

The Numerical Range of 6×6 Irreducible Matrices

Ahmed M. Sabir
College of Sciences
University of Salahaddin

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الملخص

في هذا البحث تمت دراسة مسألة خواص المدى العددي للمصفوفات اللااخترالية من الرتبة 6×6 التي لها مستقيمات مقطعية على تخومها.

ABSTRACT

In this paper, we consider the problem of characterizing the numerical range of 6 by 6 irreducible matrices which have line segments on their boundary.

1. Introduction

Let $M_n(C)$ be the algebra of $n \times n$ complex matrices. The numerical range of $A \in M_n$ is defined by $W(A) = \{x^*Ax : x \in C^n, x^*x=1\}$ [4], where x^* the adjoint of $x \in C$ is defined by $x^* = \bar{x}^T$ where $\bar{x}$ is the component-wise conjugate, and x^T is the transpose of x [4]. As pointed out by many authors, for 2×2 matrices A a complete description of the numerical range $W(A)$ is well-known. Namely, $W(A)$ is an ellipse with foci at the eigenvalues λ_1, λ_2 of A and a minor axis of the length $s = (\text{trace}(A^*A) - |\lambda_1|^2 - |\lambda_2|^2)^{\frac{1}{2}}$. In [4], of course, $s=0$ for normal A , and the ellipse in this case degenerates into a line segments connecting λ_1 with λ_2 . On the other hand, for 2×2 matrices A with coinciding eigenvalues the ellipse $W(A)$ degenerates into a disk. For 3×3 matrix A , this was first done by Kippenhahn. In [6], his characterization is based on the factorability of the associated polynomial $P_A(x,y,z) = \det(x\text{Re}A + y\text{Im}A + zI_3)$. This was improved in [5] by expressing the condition in terms of entries of A , also for 4×4 and 5×5 matrices A , this was improved in [2] by expressing the conditions in terms of entries of A . The aim of this paper is to give a sufficient and necessary condition for numerical range of 6 by 6 matrix with a line segment on its boundary.

2. Preliminaries

In the following, we give some definitions and results on $W(A)$ that are useful in this study.

Definition 2.1 [4] A matrix $A \in M_n(C)$ is said to be irreducible if either $n=1$ or $n \geq 2$ and there does not exist a permutation matrix $P \in M_n(C)$ such that $P^T A P = \begin{bmatrix} B & C \\ 0 & D \end{bmatrix}$ where B, D are nonempty square matrices.

Definition 2.2 [4] A matrix $B \in M_n(C)$ such that $x^* B x \geq 0$ for all $x \in C^n$, is said to be positive semidefinite .

Proposition 2.3 [4] The numerical range of $A \in M_n$ is always a compact convex set in C . It contains the spectrum $\sigma(A)$ of A and is equal to the convex hull of $\sigma(A)$ if A is normal.

Proposition 2.4[4] Let $A \in M_n(C)$

- (a) $W(A) = W(A^t)$
- (b) $W(A) = W(U^* A U)$ for any unitary U .
- (c) $W(\lambda A) = \lambda W(A)$ for any $\lambda \in C$.
- (d) $W(\lambda I + A) = \lambda + W(A)$ for any $\lambda \in C$.
- (e) $W(A^*) = \{z: z \in W(A)\}^c$
- (f) $W(\text{Re} A) = \text{Re} W(A)$ and $W(\text{Im} A) = \text{Im} W(A)$. Here $\text{Re} A = (A^* + A)/2$ and $\text{Im} A = (A^* - A)/2i$ are the real and imaginary part of A respectively.

Proposition 2.5 [4] Let $A \in M_n(C)$ and a, b be scalars

- (a) $W(A) = \{\lambda\}$ if and only if $A = \lambda I$.
- (b) $W(A)$ is contained in a straight line of the plane if and only if $A = aB + bI$ for some Hermitian matrix B . In particular in this case A is normal.
- (c) $W(A) \geq 0$ if and only if A is positive semidefinite.

Proposition 2.6 [4] Suppose that B is a principal submatrix of $A \in M_n(C)$. Then $W(B) \subseteq W(A)$.

We now relate the numerical range of an $n \times n$ matrix to an algebraic curve of class n . The next proposition indicates how the characteristic polynomial of some pencil associated with the matrix arises in this connection.

