Raf. J. of Comp. & Math's. , Vol. 6, No. 2, 2009

On Representation Theorem for Algebras with Three Commuting Involutions

Amir A. Mohammed **Baida S. Abdullah** **Nadwa S. Younis**

College of Education

University of Mosul

Received on:26/8/2007 **Accepted on:30/1/2008**

الملخص

تم تقديم الجبريات مع ثلاثة تشابكات ابدالية كمولد واحد للجبور الجزئية ¯ – £ – ™ من جبور فضاء متجه التطبيقات الخطية حيث ¯ – £ – ™ رموز التشابكات المثبتة المذكورة آنفاً.

ABSTRACT

Algebras with three commuting involutions are represented as commutants of one-generated ¯ – £ – ™ subalgebras of algebras of vector-space endomorphisms where ¯ – £ and ™ are involutions of a prefixed type.

Introduction and Preliminaries

Throughout this paper *(k, –)* denotes a field with an involution and the terminology of algebra and algebra involution is relative to *(k, –)*. A systematic study of representation theory for algebras with involutions was given in [6] by Quebbemann and he proved that (involutive unital finite-dimensional algebras can be represented as commutants of one-generated self-adjoint subalgebras of algebras of vector-space endomorphisms) and later the representation theory for algebras with involutions has been extend to algebras with two commuting involutions by Cabrera and Mohammed see [1].

We begin by summarizing some definitions and fundamental concepts. An involution * in an algebra *A* is a mapping *a* à *a** of *A* into it self satisfying $(a+b)^* = a^* + b^*$, $(\alpha a)^* = \bar{\alpha} a^*$ (where – denote the conjecate of complex number), $(ab)^*=b^*a^*$ and $a^{**}=a$ for all *a*, *b* in *A* and α in *k* see[1]. A subalgebra of *A* globally invariant by * is called a * – subalgebra. If *B* is a * – subalgebra of *A*, then its centralizer in *A* given by

{ *a* Î *A* : *ab=ba for all b in B* } ,

is also a * – subalgebra of *A* see [1].

Involutive algebras can be constructed from the consideration of nondegenerate hermitian spaces. Recall that, for S in k satisfying $S\bar{\Sigma}=1$, a nondegenerate Σ–hermitian form in a vector – space M over k is a mapping <. , .> from M x M into k satisfying

$$< m_1 + m_2, m' > = < m_1, m' > + < m_2, m' >,\ < \alpha m, m' > = \alpha < m, m' >$$
$$< m , m' > = S\,\overline{< m', m >}$$

for all m_1, m_2, m, m' in M and α in k, and $< m, m' > = 0$ for all m' implies $m=0$ see [1]. If M has finite dimension, then the algebra $\mathrm{End}_k(M)$ of all endomorphisms of M with the adjoint involution F **à** F^{-} given by

$$< F(m), m' > = < m, F^{-}(m') >$$

for all m, m' in M see [1].

If $(A, *, \#, d)$, $(B,\ ^{-},\ £\ ,™)$ are algebras with three commuting involutions, an isomorphism between $(A, *, \#, d)$ and $(B,\ ^{-},\ £,\ ™)$ is an algebra isomorphism f from A onto B satisfying $f(a^*) = f(a)^{-}$, $f(a^{\#}) = f(a)^{G}$ and $f(a^{d}) = f(a)^{F}$ for all a in A. In this case $(A, *, \#, d)$ and $(B,\ ^{-},\ £\ ,™)$ are said to be isomorphic see [1].

Our main result is the following :

Theorem 1. Let $(A, *, \#, d)$ be a unital finite-dimensional algebra with commuting involutions over $(k, –)$ and let S, S', S'' in k such that $S\bar{\Sigma}=S'\bar{\Sigma}'=S''\bar{\Sigma}''=1$. Then there exist a finite-dimensional vector space w over k, a nondegenerate Σ-hermitian form <. , .>, a nondegenerate Σ'-hermitian form [. , .], a nondegenerate Σ''-hermitian form [. , .], and F in $\mathrm{End}_k(w)$ such that $(A, *, \#, d)$ is isomorphic to the centralizer of the $^{-}$– £ – ™– subalgebra of $\mathrm{End}_k(w)$ generated by F, where $^{-}$, £ and ™ are adjoint involutions in $\mathrm{End}_k(w)$ determined by <. , .> , [. , .] and (. , .), respectively.

