The Approximate solutions of nonlinear Sturm-Loiuville Eigenvalue Problem using Partition Method

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Abstract:

In this work, we consider the Sturm-Loiuville eigenvalue problem related to nonlinear differential equation. We solve this type of problems using one of the weighted residual methods namely the partition method. This method is illustrated using two numerical examples, we use Math CAD software package to solve the nonlinear systems, we tabulate the exact and approximate results and compare between them using Matlab program.

الخلاصة:

تناولنا في هذا البحث مسألة القيم الذاتية اللاخطية من نوع ستيرم-ليوفيل للمعادلة التفاضلية. وقمنا بحل هذه المسألة بأستخدام واحدة من طرق الأوزان المتبقية وهي طريقة التجزئة . وقد وضحنا هذه الطريقة باستخدام مثالين عدديين وقد استعنا ببرنامج Math CAD software package لحل النظام اللاخطي والنتائج التي حصلنا عليها قدمناها في جداول وقمنا برسم النتائج التقريبية والصريحة بأستخدام برنامج Matlab

1. Introduction:

Nonlinear Sturm-Liouville eigenvalue problems can be found in a wide variety of scientific and engineering applications, including the buckling of a column and joule losses in electrically

conducting solids and many important mathematical models can be expressed in terms of nonlinear Sturm-Liouville eigenvalue problems. So in recent years much more research has been focused on finding the numerical solution of this type of problems.

Recall that, the linear Sturm-Liouville eigenvalue problem has the form:

$$[p(x)y'(x)]' + z(x)y(x) + \lambda r(x)y(x) = 0 \qquad a \le x \le b$$

together with the boundary conditions

$$\alpha_1 y(a) - \beta_1 y'(a) = 0,\ \alpha_2 y(b) + \beta_2 y'(b) = 0$$

where p, p', z, and r are continuous functions on the closed interval [a,b] and $p(x) > 0$, $r(x) > 0$ over [a,b]. The values of λ for which the above equation has a nontrivial solution are called eigenvalues. This solution is called an eigenfunction, [Keller H., 1968].

Keller H., in 1968 solve the Sturm-Liouville boundary value problem using collocation method, Galerkin method.

Granas A., in 1978 discuss the existence of Sturm-Liouville boundary value problem related to differential equations.

Agarwal R., in 1998 gives the necessary conditions for the existence of the solutions of nonlinear Sturm-Liouville eigenvalue problem:

$$\left[p(x)y'(x)\right]' + \lambda f\left(x, y(x)\right) = 0, \qquad 0 \le x \le 1 \qquad (1)$$

together with the boundary conditions

$$\alpha_1 y(0) - \beta_1 y'(0) = 0, \qquad \alpha_2 y(1) + \beta_2 y'(1) = 0 \qquad (2)$$

In this paper, we apply the partition method to approximate the solution of the above nonlinear Sturm-Liouville eigenvalue problem (1), (2).

2. Partition method**:**

This method based on approximating the unknown function y(x) as:

$$y(x) \cong \sum_{i=0}^{n} a_i \phi_i(x) \tag{3}$$

where $a_i,\ i = 0,1,...,n$ are unknown constants that must be determined, and $\varphi_i(x),\ i = 0,1,...,n$ form a linearly independent set of functions. Also this approximated solution must satisfy the boundary conditions, and by

substituting the resulting approximated solution into eq.(1) one can get:

$$\sum_{i=0}^{n} a_i \phi_i''(x) + \lambda f\left(x, \sum_{i=0}^{n} a_i \phi_i(x) \right) = \varepsilon(x, \lambda, a_i) \tag{4}$$

where $\varepsilon(x, \lambda, a_i)$ is said to be the error term (or residue) in the approximation of the solution. The exact solution is obtained when the residue is identically zero. Since it is difficult to make $\varepsilon(x, \lambda, a_i)$ identically zero, we shall try to make it as small as possible in some sense, to do this in partition method we divide the domain $\Re$ $(0 \le x \le 1)$ into N non-over lapping sub domains $\Re_j$ where j=1,2,...,N to find the coefficients (a_i and λ), so the required equation for partition method becomes

$$\int_{\Re_j} \varepsilon(x, \lambda, a_i)\, dx = 0$$

(5)

for j=1,2,...,N, one can get a system of n-1 nonlinear equations which can be solved using Math CAD software package to obtain

the n coefficients (a_i and λ). To illustrate the use of this method we give the following examples:

3. Numerical examples**:**

Example (1):