Proposition 2.7 [7] Let $A \in M_n(C)$. If $ax + by + c = 0$ is a supporting line of $W(A)$, then $\det(a \text{Re} A + b \text{Im} A + c I_n) = 0$

It follows from the above proposition that when studying the numerical range of A it is sensible to consider the algebraic curve $C(A)$ which is dual to the one given by $P_A(x \text{Re} A + y \text{Im} A + z I_n) = 0$.

Note that $P_A(-x, -y, z) = \det(zI_n - x\operatorname{Re}A - y\operatorname{Im}A)$ is nothing, but the characteristic polynomial of the pencil $x\operatorname{Re}A + y\operatorname{Im}A$. It is easily to see that P_A is a homogenous polynomial of degree n with real coefficients. Thus, in particular, the curve $C(A)$ is of class n and has n real eigenvalues of A .

Proposition 2.8 [7] If the eigenvalues of the $n \times n$ matrix A are $a_j + ib_j$, $j=1, \dots, n$ where the a_j and b_j are real, then the real foci of the algebraic curve $C(A)$ are exactly the points (a_j, b_j) , $j=1, \dots, n$.

Note that the proposition (2.7) together with the duality, implied that any supporting line of $W(A)$ is tangent to $C(A)$. The following proposition gives a more precise relation between $W(A)$ and $C(A)$.

Proposition 2.9 [6] If A is a $n \times n$ matrix, then its numerical range $W(A)$ is the convex hull of the real points of the curve $C(A)$.

The real part of the curve $C(A)$ in the complex plane namely the set $\{a + ib \in \mathbb{C} : a, b \in \mathbb{R} \text{ and } ax + by + z = 0 \text{ is tangent to } P_A(x, y, z) = 0\}$, will be denoted by $C_R(A)$ and is called the Kippenhahn curve of A .

3. Line segments of the Boundary of Numerical Range

In the following we will restrict ourselves to the irreducible matrix. The next theorem gives conditions for the numerical range to have a line segments on its boundary.

Theorem 3.1 Let A be an irreducible matrix. Then the following statements are equivalent:

- (a) $W(A)$ has a line segments on its boundary.
- (b) $0 < \operatorname{rank}(u\operatorname{Re}A + v\operatorname{Im}A + wI_6) \leq 4$ for some real numbers u, v and w . (u and v not both zero).
- (c) $u\operatorname{Re}A + v\operatorname{Im}A$ has a multiple eigenvalue for some real u and v which are not both zero.

Proof: $W(A)$ has a line segments on the boundary of $C_R(A)$ has a double or triple or quadripartite or tangent $ux + vy + w = 0$ (u and v not both zero). This corresponds to a root w of the equation $\det(u\operatorname{Re}A + v\operatorname{Im}A + zI_6) = 0$ with multiplicity 2 or 3 or 4 or 5, which is the same as saying that $u\operatorname{Re}A + v\operatorname{Im}A + zI_6$ has rank 4 or 3 or 2 or 1. This proves the equivalence of (a) and (b).

(b) $\Rightarrow$ (c) : Suppose (b) holds then $-w$ is an eigenvalue of $u\operatorname{Re}A + v\operatorname{Im}A$ with multiplicity 5, and we get double or triple or quadripartite or tangent. Thus (b) implies (c). To prove (c) $\Rightarrow$ (b), assume that $-w$ is a multiple eigenvalue of $u\operatorname{Re}A + v\operatorname{Im}A$. If $-w$ is of multiplicity 6, then the Hermitian $u\operatorname{Re}A + v\operatorname{Im}A$

would be a scalar matrix. In this case, $\text{Re}A$ and $\text{Im}A$ would commute and hence A would be normal, contradicting the irreducibility of A . Thus (c) implies (b).

Corollary 3.2 Let A be an irreducible matrix and unitary similar to real matrix. Then $W(A)$ has a line segment on its boundary if and only if $\text{Re}A$ has a multiple eigenvalue.

Proof: If A is a real matrix. Then $W(A)$ is symmetric about the real axis. Hence the line segment of the boundary of $W(A)$ must be a vertical line, by theorem(3.1) the matrix $\frac{1}{2}(\text{Re}A + i\text{Im}A) = \text{Re}A$ has a multiple eigenvalue.

