpha} = \mathbf{0.01}, \mathrm{a} = \mathbf{0.5}$

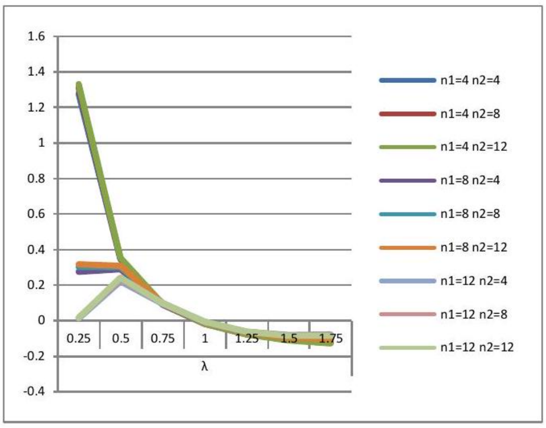

FIGURE 6. **Bias ratio** $\widetilde{\mathbf{R}}_{\mathrm{D1}}\boldsymbol{\alpha} = \mathbf{0.01}$

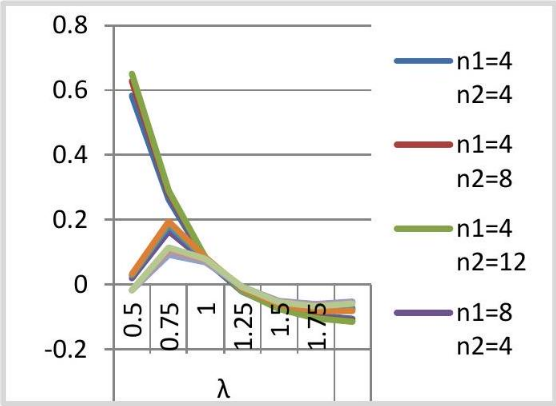

FIGURE 7. **Bias ratio $\widetilde{\mathbf{R}}_{\mathrm{D1}}$ k = 0.5α = 0.05, a = 0.5**

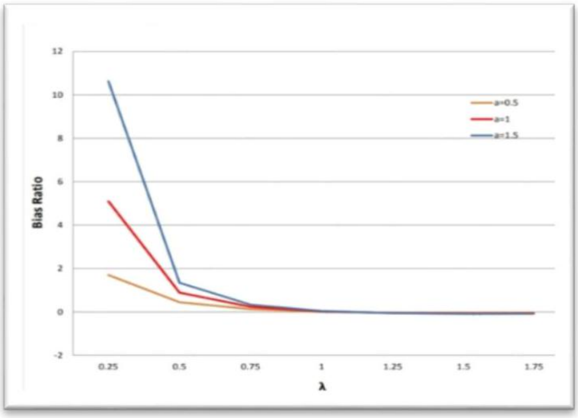

FIGURE 8. **Bias ratio** $\widetilde{R}_{D2} a = 0.5, 1, 1.5$

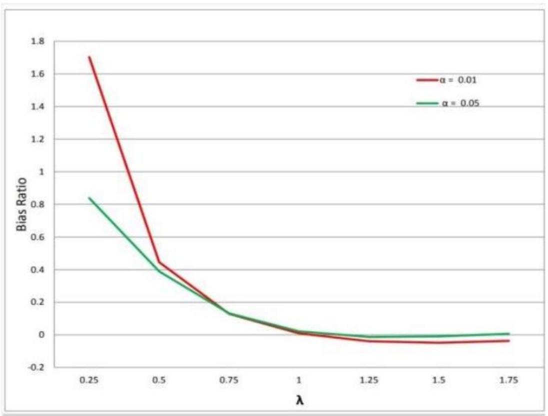

FIGURE 9. **Bias ratio $\widetilde{\mathbf{R}}_{\mathbf{D2}}\alpha = \mathbf{0.01, 0.05}$**