Consider the nonlinear Sturm-Liouville eigenvalue problem

$$y'' + \lambda\left(-y^3 + (x^3 - x^4)^3 + (2x^2 - x)\right) = 0 \qquad 0 \le x \le 1$$

together with the boundary conditions

$$y(0) = 0,\ y(1) = 0$$

First, approximate the solution of this problem using eq.(3) with $n = 5$ and $\phi_i(x) = x^i$ to get:

$$y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5$$

using the boundary conditions

$$y(x) = -(a_2 + a_3 + a_4 + a_5)x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5$$

substituting this resulting approximated solution into the above differential equation to get:

$$\varepsilon(x, a_2, a_3, a_4, a_5, \lambda) = 2a_2 + 6a_3 x + 12a_4 x^2 + 20a_5 x^3$$
$$+\lambda\left[-(-(a_2 + a_3 + a_4 + a_5)x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5)^3 + (x^3 - x^4)^3 + (2x^2 - x)\right]$$

next, to find a_2, a_3, a_4, a_5 and λ we divide the domain $0 \le x \le 1$ into 5 sub intervals say [0,0.2], [0.2,0.4], [0.4,0.6], [0.6,0.8], [0.8,1] and observe that the integration of the error term over these sub intervals vanishes to get the following nonlinear system of equations

$(1.188e\text{-}3)\lambda.a_4^2.a_5 - (0.0147)\lambda + (1.006e\text{-}3)\lambda.a_2.a_5^2 + (1.166e\text{-}4)\lambda$
$a_3.a_5^2 + (0.4)a_2 +$
$(0.008)a_5 + (0.032)a_4 + (0.12)a_3 + (1.1926e\text{-}3)\lambda.a_4.a_5^2 + (8.2601e\text{-}4)\lambda$
$.a_3.a_2^2 +$

$(9.5576e\text{-}4)\lambda.a_2.a_3^2+(1.1318e\text{-}3)\lambda.a_3^2.a_4+(1.13606e\text{-}3)\lambda$
$.a_3^2.a_5+(1.1574e\text{-}3)$
$\lambda.a_3.a_4^2+(8.4426e\text{-}4)\lambda.a_2^2.a_4+(8.4735e\text{-}4)\lambda.a_2^2.a_5+(9.9897e\text{-}4)\lambda$
$.a_2.a_4^2$
$+(2.3235e\text{-}3)\lambda.a_3.a_4.a_5+(2.0054e\text{-}3)\lambda.a_2.a_4.a_5+(1.9542e\text{-}3)\lambda$
$.a_3.a_2.a_4+$
$(1.9614e\text{-}3)\lambda.a_3.a_2.a_5+(3.9904e\text{-}4)\lambda.a_5^3+(3.6895e\text{-}4)\lambda$
$.a_3^3+(3.9454e\text{-}3)\lambda a_4^3+$
$(2.3817e\text{-}4)\lambda$
$.a_2^3=0$……(6a)

$(0.01642)\lambda.a_4^2.a_5-(0.02266)\lambda+(0.0117)\lambda.a_2.a_5^2+(0.0156)\lambda$
$a_3.a_5^2+(0.4)a_2+$
$(0.12)a_5+(0.224)a_4+(0.36)a_3+(0.0168)\lambda.a_4.a_5^2+(7.2446e\text{-}3)\lambda$
$.a_3.a_2^2+$
$(9.56696e\text{-}3)\lambda.a_2.a_3^2+(0.0137)\lambda.a_3^2.a_4+(0.01403)\lambda$
$.a_3^2.a_5+(0.01479)\lambda.a_3.a_4^2+$
$(7.8177e\text{-}3)\lambda.a_2^2.a_4+(8.01019e\text{-}3)\lambda.a_2^2.a_5+(0.0112)\lambda$
$.a_2.a_4^2+(0.0303)\lambda.a_3.a_4.a_5+$
$(0.0229)\lambda.a_2.a_4.a_5+(0.0207)\lambda.a_3.a_2.a_4+(0.0212)\lambda.a_3.a_2.a_5+$
$(0.05759)\lambda.a_5^3+(4.2183e\text{-}3)\lambda.a_3^3+(5.3342e\text{-}3)\lambda a_4^3+(1.83177e\text{-}3)\lambda$
$.a_2^3=0$……(6b)

$(0.0529)\lambda.a_4^2.a_5-(1.38955)\lambda+(0.0327)\lambda.a_2.a_5^2+(0.0486)\lambda$
$a_3.a_5^2+(0.4)a_2+$
$(0.52)a_5+(0.608)a_4+(0.6)a_3+(0.057002)\lambda.a_4.a_5^2+(0.01351)\lambda.a_3.a_2^2+$
$(0.0203)\lambda.a_2.a_3^2+(0.03576)\lambda.a_3^2.a_4+(0.03847)\lambda.a_3^2.a_5+(0.04192)\lambda$
$.a_3.a_4^2+$

$(0.01579)\lambda.a_2^2.a_4+(0.01696)\lambda.a_2^2.a_5+(0.02781)\lambda$
$.a_2.a_4^2+(9.02093e\text{-}3)\lambda.a_3.a_4.a_5+$
$(0.05981)\lambda.a_2.a_4.a_5+(0.0475)\lambda.a_3.a_2.a_4+(0.05105)\lambda.a_3.a_2.a_5+$
$(0.02047)\lambda.a_5^3+(0.01018)\lambda.a_3^3+(0.01639)\lambda a_4^3+(3.00297e\text{-}3)\lambda$
$.a_2^3=0$…….(6c)