We begin by deriving a canonical form for an irreducible form for an irreducible 6×6 matrix with a line segment on the boundary of its numerical range, if $W(A)$ has a line segment on its boundary. After rotation, shifting, and multiplication by a positive number, we may assume that a line segment stretches from 0 to i . Since $W(A)$ is convex, it must be contained entirely in the right or left half-plane. Applying yet another rotation and translation, if necessary we may assume that $W(A)$ is in the right half-plane. By theorem(3.1) we have $\text{rank}A=1$ or 2 or 3. Therefore we will discuss these two cases, respectively.

Theorem 3.3 Let A be an irreducible matrix. Then $W(A)$ has a line segment extending from 0 to i on its boundary and $\text{rank}A=1$ if and only if A may be

$$\text{written in the form } A = \begin{bmatrix} i & 0 & 0 & 0 & 0 & -c_1 \\ 0 & 0 & 0 & 0 & 0 & -c_2 \\ 0 & 0 & 0 & 0 & e_1 & -e_2 \\ 0 & 0 & 0 & e_1 & e_2 & -e_3 \\ 0 & 0 & e_1 & e_2 & e_3 & -e_4 \\ c_1 & c_2 & e_2 & e_3 & e_4 & -e_6 \end{bmatrix}, \text{ where}$$

c_1, c_2, e_2, e_3, e_4 , and real part of e_6 are positive and $e_1, e_2, e_3 \in [0, i]$.

Proof: Since $W(A)$ is contained in the closed right half-plane, $\text{Re}A$ is positive semi-definite. By assumption, $\ker \text{Re}A$ is 5-dimensional subspace, we may represent the line transformation of A restricted to $\ker \text{Re}A$, $A|_{\ker \text{Re}A} = A'$ by a 5×5 matrix. By choosing a proper basis for A , A' is the leading principal submatrix of A . since 0 and i are in $W(A)$, there exists unit vectors $x_1, x_2 \in \mathbb{C}^6$ such that $\langle Ax_1, x_1 \rangle = i$ and $\langle Ax_2, x_2 \rangle = 0$. It is clear that $x_1, x_2 \in \ker \text{Re}A$, since $\text{Re}A \geq 0$ and $\langle (\text{Re}A)x_j, x_j \rangle = 0$, $j=1, 2$. It follows that the line segment $[0, i]$ is contained in $W(A')$. Also $W(A') \subseteq W(A)$. since

$A|_{\ker \operatorname{Re} A} = A'$, we have $\operatorname{Re} A' = 0$ and $\operatorname{Re} W(A') = W(\operatorname{Re} A') = \{0\}$. We thus get $W(A') = [0, i]$. This implies that A' is normal with eigenvalues 0 and i so

$$\text{with proper basis } A' = \begin{bmatrix} i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e_1 \\ 0 & 0 & 0 & e_1 & e_2 \\ 0 & 0 & e_1 & e_2 & e_3 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} i & 0 & 0 & 0 & 0 & v_1 \\ 0 & 0 & 0 & 0 & 0 & v_2 \\ 0 & 0 & 0 & 0 & e_1 & e_2 \\ 0 & 0 & 0 & e_1 & e_2 & e_3 \\ 0 & 0 & e_1 & e_2 & e_3 & e_4 \\ c_1 & c_2 & e_2 & e_3 & e_4 & e_6 \end{bmatrix}$$

where $e_1, e_2, e_3 \in [0, i]$ are pure imaginary.

Since $\operatorname{Re} A$ is positive semi definite, a calculation shows that:

$$2\operatorname{Re} A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & v_1 + \bar{c}_1 \\ 0 & 0 & 0 & 0 & 0 & v_2 + \bar{c}_2 \\ 0 & 0 & 0 & 0 & e_1 + \bar{e}_2 & e_2 + \bar{e}_3 \\ 0 & 0 & 0 & e_1 + \bar{e}_2 & e_2 + \bar{e}_3 & e_3 + \bar{e}_4 \\ 0 & 0 & e_1 + \bar{e}_2 & e_2 + \bar{e}_3 & e_3 + \bar{e}_4 & e_4 + \bar{e}_5 \\ c_1 + \bar{v}_1 & c_2 + \bar{v}_2 & e_2 + \bar{e}_3 & e_3 + \bar{e}_4 & e_4 + \bar{e}_5 & 2\operatorname{Re} e_6 \end{bmatrix},$$

And therefore $\operatorname{Re} e_6 \geq 0, v_1 = -\bar{c}_1, v_2 = -\bar{c}_2, e_2 = -\bar{e}_3, e_3 = -\bar{e}_4$ and $e_4 = -\bar{e}_5$