We will follow the lines of the following lemma : Let $(A, *, \#)$ be a unital finite-dimensional algebra with commuting involutions over $(K, –)$ and let S , S' in K such that $S\bar{\Sigma} = S'\bar{\Sigma}' = 1$. Then there exist a finite-dimensional vector space W over K, a nondegenerate S-hermitian form <.,.> in W, a nodegenerate S'-hermitian from $[. , .]$ in W, and F in $End_K(W)$ such that $(A, * , \#)$ is isomorphic to the centralizer of the $^{-}$– £ – subalgebra of $End_K(W)$ generated by F, where $^{-}$ and £ are adjoint involutions in $End_K(W)$ determined by < . , . > and $[. , .]$ respectively.

Proof : see [1, Theorem 1]

The first part of the proof consists in finding a nondegenerate $S – S' – S''$-hermitian space w_o over k such that $(A, *, \#, d)$ is embedded into $\mathrm{End}_k(w_o)$ in such a way that w_o is a balanced A-module (that is, $A = \mathrm{End}_B(w_o)$ if $B=\mathrm{End}_A(w_o)$). Our construction involves the three commuting involutions of A and consists in a convenient triple of the representation used in [1].

Theorem 2. let (A, *, #, d) be a unital finite-dimensional algebra with commuting involutions over (k, –) and let S, S', S'' in k such that $S\overline{\Sigma} = S'\overline{\Sigma}' = S''\overline{\Sigma}'' = 1$. Then there exists ($w_o$, <. , . > , [. , .] , (. , .)), where w_o is a finite-dimensional vector space over k which is a balanced left A-module (in fact, w_o contains A as a direct summand) and <. , .> , [. , .] , (. , .) are nondegenerate Σ-hermitian, Σ'-hermitian and Σ''-hermitian forms in w_o, respectively, in such a way that the associated representation of A in w_o becomes an isomorphism of algebras with three involutions of (A, *, #, d) into ($\mathrm{End}_k(w_o)$, ¯, **£** ,**™**), where ¯, **£** and **™** are adjoint involutions in $\mathrm{End}_k(w_o)$ determined by <. , .> , [. , .] and (. , .), respectively.

Proof. Consider the vector space w_o : = U_1 Å U_2 Å U_3 Å U_4 Å U_5 Å U_6, where $U_1 = U_3 = U_5 = A$ and $U_2 = U_4 = U_6 = Hom_k(A, k)$. Endow w_o with the structure of faithful left A-module given by :
$a\,(x_1, f_1, x_2, f_2, x_3, f_3) := (ax_1, f_1 o L_{a*}, a^{*\#}x_2, f_2 o L_{a\#}, a^{*\#d}x_3, f_3 o L_{ad})$ for all a in A and $(x_1, f_1, x_2, f_2, x_3, f_3)$ in w_o. The mapping <. , .> from w_o x w_o in to k defined by

$$< (x_1, f_1, x_2, f_2, x_3, f_3), (y_1, g_1, y_2, g_2, y_3, g_3) > := \overline{f_1(y_1) + f_2(y_2) + f_3(y_3)} + S\,(g_1(x_1) + g_2(x_2) + g_3(x_3))$$

is a nondegenerate Σ-hermitian form satisfying

$$< a\,(x_1, f_1, x_2, f_2, x_3, f_3), (y_1, g_1, y_2, g_2, y_3, g_3) > = < (x_1, f_1, x_2, f_2, x_3, f_3), a^*(y_1, g_1, y_2, g_2, y_3, g_3) > ,$$

and therefore the representation of A on w_o becomes an isomorphism of involutive algebras from $(A, *)$ into ($\mathrm{End}_k\,(w_o)$, ¯), where ¯ denotes the adjoint involution with respect to <. , .>. Furthermore, the mapping [. , .] from w_o x w_o into k defined by

$$[(x_1, f_1, x_2, f_2, x_3, f_3), (y_1, g_1, y_2, g_2, y_3, g_3)] := \overline{f_1(y_2) + f_2(y_3) + f_3(y_1)} + S'\,(g_1(x_2) + g_2(x_3) + g_3(x_1))$$