$(0.06364)\lambda.a_4^2.a_5-$ $(0.05754)\lambda+(0.03383)\lambda.a_2.a_5^2+(0.05712)\lambda$
$a_3.a_5^2+(0.4)a_2+$
$(1.4)a_5+(1.184)a_4+(0.84)a_3+(0.07346)\lambda.a_4.a_5^2+(9.24137e\text{-}3)\lambda$
$.a_3.a_2^2+$
$(0.01556)\lambda.a_2.a_3^2+(0.03357)\lambda.a_3^2.a_4+(0.03867)\lambda.a_3^2.a_5+(0.04302)$
$\lambda.a_3.a_4^2+$
$(0.01181)\lambda.a_2^2.a_4+(0.01359)\lambda.a_2^2.a_5+(0.02548)\lambda.a_2.a_4^2+(0.09918)$
$\lambda.a_3.a_4.a_5+$
$(0.058699)\lambda.a_2.a_4.a_5+(0.0398)\lambda.a_3.a_2.a_4+(0.04582)\lambda.a_3.a_2.a_5+$
$(0.02828)\lambda.a_5^3+(8.73924e\text{-}3)\lambda.a_3^3+(0.01839)\lambda a_4^3+(1.83177e\text{-}3)\lambda$
$.a_2^3=0$……(6d)

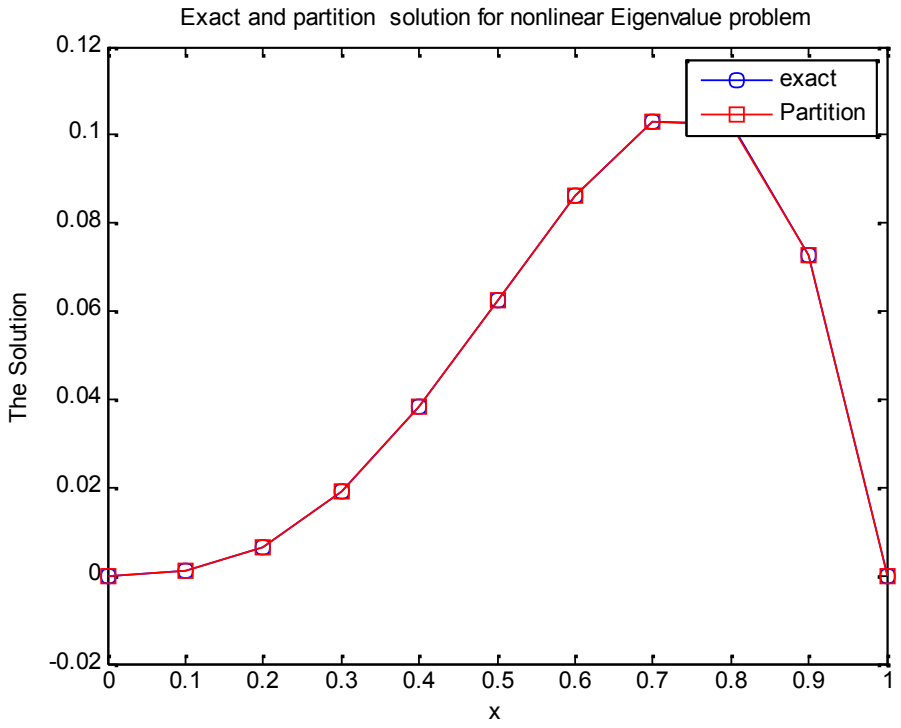
$(0.01485)\lambda.a_4^2.a_5-(0.14542)\lambda+(7.16287e\text{-}3)\lambda.a_2.a_5^2+(0.013235)\lambda$
$a_3.a_5^2+(0.4)a_2+$
$(2.952)a_5+(1.952)a_4+(1.08)a_3+(0.01839)\lambda.a_4.a_5^2+(1.31753e\text{-}3)\lambda$
$.a_3.a_2^2+$
$(2.43032e\text{-}3)\lambda.a_2.a_3^2+(6.22396e\text{-}3)\lambda.a_3^2.a_4+(7.69936e\text{-}3)\lambda.a_3^2.a_5+$
$(8.64082e\text{-}3)\lambda.a_3.a_4^2+(1.8273e\text{-}3)\lambda.a_2^2.a_4+(2.25899e\text{-}3)\lambda.a_2^2.a_5+$
$(4.6795e\text{-}3)\lambda.a_2.a_4^2+(0.02139)\lambda.a_3.a_4.a_5+(0.01158)\lambda$
$.a_2.a_4.a_5+(6.74357e\text{-}3)$
$\lambda.a_3.a_2.a_4+(8.3394e\text{-}3)\lambda.a_3.a_2.a_5+(7.5939e\text{-}3)\lambda.a_5^3+(1.4949e\text{-}3)\lambda$
$.a_3^3+(4.00007e\text{-}3)$