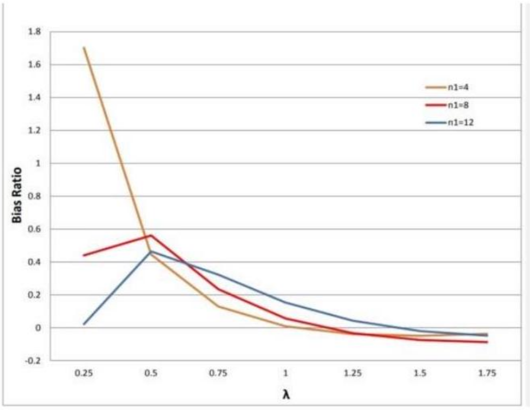

FIGURE 10. **Bias ratio $\widetilde{R}_{D2}$ n1 = 4, 8,** 12

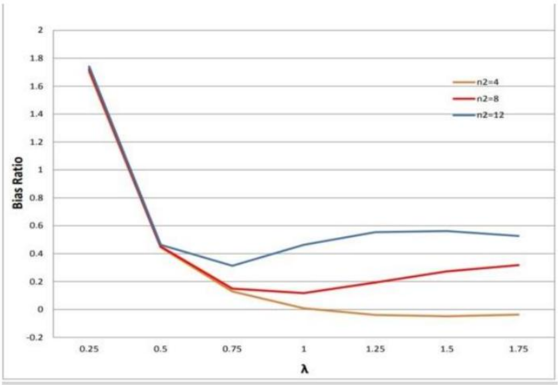

FIGURE 11. **Bias ratio** $\widetilde{R}_{D2}$n2 = **4, 6,** 12

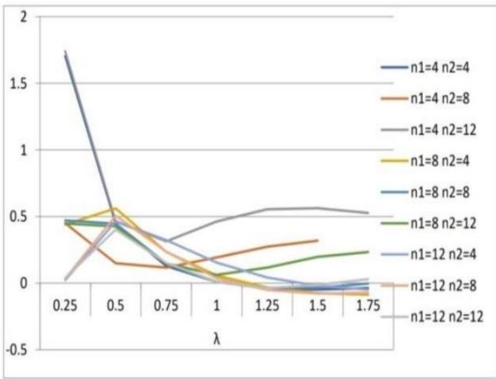

FIGURE 12. **Bias ratio $R_{D2}\alpha$ = 0.01**

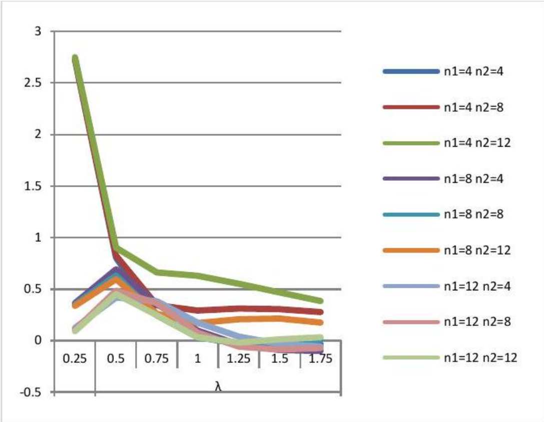

FIGURE 13. **Bias ratio $R_{D2}\alpha = 0.05$**

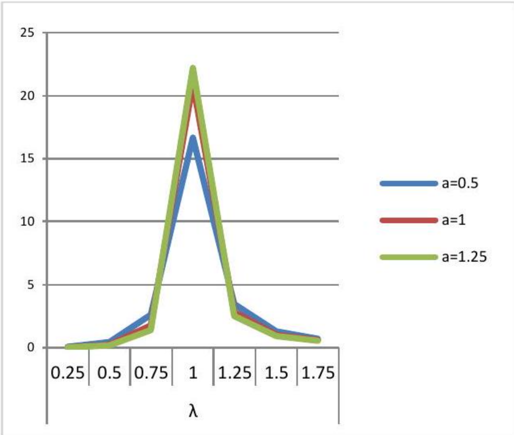

FIGURE 14. **Relative efficiency of the estimator $\widetilde{R}_{D1}$ for $k = 0 : 1$**

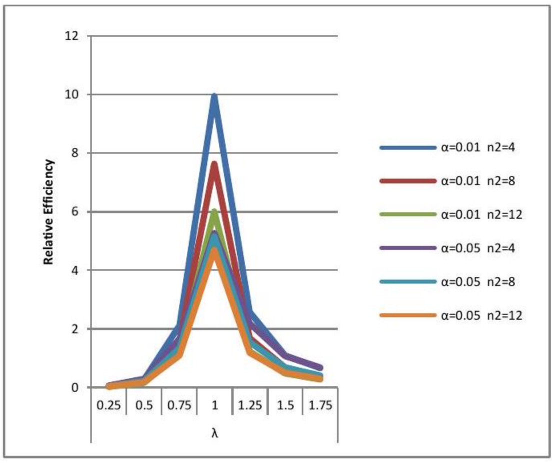

FIGURE 15. **Relative efficiency of the estimator $\widetilde{R}_{D1}$ for $k = 0 : 1; n1 = 4; a = 1$**