is a nondegenerate Σ'-hermitian form satisfying

$$[a\,(x_1, f_1, x_2, f_2, x_3, f_3), (y_1, g_1, y_2, g_2, y_3, g_3)] = [(x_1, f_1, x_2, f_2, x_3, f_3), a^{\#}(y_1, g_1, y_2, g_2, y_3, g_3)] ,$$

and so the representation of A on w_o also becomes an isomorphism of involutive algebras from (A, #) into ($\mathrm{End}_k\,(w_o)$, ***G***), where ***G*** denotes the adjoint involution with respect to [.,.]. furthermore, the mapping (. , .) from w_o x w_o into k defined by

$((x_1, f_1, x_2, f_2, x_3, f_3), (y_1, g_1, y_2, g_2, y_3, g_3)) : =$
$\overline{f_1(y_3) + f_2(y_1) + f_3(y_2)} + S'' (g_1(x_3) + g_2(x_1) + g_3(x_2))$
is a nondegenerate Σ"-hermitian form satisfying
$[a\ (x_1, f_1, x_2, f_2, x_3, f_3), (y_1, g_1, y_2, g_2, y_3, g_3)] =$
$[(x_1, f_1, x_2, f_2, x_3, f_3), a^d(y_1, g_1, y_2, g_2, y_3, g_3)]$,
and so the representation of A on w_o also becomes an isomorphism of involutive algebras from (A, d) into (End$_k$(w_o), ***F***), where ***F*** denotes the adjoint involution with respect to (. , .). Therefore, the representation of A on w_o is an isomorphism of algebras with three involutions. Since w_o contains the "regular" A-module A as a direct summand, it is balanced (see [4, P. 451]). **¢**

Remark 1. The involutions ¯, **£** and **™** in End$_k$(w_o) obtained in the above proof are not necessarily commuting. Since w_o = U_1 Å U_2 Å U_3 Å U_4 Å U_5 Å U_6 we can represent each T in End$_k$ (w_o) as a 6x6 homomorphism matrix.

$$T = \begin{pmatrix} T_{11} & T_{12} & T_{13} & T_{14} & T_{15} & T_{16} \\ T_{21} & T_{22} & T_{23} & T_{24} & T_{25} & T_{26} \\ T_{31} & T_{32} & T_{33} & T_{34} & T_{35} & T_{36} \\ T_{41} & T_{42} & T_{43} & T_{44} & T_{45} & T_{46} \\ T_{51} & T_{52} & T_{53} & T_{54} & T_{55} & T_{56} \\ T_{61} & T_{62} & T_{63} & T_{64} & T_{65} & T_{66} \end{pmatrix}$$

Where T_{ij} Î Hom_k (U_j, U_i) for i , j Î { $1, 2, 3, 4, 5, 6$ }. It is easy to verify that

$$T^{-} = \begin{pmatrix} T'_{22} & \overline{\Sigma}T'_{12} & T'_{42} & \overline{\Sigma}T'_{32} & T'_{62} & \overline{\Sigma}T'_{52} \\ \Sigma T'_{21} & T'_{11} & \Sigma T'_{41} & T'_{31} & \Sigma T'_{61} & T'_{51} \\ T'_{24} & \overline{\Sigma}T'_{14} & T'_{44} & \overline{\Sigma}T'_{34} & T'_{64} & \overline{\Sigma}T'_{54} \\ \Sigma T'_{23} & T'_{13} & \Sigma T'_{43} & T'_{33} & \Sigma T'_{63} & T'_{53} \\ T'_{26} & \overline{\Sigma}T'_{16} & T'_{46} & \overline{\Sigma}T'_{36} & T'_{66} & \overline{\Sigma}T'_{56} \\ \Sigma T'_{25} & T'_{15} & \Sigma T'_{45} & T'_{35} & \Sigma T'_{65} & T'_{55} \end{pmatrix} ,$$