λa_4^3+(2.3817e-4)λ

.a_2^3=0…..……..…(6e)

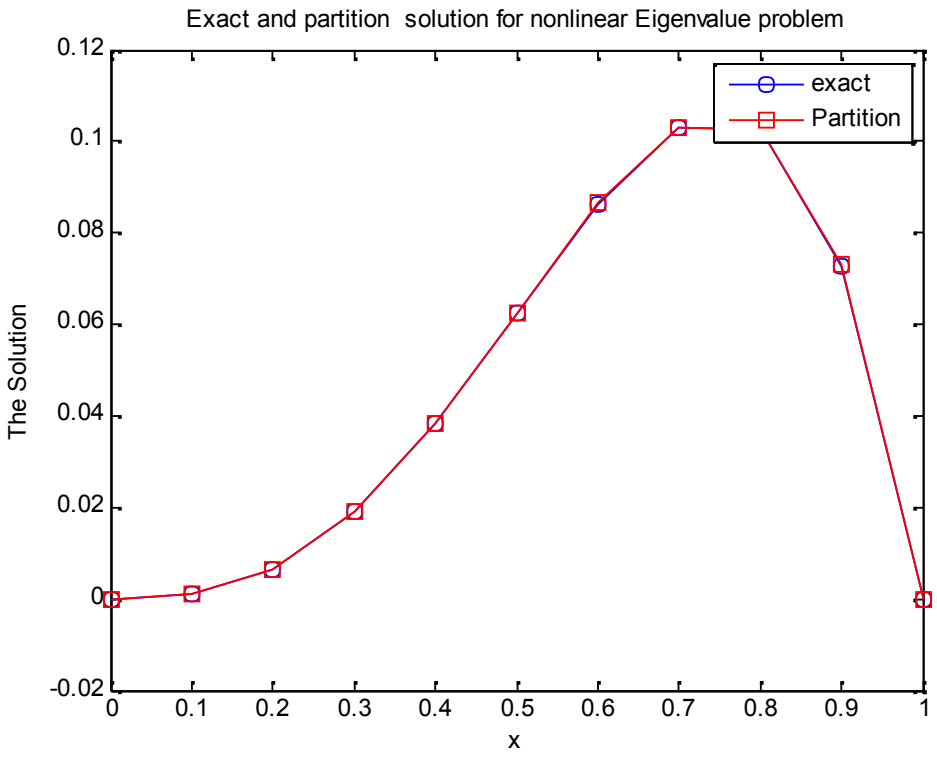
and by using Math CAD software package to solve this system with different initial values of ($a_2, a_3, a_4, a_5, \lambda$) and direct Ritz method we obtain the following tabulated results:

x	The initial values of a_2, a_3, a_4, a_5 and λ	The approximated values of a_2, a_3, a_4, a_5 and λ	The exact solution	The approximate solution	Absolute error
0	8, 2,-3, 0, 7	-1.811×10^{-6}, 1, -1, 7.758×10^{-6}, 5.997	0	0	0
0.1	=	=	0.0009	0.0009	0.0613×10^{-5}
0.2	=	=	0.0064	0.0064	0.1259×10^{-5}
0.3	=	=	0.0189	0.0189	0.1928×10^{-5}
0.4	=	=	0.0384	0.0384	0.2589×10^{-5}
0.5	=	=	0.0625	0.0625	0.3184×10^{-5}
0.6	=	=	0.0864	0.0864	0.3617×10^{-5}
0.7	=	=	0.1029	0.1029	0.3746×10^{-5}
0.8	=	=	0.1024	0.1024	0.3374×10^{-5}
0.9	=	=	0.0729	0.0729	0.2238×10^{-5}
1	=	=	0	0.0000	0.0000