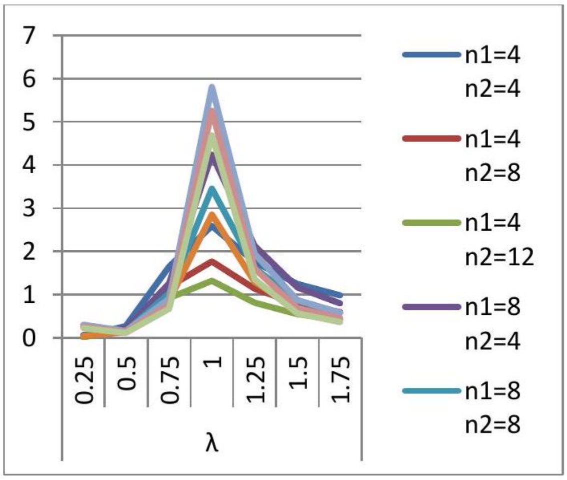

FIGURE 16. **Relative efficiency of the estimator $\widetilde{R}_{D1}$ for** $k = 0 : 2; n1 = 4$**;** $a = 1$

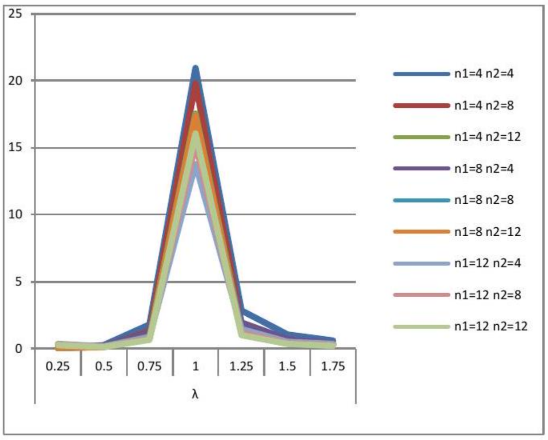

FIGURE 17. **Relative efficiency of the estimator $\widetilde{R}_{D1}$ for** $k_1 = 0 : 1; \alpha = 0.01$

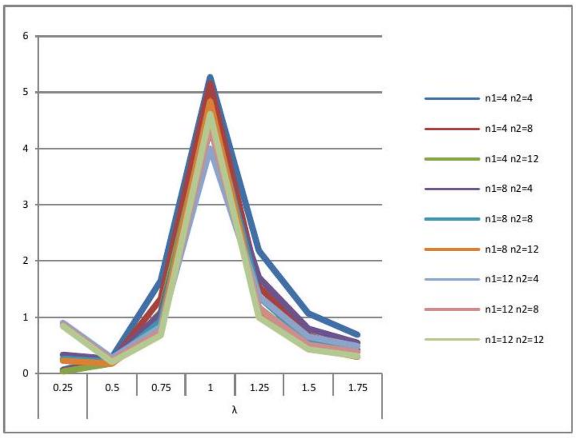

FIGURE 18. **Relative efficiency of the estimator $\widetilde{\mathbf{R}}_{D1}$ for** $k_1 = \mathbf{0} : \mathbf{2}; \alpha = \mathbf{0.05}$

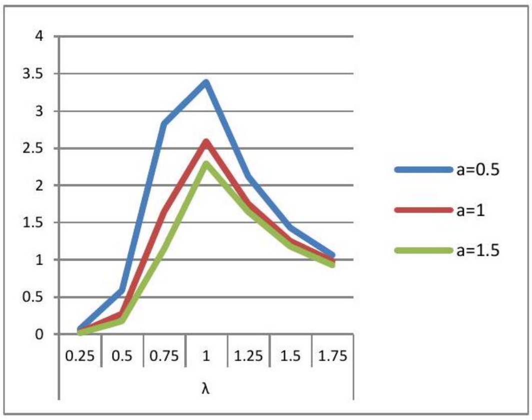

FIGURE 19. Relative efficiency of the estimator $\widetilde{R}_{D2}$ for $k = 0 : 1$

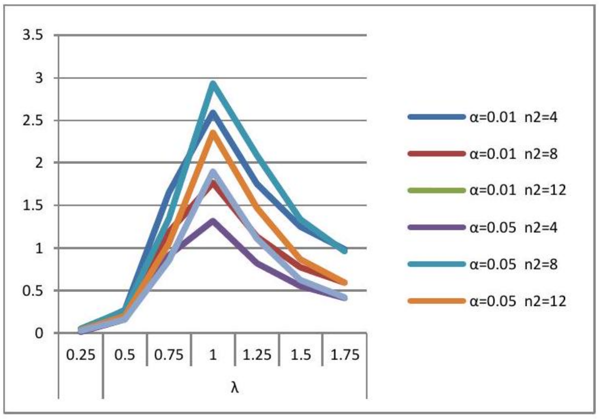

FIGURE 20. Relative efficiency of the estimator $\widetilde{R}_{D2}$ for k_2 for $n1 = 4$; $a = 1$