$$T^{\pounds} = \begin{pmatrix} T'_{66} & \overline{\Sigma'}\,T'_{56} & T'_{46} & \overline{\Sigma'}\,T'_{36} & T'_{26} & \overline{\Sigma'}\,T'_{16} \\ \Sigma' T'_{65} & T'_{55} & \Sigma' T'_{45} & T'_{35} & \Sigma' T'_{25} & T'_{15} \\ T'_{64} & \overline{\Sigma'}\,T'_{54} & T'_{44} & \overline{\Sigma'}\,T'_{34} & T'_{24} & \overline{\Sigma'}\,T'_{14} \\ \Sigma' T'_{63} & T'_{53} & \Sigma' T'_{43} & T'_{33} & \Sigma' T'_{23} & T'_{13} \\ T'_{62} & \overline{\Sigma'}\,T'_{52} & T'_{42} & \overline{\Sigma'}\,T'_{32} & T'_{22} & \overline{\Sigma'}\,T'_{12} \\ \Sigma' T'_{61} & T'_{51} & \Sigma' T'_{41} & T'_{31} & \Sigma' T'_{21} & T'_{11} \end{pmatrix}$$

And

$$T^{\text{™}} = \begin{pmatrix} T'_{44} & \overline{\Sigma''}\,T'_{14} & T'_{64} & \overline{\Sigma''}\,T'_{34} & T'_{24} & \overline{\Sigma''}\,T'_{54} \\ \Sigma'' T'_{41} & T'_{11} & \Sigma' T'_{61} & T'_{31} & \Sigma'' T'_{21} & T'_{51} \\ T'_{46} & \overline{\Sigma''}\,T'_{16} & T'_{66} & \overline{\Sigma''}\,T'_{36} & T'_{26} & \overline{\Sigma''}\,T'_{56} \\ \Sigma'' T'_{43} & T'_{13} & \Sigma'' T'_{63} & T'_{33} & \Sigma'' T'_{23} & T'_{53} \\ T'_{42} & \overline{\Sigma''}\,T'_{12} & T'_{62} & \overline{\Sigma''}\,T'_{32} & T'_{22} & \overline{\Sigma''}\,T'_{52} \\ \Sigma'' T'_{45} & T'_{15} & \Sigma'' T'_{65} & T'_{35} & \Sigma'' T'_{25} & T'_{55} \end{pmatrix}$$

Therefore

$$T^{-\pounds} = \begin{pmatrix} T'_{55} & \overline{\Sigma'}\,\Sigma T'_{56} & T'_{53} & \overline{\Sigma'}\,\Sigma T'_{54} & T'_{51} & \overline{\Sigma'}\,\Sigma T'_{52} \\ \Sigma'\,\overline{\Sigma}\,T'_{65} & T'_{66} & \Sigma'\,\overline{\Sigma}\,T'_{63} & T'_{64} & \Sigma'\,\overline{\Sigma}\,T'_{61} & T'_{62} \\ T'_{35} & \overline{\Sigma'}\,\Sigma T'_{36} & T'_{33} & \overline{\Sigma'}\,\Sigma T'_{34} & T'_{31} & \overline{\Sigma'}\,\Sigma T'_{32} \\ \Sigma'\,\overline{\Sigma}\,T'_{45} & T'_{46} & \Sigma'\,\overline{\Sigma}\,T'_{43} & T'_{44} & \Sigma'\,\overline{\Sigma}\,T'_{41} & T'_{42} \\ T'_{15} & \overline{\Sigma'}\,\Sigma T'_{16} & T'_{13} & \overline{\Sigma'}\,\Sigma T'_{14} & T'_{11} & \overline{\Sigma'}\,\Sigma T'_{12} \\ \Sigma'\,\overline{\Sigma}\,T'_{25} & T'_{26} & \Sigma'\,\overline{\Sigma}\,T'_{23} & T'_{24} & \Sigma'\,\overline{\Sigma}\,T'_{21} & T'_{22} \end{pmatrix}$$