Moreover, $\operatorname{Re} e_6$ must be positive, because $\operatorname{rank} \operatorname{Re} A = 1$. By a diagonal unitary similarity, we may assume that c_1, c_2, e_2, e_3, e_4 are non-negative. If one of them is 0, then A is reducible, so c_1, c_2, e_2, e_3, e_4 are positive. Now suppose that A is in the form expressed in the theorem. Consider the principal submatrix A' from the first five rows and columns of A , $W(A')$ is a line segment from 0 to i . clearly, $W(A') \subseteq W(A)$. But since $\operatorname{Re} A$ is positive semidefinite, $W(A)$ lies entirely in the right half-plane. So the line segment from 0 to i must be on the boundary of $W(A)$. To see that the line segment does not go beyond 0 or i , note that any point α on that line must be pure imaginary. So if

$$\alpha = \langle Ax, x \rangle = \langle (\operatorname{Re} A)x, x \rangle + i \langle (\operatorname{Im} A)x, x \rangle, \text{ then } \langle (\operatorname{Re} A)x, x \rangle = 0.$$

Hence

$$x \in \ker \operatorname{Re} A = \operatorname{span}\{[1, 0, 0, 0, 0]^T, [0, 1, 0, 0, 0]^T, [0, 0, 1, 0, 0]^T, [0, 0, 0, 1, 0]^T, [0, 0, 0, 0, 1]^T\} \text{ if } \|x\| = 1, \text{ then } x = [v_1, v_2, v_3, v_4, v_5]^T$$

with $|v_1|^2 + |v_2|^2 + |v_3|^2 + |v_4|^2 + |v_5|^2 = 1$, and

$$0 \leq \langle (\text{Im } A)x, x \rangle = |v_1|^2 + |e_1 v_3|^2 + |e_2 v_4|^2 + |e_5 v_5|^2 \leq$$

$$|v_1|^2 + |v_3|^2 + |v_4|^2 + |v_5|^2 \leq 1.$$

Theorem 3.4 Let A be an irreducible matrix written in the form

$$A = \begin{bmatrix} h_1 & h_{12} & h_{13} & h_{14} & h_{15} & h_{16} \\ \bar{h}_{12} & h_2 & h_{23} & h_{24} & h_{25} & h_{26} \\ \bar{h}_{13} & \bar{h}_{23} & h_3 & h_{34} & h_{35} & h_{36} \\ \bar{h}_{14} & \bar{h}_{24} & \bar{h}_{34} & h_4 & h_{45} & h_{46} \\ \bar{h}_{15} & \bar{h}_{25} & \bar{h}_{35} & \bar{h}_{45} & h_5 & h_{56} \\ \bar{h}_{16} & \bar{h}_{26} & \bar{h}_{36} & \bar{h}_{46} & \bar{h}_{56} & h_6 \end{bmatrix} + i \begin{bmatrix} k_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & k_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & k_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & k_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & k_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & k_6 \end{bmatrix}$$

where $k_1, k_2, k_3, k_4, k_5, k_6$ are distinct. Then $W(A)$ has a line segment on its boundary if and only if

$$h_6(k_4 - k_5) + h_5(k_6 - k_4) + h_4(k_5 - k_6) = (k_4 - k_5) \left(\frac{h_{45} \bar{h}_{46}}{h_{45}} \right) + (k_6 - k_4) \left(\frac{h_{35} \bar{h}_{45}}{h_{36}} \right) + (k_5 - k_6) \left(\frac{h_{36} \bar{h}_{56}}{h_{35}} \right) \quad (3.1)$$

$$h_6(k_3 - k_4) + h_4(k_6 - k_3) + h_3(k_4 - k_6) = (k_3 - k_4) \left(\frac{h_{36} \bar{h}_{34}}{h_{46}} \right) + (k_6 - k_3) \left(\frac{h_{24} \bar{h}_{36}}{h_{23}} \right) + (k_4 - k_6) \left(\frac{h_{46} \bar{h}_{46}}{h_{45}} \right) \quad (3.2)$$

$$h_6(k_2 - k_3) + h_3(k_6 - k_2) + h_2(k_3 - k_6) = (k_2 - k_3) \left(\frac{h_{26} \bar{h}_{23}}{h_{56}} \right) + (k_6 - k_2) \left(\frac{h_{24} \bar{h}_{26}}{h_{34}} \right) + (k_3 - k_6) \left(\frac{h_{23} \bar{h}_{26}}{h_{23}} \right) \quad (3.3)$$