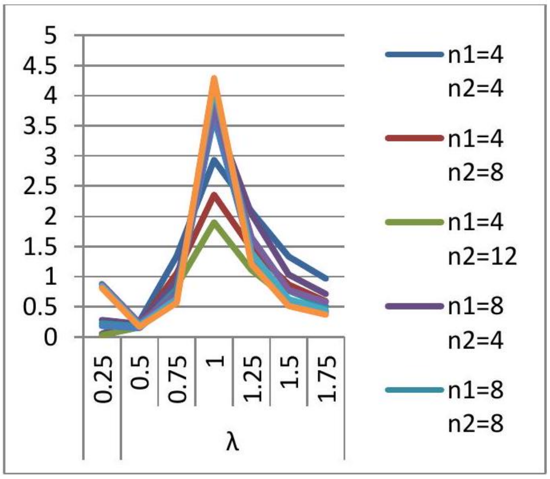

FIGURE 21. Relative efficiency of the estimator $\widetilde{R}_{D2}$ at $K_2\alpha = 0.01$

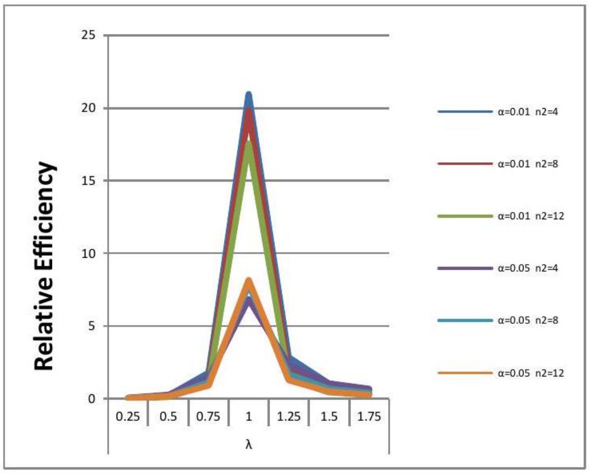

FIGURE 22. Relative efficiency of the estimator $\widetilde{R}_{D2}$ at $K_2\alpha = 0.05$

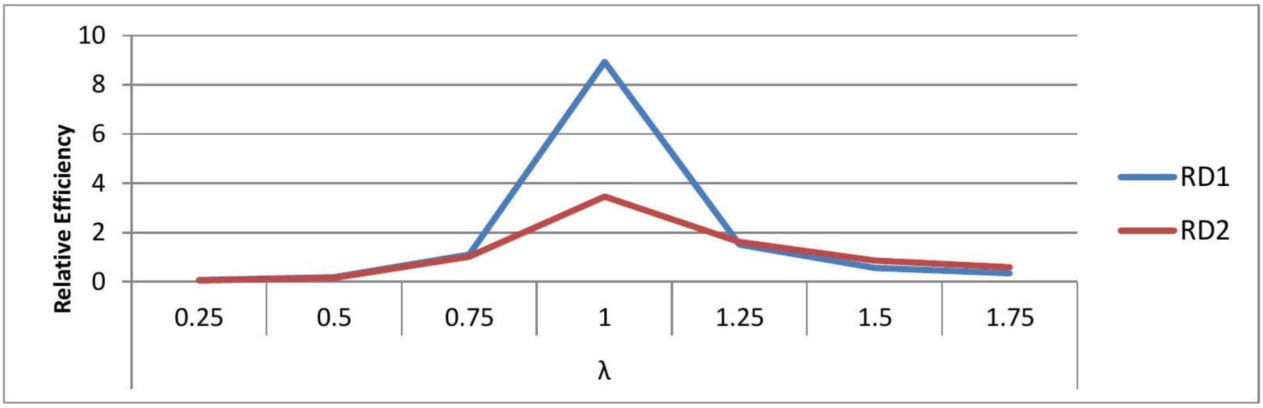

FIGURE 23. Comparison between estimates $\widetilde{\mathbf{R}}_{D1}$ and $\widetilde{\mathbf{R}}_{D2}$

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CONFLICTS OF INTEREST

The authors declare no conflict of interest.

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