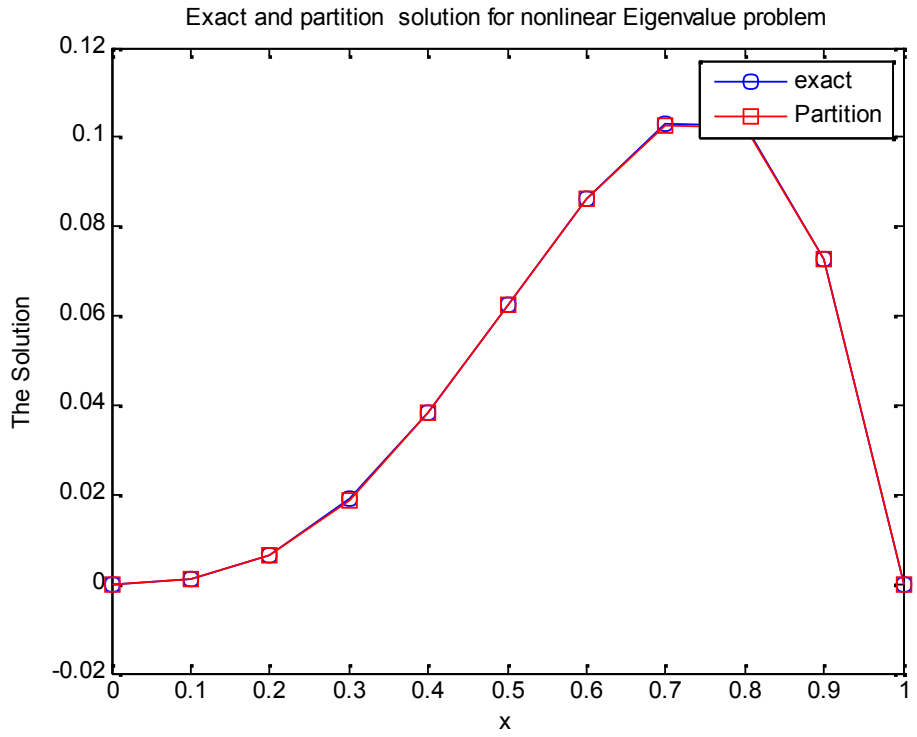
x	The initial values of a_2, a_3, a_4, a_5 and λ	The approximated values of a_2, a_3, a_4, a_5 and λ	The exact solution	The approximate solution	Absolute error
0	-2, 2,-3, -1, 8	9.926×10^{-7}, 1.001, -1.001, -3.198×10^{-6}, 6.008	0	0	0
0.1	=	=	0.0009	0.0009	0.0011×10^{-3}
0.2	=	=	0.0064	0.0064	0.0069×10^{-3}
0.3	=	=	0.0189	0.0189	0.0196×10^{-3}
0.4	=	=	0.0384	0.0384	0.0394×10^{-3}
0.5	=	=	0.0625	0.0626	0.0638×10^{-3}
0.6	=	=	0.0864	0.0865	0.0878×10^{-3}
0.7	=	=	0.1029	0.1030	0.1044×10^{-3}
0.8	=	=	0.1024	0.1025	0.1038×10^{-3}
0.9	=	=	0.0729	0.0730	0.0738×10^{-3}

1	=	=	0	0.0000	0.0000



x	The initial values of a_2, a_3, a_4, a_5 and λ	The approximated values of a_2, a_3, a_4, a_5 and λ	The exact solution	The approximate solution	Absolute error
0	2, 7,-4 6, 5	-1.204×10^{-5}, 0.999, -0.999, 3.247×10^{-5}, 5.991	0	0	0
0.1	=	=	0.0009	0.0009	0.0031×10^{-3}
0.2	=	=	0.0064	0.0064	0.0110×10^{-3}
0.3	=	=	0.0189	0.0189	0.0260×10^{-3}
0.4	=	=	0.0384	0.0384	0.0482×10^{-3}
0.5	=	=	0.0625	0.0624	0.0747×10^{-3}
0.6	=	=	0.0864	0.0863	0.1005×10^{-3}
0.7	=	=	0.1029	0.1028	0.1176×10^{-3}
0.8	=	=	0.1024	0.1023	0.1158×10^{-3}

0.9	=	=	0.0729	0.0728	0.0819×10^{-3}
1	=	=	0	0.0000	0.0000



Example (2):

Consider the nonlinear Sturm-Liouville eigenvalue problem

$$y'' + \lambda\left(y^2 - (x^2 - x^3)^2 + (3x-1)\right) = 0 \qquad 0 \le x \le 1$$

$$y(0) = 0,\ y(1) = 0$$

similarly, as in the previous example, using eq.(3) with n=5 and the boundary conditions given in this example to get the following error term equation

$$\varepsilon(x, a_2, a_3, a_4, a_5, \lambda) = 2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3$$
$$+\lambda\left[(-(a_2 + a_3 + a_4 + a_5)x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5)^2 - (x^2 - x^3)^2 + (3x-1)\right]$$

similarly, as in the previous example, to find a_2, a_3, a_4, a_5 and λ we divide the domain $0 \le x \le 1$ into 5 sub intervals say [0,0.2], [0.2,0.4], [0.4,0.6], [0.6,0.8], [0.8,1] and observe that the

integration of the error term over these sub intervals vanishes to get the following nonlinear system of equations

$$(0.4)\ a_2+(0.12)\ a_3+(0.032)\ a_4+(0.008)\ a_5+(1.93067e\text{-}3)\lambda .a_2^2+(4.4267e\text{-}3)\lambda .a_2.a_3+(4.53032e\text{-}3)\lambda .a_2.a_5+(5.18464e\text{-}3)\lambda .a_3.a_4+(2.6454e\text{-}3)\lambda .a_4^2+(5.20179e\text{-}3)\lambda .a_3.a_5+(2.66301e\text{-}3)\lambda .a_5^2+(5.3084e\text{-}3)\lambda .a_4.a_5+(2.5405e\text{-}3)\lambda .a_3^2+(4.5157e\text{-}3)\ \lambda .a_2.a_4-(0.14004)\lambda=0$$