$$h_1(k_5 - k_6) + h_5(k_6 - k_1) + h_6(k_1 - k_5) = (k_5 - k_6) \left(\frac{h_{35} \bar{h}_{46}}{h_{35}} \right) + (k_6 - k_1) \left(\frac{h_{16} \bar{h}_{15}}{h_{56}} \right) + (k_1 - k_5) \left(\frac{h_{56} \bar{h}_{56}}{h_{26}} \right) \quad (3.4)$$

$$h_{12} h_{26} \bar{h}_{16}, h_{13} h_{36} \bar{h}_{16}, h_{14} h_{46} \bar{h}_{16}, h_{15} h_{56} \bar{h}_{16}, h_{35} h_{56} \bar{h}_{36}, h_{34} h_{45} \bar{h}_{36}, h_{24} h_{35} \bar{h}_{26}, h_{23} h_{34} \bar{h}_{26},$$

$$h_{25} h_{36} \bar{h}_{26}, h_{45} h_{56} \bar{h}_{46} \text{ are real with } h_{12} h_{13} h_{14} h_{15} h_{16} h_{23} h_{24} h_{25} h_{26} h_{34} h_{35} h_{36} h_{45} h_{46} h_{56} \neq 0. \quad (3.5)$$

Proof: suppose $\text{rank } B = \text{rank}(u \text{Re} A + v \text{Im } A + w I_6) = 1$, since the eigenvalues of $\text{Im } A$ are all distinct, it is possible only when u is non zero. With-out loss of generality we may assume that $u=1$. To simplify further calculations, rewrite B in the form

$$B = \begin{bmatrix} h'_1 + vk'_1 + w' & h_{12} & h_{13} & h_{14} & h_{15} & h_{16} \\ \bar{h}_{12} & \bar{h}_2 + vk'_2 + w' & h_{23} & h_{24} & h_{25} & h_{26} \\ \bar{h}_{13} & \bar{h}_{23} & \bar{h}_3 + vk'_3 + w' & h_{34} & h_{35} & h_{36} \\ \bar{h}_{14} & \bar{h}_{24} & \bar{h}_{34} & \bar{h}_4 + vk'_4 + w' & h_{45} & h_{46} \\ \bar{h}_{15} & \bar{h}_{25} & \bar{h}_{35} & \bar{h}_{45} & \bar{h}_5 + vk'_5 + w' & h_{56} \\ \bar{h}_{16} & \bar{h}_{26} & \bar{h}_{36} & \bar{h}_{46} & \bar{h}_{56} & w' \end{bmatrix}$$

Where $w' = w + h_6 + vk_6$, $h'_i = h_i - h_6$, $k'_i = k_i - k_6$ ($i=1, 2, 3, 4, 5$), since by assumption $\text{rank}(B)=1$, then

$h_{12}h_{13}h_{14}h_{15}h_{16}h_{23}h_{24}h_{25}h_{26}h_{34}h_{35}h_{36}h_{45}h_{46}h_{56} \neq 0$ and

$$\frac{w'}{\bar{h}_{56}} = \frac{h_{56}}{h'_5 + vk'_5 + w'} = \frac{h_{16}}{h_{15}} = \frac{h_{26}}{h_{25}} = \frac{h_{36}}{h_{35}} = \frac{h_{46}}{h_{45}} \dots\dots\dots (3.6)$$

$$\frac{w'}{\bar{h}_{46}} = \frac{h_{56}}{\bar{h}_{45}} = \frac{h_{46}}{h'_4 + vk'_4 + w'} = \frac{h_{36}}{h_{34}} = \frac{h_{26}}{h_{24}} = \frac{h_{16}}{h_{14}} \dots\dots\dots (3.7)$$

$$\frac{w'}{\bar{h}_{36}} = \frac{h_{56}}{h'_{35}} = \frac{h_{46}}{h'_{34}} = \frac{h_{36}}{h'_3 + vk'_3 + w'} = \frac{h_{26}}{h_{23}} = \frac{h_{16}}{h_{13}} \dots\dots\dots (3.8)$$

$$\frac{w'}{\bar{h}_{26}} = \frac{h_{56}}{h'_{25}} = \frac{h_{46}}{h'_{24}} = \frac{h_{36}}{h'_{23}} = \frac{h_{26}}{h'_2 + vk'_2 + w'} = \frac{h_{16}}{h_{12}} \dots\dots\dots (3.9)$$

$$\frac{w'}{\bar{h}_{16}} = \frac{h_{56}}{h'_{16}} = \frac{h_{46}}{h'_{14}} = \frac{h_{36}}{h'_{13}} = \frac{h_{26}}{h'_{12}} = \frac{h_{16}}{h'_1 + vk'_1 + w'} \dots\dots\dots(3.10)$$

Solving (3.6), (3.7), (3.8), (3.9) and (3.10) with respect to v , w' we find that

$$\begin{aligned} w' &= \frac{\bar{h}_{56} h_{46}}{h_{45}} = \frac{\bar{h}_{56} h_{36}}{h_{35}} = \frac{\bar{h}_{56} h_{26}}{h_{25}} = \frac{h_{56} \bar{h}_{16}}{h_{15}} = \frac{h_{16} \bar{h}_{46}}{h_{14}} = \frac{h_{26} \bar{h}_{46}}{h_{24}} = \frac{\bar{h}_{46} h_{36}}{h'_{34}} = \frac{\bar{h}_{46} h_{56}}{\bar{h}_{45}} \\ &= \frac{\bar{h}_{36} h_{16}}{h_{13}} = \frac{\bar{h}_{36} h_{26}}{\bar{h}_{23}} = \frac{h_{36} \bar{h}_{46}}{\bar{h}_{34}} = \frac{\bar{h}_{36} h_{56}}{\bar{h}_{35}} = \frac{\bar{h}_{25} h_{16}}{\bar{h}_{12}} = \frac{h_{26} h_{36}}{h'_{23}} = \frac{\bar{h}_{26} h_{46}}{h'_{24}} = \frac{\bar{h}_{26} h_{56}}{h'_{25}} \\ &= \frac{h'_{16} h_{26}}{h'_{12}} = \frac{h'_{16} h_{36}}{h'_{13}} = \frac{\bar{h}_{16} h_{46}}{h'_{14}} = \frac{\bar{h}_{16} h_{56}}{h'_{15}} \dots\dots\dots(3.11) \end{aligned}$$

and

$$\begin{aligned}
 v &= \frac{1}{k'_1} \left(\frac{\bar{h}_{15} h_{16}}{h_{56}} - h'_1 - w' \right) = \frac{1}{k'_1} \left(\frac{h_{16} \bar{h}_{14}}{h_{46}} - h'_1 - w' \right) = \frac{1}{k'_1} \left(\frac{h_{16} \bar{h}_{13}}{h_{36}} - h'_1 - w' \right) = \frac{1}{k'_1} \left(\frac{h_{16} \bar{h}_{12}}{h_{26}} - h'_1 - w' \right) \\
 &= \frac{1}{k'_2} \left(\frac{\bar{h}_{25} h_{26}}{h_{56}} - h'_2 - w' \right) = \frac{1}{k'_2} \left(\frac{\bar{h}_{24} h_{26}}{h_{46}} - h'_2 - w' \right) = \frac{1}{k'_2} \left(\frac{\bar{h}_{23} h_{26}}{h_{36}} - h'_2 - w' \right) = \frac{1}{k'_2} \left(\frac{h_{16} h_{26}}{h_{16}} - h'_2 - w' \right) \\
 &= \frac{1}{k'_3} \left(\frac{\bar{h}_{35} h_{36}}{h_{56}} - h'_3 - w' \right) = \frac{1}{k'_3} \left(\frac{\bar{h}_{34} h_{36}}{h_{46}} - h'_3 - w' \right) = \frac{1}{k'_3} \left(\frac{h_{23} h_{26}}{h_{26}} - h'_3 - w' \right) = \frac{1}{k'_3} \left(\frac{h_{36} h_{13}}{h_{16}} - h'_3 - w' \right) \\
 &= \frac{1}{k'_4} \left(\frac{\bar{h}_{45} h_{46}}{h_{56}} - h'_4 - w' \right) = \frac{1}{k'_4} \left(\frac{h_{34} h_{46}}{h_{36}} - h'_4 - w' \right) = \frac{1}{k'_4} \left(\frac{h_{46} h_{24}}{h_{26}} - h'_4 - w' \right) = \frac{1}{k'_4} \left(\frac{h_{46} h_{14}}{h_{16}} - h'_4 - w' \right) \\
 &= \frac{1}{k'_5} \left(\frac{h_{45} h_{56}}{h_{46}} - h'_5 - w' \right) = \frac{1}{k'_5} \left(\frac{h_{56} h_{36}}{h_{36}} - h'_5 - w' \right) = \frac{1}{k'_5} \left(\frac{h_{56} h_{25}}{h_{26}} - h'_5 - w' \right) = \frac{1}{k'_5} \left(\frac{h_{56} h_{15}}{h_{16}} - h'_5 - w' \right) \dots (3.12)
 \end{aligned}$$