……(7a)

$$(0.4)\ a_2+(0.36)\ a_3+(0.224)\ a_4+(0.12)\ a_5+(8.65067e\text{-}3)\lambda .a_2^2+(0.02271)\lambda .a_2.a_3+(0.02503)\lambda .a_2.a_5+(0.03218)\lambda .a_3.a_4+(0.017352)\lambda .a_4^2+(0.03296)\lambda .a_3.a_5+(0.018206)\lambda .a_5^2+(0.03555)\lambda .a_4.a_5+(0.01493)\lambda .a_3^2+(0.02445)\ \lambda .a_2.a_4-(0.02087)\lambda=0$$

……(7b)

$$(0.4)\ a_2+(0.6)\ a_3+(0.608)\ a_4+(0.52)\ a_5+(0.01217)\lambda .a_2^2+(0.036512)\ \lambda .a_2.a_3+(0.04584)\lambda .a_2.a_5+(0.06417)\lambda .a_3.a_4+(0.03757)\lambda .a_4^2+(0.06898)\lambda .a_3.a_5+(0.04346)\lambda .a_5^2+(0.080805)\lambda .a_4.a_5+(0.02743)\lambda .a_3^2+(0.042677)\ \lambda .a_2.a_4-(0.09692)\lambda=0$$

……(7c)

$(0.4)\ a_2+(0.84)\ a_3+(1.184)\ a_4+(1.4)\ a_5+(8.65067e\text{-}3)\lambda .a_2^2+(0.029195)\lambda .a_2.a_3+(0.043157)\lambda .a_2.a_5+(0.06328)\lambda .a_3.a_4+(0.04063)\lambda .a_4^2+(0.073032)\lambda .a_3.a_5+(0.054225)\lambda .a_5^2+(0.093849)\lambda .a_4.a_5+(0.024659)\lambda .a_3^2+(0.03742)\ \lambda .a_2.a_4-(0.21589)\lambda =0$

……(7d)

$(0.4)\ a_2+(1.08)\ a_3+(1.952)\ a_4+(2.952)\ a_5+(1.93067e\text{-}3)\lambda .a_2^2+(7.1573e\text{-}3)\lambda .a_2.a_3+(0.012395)\lambda .a_2.a_5+(0.01851)\lambda .a_3.a_4+(0.012913)\lambda .a_4^2+(0.023006)\ \lambda .a_3.a_5+(0.01997)\lambda .a_5^2+(0.03211)\lambda .a_4.a_5+(6.636495e\text{-}3)\lambda .a_3^2+(9.97699e\text{-}3)\ \lambda .a_2.a_4-(0.33859)\lambda =0$

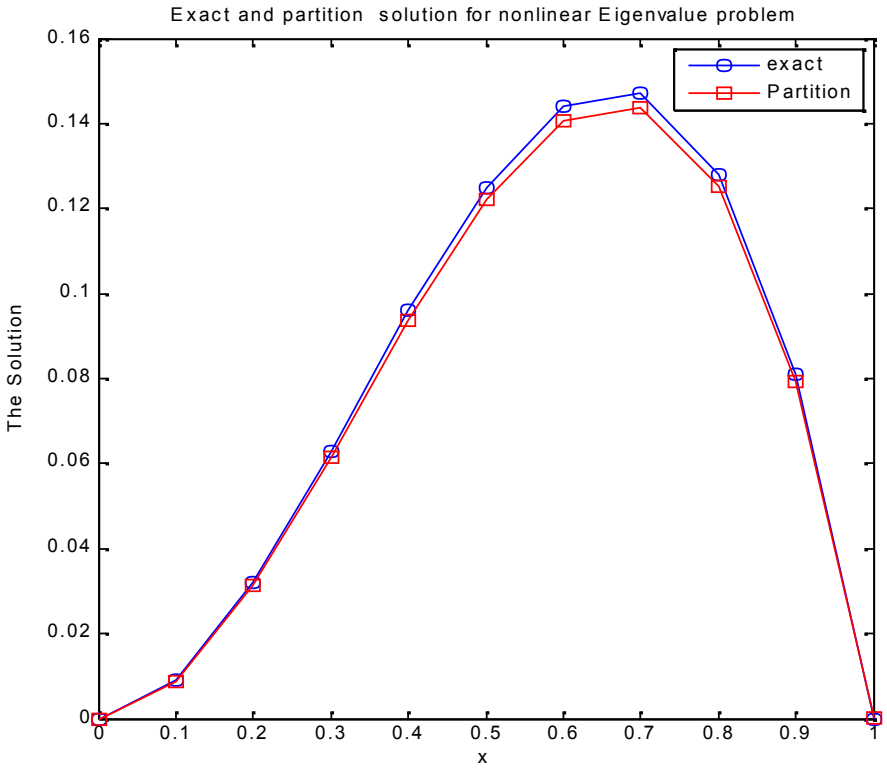
……(7e)

and by using Math CAD software package to solve this system with different initial values of $(a_2, a_3, a_4, a_5, \lambda)$ we obtain the following tabulated results:

x	The initial values of a_2, a_3, a_4, a_5 and λ	The approximated values of a_2, a_3, a_4, a_5 and λ	The exact solution	The approximate solution	Absolute error