For convenience, let $w' = \frac{\bar{h}_{56} h_{46}}{h_{45}}$ in the remaining proof, since

$h_{12} h_{13} h_{14} h_{15} h_{16} h_{23} h_{24} h_{25} h_{26} h_{34} h_{35} h_{36} h_{45} h_{46} h_{56} \neq 0$ and w' is real, the equality (3.11) yields, (3.5) and (3.6) from (3.12) we have

$$k'_4 \left(\frac{h_{45} h_{56}}{h_{46}} - \frac{\bar{h}_{56} h_{46}}{h_{45}} - h'_5 \right) = k'_5 \left(\frac{\bar{h}_{45} h_{46}}{h_{56}} - \frac{\bar{h}_{56} h_{46}}{h_{45}} - h'_4 \right) \dots (3.13)$$

$$k'_3 \left(\frac{\bar{h}_{45} h_{46}}{h_{56}} - \frac{\bar{h}_{56} h_{46}}{h_{45}} - h'_4 \right) = k'_4 \left(\frac{\bar{h}_{35} h_{36}}{h_{56}} - \frac{\bar{h}_{56} h_{46}}{h_{45}} - h'_3 \right) \dots (3.14)$$

$$k'_2 \left(\frac{\bar{h}_{35} h_{36}}{h_{56}} - \frac{\bar{h}_{56} h_{46}}{h_{45}} - h'_3 \right) = k'_3 \left(\frac{\bar{h}_{25} h_{26}}{h_{56}} - \frac{\bar{h}_{56} h_{46}}{h_{45}} - h'_2 \right) \dots (3.15)$$

$$k'_1 \left(\frac{h_{56} h_{35}}{h_{36}} - \frac{\bar{h}_{56} h_{46}}{h_{45}} - h'_5 \right) = k'_4 \left(\frac{\bar{h}_{15} h_{16}}{h_{56}} - \frac{\bar{h}_{56} h_{46}}{h_{45}} - h'_1 \right) \dots (3.16)$$

It is easy to check that (3.1), (3.2), (3.3), (3.4) and (3.5) follow from (3.13), (3.14) and (3.15) and (3.16).

To prove the converse assume that (3.1), (3.2), (3.3), (3.4), (3.5) hold, $w' = \frac{\bar{h}_{56} h_{46}}{h_{45}}$ by (3.1), (3.2), (3.3), (3.4) and (3.5) we obtain

(3.13), (3.14), (3.15) and (3.16) where $h'_i = h_i - h_6$, $k'_i = k_i - k_6$ $i=(1,2,3,4,5)$, moreover we set

$$\begin{aligned}
 v &= \frac{1}{k'_1} \left(\frac{\bar{h}_{15} h_{16}}{h_{56}} - w' - h'_1 \right) = \frac{1}{k'_2} \left(\frac{\bar{h}_{25} h_{26}}{h_{56}} - w' - h'_2 \right) = \frac{1}{k'_3} \left(\frac{\bar{h}_{35} h_{36}}{h_{56}} - w' - h'_3 \right) \\
 &= \frac{1}{k'_4} \left(\frac{\bar{h}_{45} h_{46}}{h_{56}} - w' - h'_4 \right) = \frac{1}{k'_5} \left(\frac{h_{45} h_{56}}{h_{46}} - w' - h'_5 \right) \dots \dots \dots (3.17)
 \end{aligned}$$

By taking $w' = \frac{\bar{h}_{56} h_{46}}{h_{45}}$ and (3.17), we conclude that (3.6), (3.7), (3.8), (3.9) and (3.10) hold.

Finally, choose $u = 1$, $w = w' - h_6 - vk_6$, then $u \operatorname{Re} A + v \operatorname{Im} A + w I_6$ is in the form (M) and has rank 1, hence by theorem (3.1) $W(A)$ has a line segment on its boundary.

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