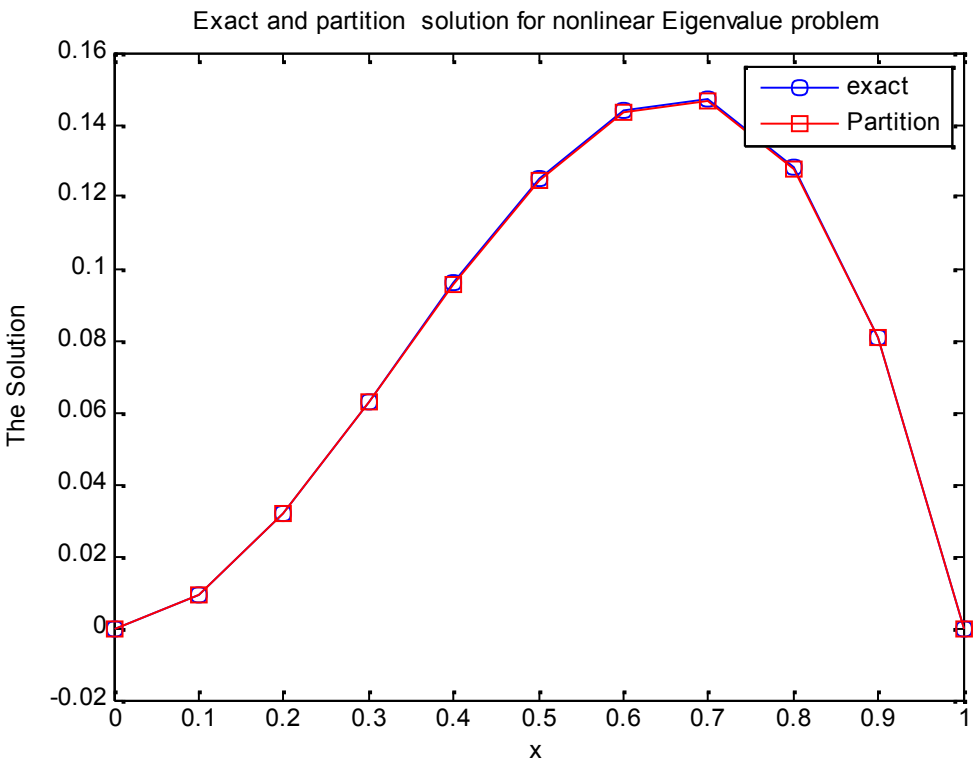
0	6.3, -2.712, 6.026, 4.8, 8	0.978, -0.979, 2.405×10^{-3} -1.174×10^{-3}, 1.956	0	0	0
0.1	=	=	0.0090	0.0088	0.0002
0.2	=	=	0.0320	0.0312	0.0008
0.3	=	=	0.0630	0.0615	0.0015
0.4	=	=	0.0960	0.0938	0.0022
0.5	=	=	0.1250	0.1221	0.0029
0.6	=	=	0.1440	0.1407	0.0033
0.7	=	=	0.1470	0.1436	0.0034
0.8	=	=	0.1280	0.1251	0.0029
0.9	=	=	0.0810	0.0792	0.0018
1	=	=	0	0.0000	0.0000

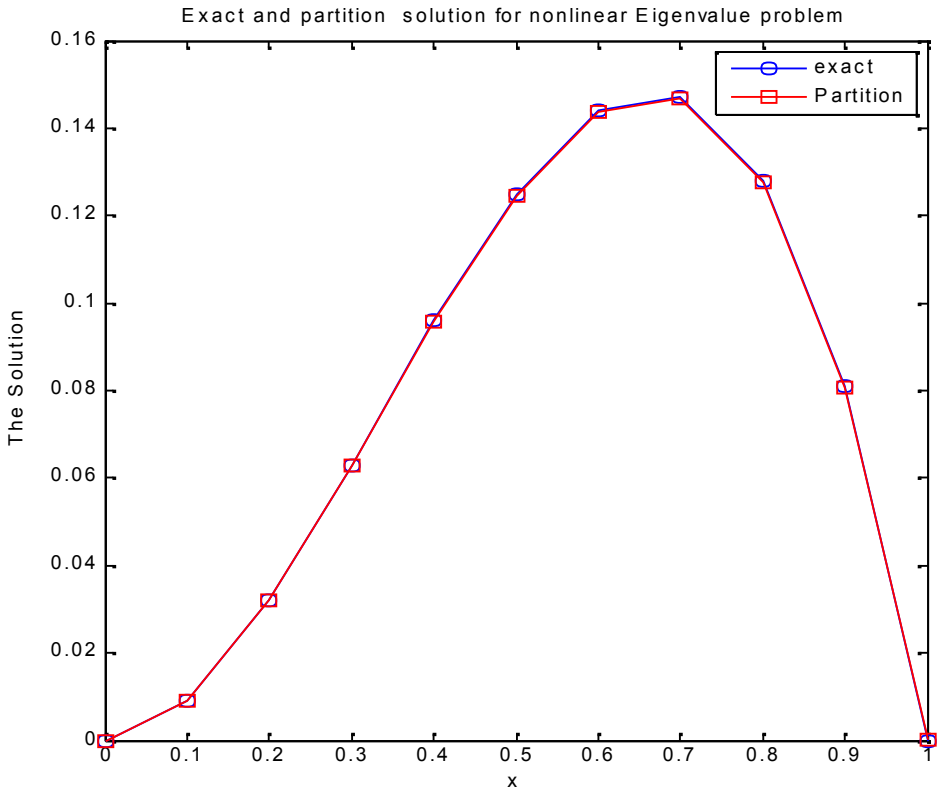
x	The initial values of a_2, a_3, a_4, a_5 and λ	The approximated values of a_2, a_3, a_4, a_5 and λ	The exact solution	The approximate solution	Absolute error



0	-3, 2, 3, 1, 5	0.997,-0.997, 4.411 $\times 10^{-4}$, -2.186$\times 10^{-4}$, 1.994	0	0	0
0.1	=	=	0.0090	0.0090	0.0492$\times 10^{-3}$
0.2	=	=	0.0320	0.0319	0.1399$\times 10^{-3}$
0.3	=	=	0.0630	0.0627	0.2527$\times 10^{-3}$
0.4	=	=	0.0960	0.0956	0.3679$\times 10^{-3}$
0.5	=	=	0.1250	0.1245	0.4655$\times 10^{-3}$
0.6	=	=	0.1440	0.1435	0.5253$\times 10^{-3}$
0.7	=	=	0.1470	0.1465	0.5276$\times 10^{-3}$
0.8	=	=	0.1280	0.1275	0.4530$\times 10^{-3}$
0.9	=	=	0.0810	0.0807	0.2829$\times 10^{-3}$
1	=	=	0	0.0000	0.0000

x	The initial values of a_2, a_3, a_4, a_5 and λ	The approximated values of a_2, a_3, a_4, a_5 and λ	The exact solution	The approximate solution	Absolute error
0	6.1, 1, 4, 2, 4.7	0.998,-0.998, 2.211 $\times 10^{-4}$ -1.038$\times 10^{-4}$,1.996	0	0	0
0.1	=	=	0.0090	0.0090	0.0297$\times 10^{-3}$
0.2	=	=	0.0320	0.0319	0.0871$\times 10^{-3}$
0.3	=	=	0.0630	0.0628	0.1597$\times 10^{-3}$
0.4	=	=	0.0960	0.0958	0.2343$\times 10^{-3}$
0.5	=	=	0.1250	0.1247	0.2981$\times 10^{-3}$
0.6	=	=	0.1440	0.1437	0.3378$\times 10^{-3}$
0.7	=	=	0.1470	0.1467	0.3405$\times 10^{-3}$
0.8	=	=	0.1280	0.1277	0.2933$\times 10^{-3}$
0.9	=	=	0.0810	0.0808	0.1838$\times 10^{-3}$
1	=	=	0	0.0000	0.0000


Exact and partition solution for nonlinear Eigenvalue problem
exact
Partition
The Solution
x
0.16
0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
-0.02
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1


Exact and partition solution for nonlinear Eigenvalue problem
exact
Partition
The Solution
x
0.16
0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

4. Conclusions**:**

1. In this method as n increases the size of sub domains decreases with the result that the eigenvalue problem is satisfied on smaller sub domains this lead to the residue approaches zero.

2. When the number of the independent functions in the approximation, partition method gives better results but the nonlinear system becomes large therefore the Math CAD software package fails to find the solution.

3. Comparing the results obtained by applying the partition method on the nonlinear Sturm-Liouville eigenvalue problem with the exact solution shows there is a very good agreement in all examples.

4. When the initial values of the independent functions a_2, a_3, a_4, a_5 and λ far away from the exact values the Math CAD program does not give convergent approximate values, it give convergent values only when the initial values is nearby the exactl values.In this case the approximation solution of non linear Sturm-Liouville eigenvalue problem convergent to exact solution .

5. Refrences**:**

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