$$T^{\pounds -} = \begin{pmatrix} T'_{55} & \overline{\Sigma}\,\Sigma' T'_{56} & T'_{53} & \overline{\Sigma}\,\Sigma' T'_{54} & T'_{51} & \overline{\Sigma}\,\Sigma' T'_{52} \\ \Sigma\,\overline{\Sigma'}\,T'_{65} & T'_{66} & \Sigma\,\overline{\Sigma'}\,T'_{63} & T'_{64} & \Sigma\,\overline{\Sigma'}\,T'_{61} & T'_{62} \\ T'_{35} & \overline{\Sigma}\,\Sigma' T'_{36} & T'_{33} & \overline{\Sigma}\,\Sigma' T'_{34} & T'_{31} & \overline{\Sigma}\,\Sigma' T'_{32} \\ \Sigma\,\overline{\Sigma'}\,T'_{45} & T'_{46} & \Sigma\,\overline{\Sigma'}\,T'_{43} & T'_{44} & \Sigma\,\overline{\Sigma'}\,T'_{41} & T'_{42} \\ T'_{15} & \overline{\Sigma}\,\Sigma' T'_{16} & T'_{13} & \overline{\Sigma}\,\Sigma' T'_{14} & T'_{11} & \overline{\Sigma}\,\Sigma' T'_{12} \\ \Sigma\,\overline{\Sigma'}\,T'_{25} & T'_{26} & \Sigma\,\overline{\Sigma'}\,T'_{23} & T'_{24} & \Sigma\,\overline{\Sigma'}\,T'_{21} & T'_{22} \end{pmatrix},$$

And

$$T^{-\text{™}} = \begin{pmatrix} T'_{55} & \overline{\Sigma''}\Sigma T'_{56} & T'_{53} & \overline{\Sigma''}\Sigma T'_{54} & T'_{51} & \overline{\Sigma''}\Sigma T'_{52} \\ \Sigma''\overline{\Sigma}\, T'_{65} & T'_{66} & \Sigma''\overline{\Sigma}\, T'_{63} & T'_{64} & \Sigma''\overline{\Sigma}\, T'_{61} & T'_{62} \\ T'_{35} & \overline{\Sigma''}\Sigma T'_{36} & T'_{33} & \overline{\Sigma''}\Sigma T'_{34} & T'_{31} & \overline{\Sigma''}\Sigma T'_{32} \\ \Sigma''\overline{\Sigma}\, T'_{45} & T'_{46} & \Sigma''\overline{\Sigma}\, T'_{43} & T'_{44} & \Sigma''\overline{\Sigma}\, T'_{41} & T'_{42} \\ T'_{15} & \overline{\Sigma''}\Sigma T'_{16} & T'_{13} & \overline{\Sigma''}\Sigma T'_{14} & T'_{11} & \overline{\Sigma''}\Sigma T'_{12} \\ \Sigma''\overline{\Sigma}\, T'_{25} & T'_{26} & \Sigma''\overline{\Sigma}\, T'_{23} & T'_{24} & \Sigma''\overline{\Sigma}\, T'_{21} & T'_{22} \end{pmatrix}$$

As a result, ¯, **£** and **™** are commuting if and only if $\overline{\Sigma'}\,\Sigma = \overline{\Sigma}\,\Sigma' = \overline{\Sigma''}\,\Sigma = \overline{\Sigma}\,\Sigma''$, or equivalently $S^2 = S'^2 = S''^2$.

Proof of Theorem 1. let (*A, *, #, d*) be a unital finite-dimensional algebra with commuting involutions over *(k, –)* and let *S, S', S''* in *k* such that $\Sigma\overline{\Sigma} = \Sigma'\overline{\Sigma'} = \Sigma''\overline{\Sigma''} = 1$. By Theorem 2 there exists (w_o, <.,.>, [.,.] , (.,.)), where w_o is a finite-dimensional vector space over k which is a balanced left A-module and <.,.>, [.,.] and (.,.) are nondegenerate Σ-hermitian, Σ'-hermitian and Σ''-hermitian forms in w_o, respectively, in such a way the associated representation of *A* in w_o becomes an isomorphism from (*A, *, #, d)* into ($\mathrm{End}_k(w_o)$, ¯, **£**, **™**), where ¯, **£** and **™** are adjoint involutions in $\mathrm{End}_k(w_o)$ determined by <.,.>, [.,.] and (.,.), respectively. Let m denote the dimension of $B = \mathrm{End}_A(w_o)$. Put (w, <.,.>, [.,.], (.,.)) : = (w_o, <.,.> , [.,.] , (.,.)) *Å* $m+2$ *Å* (w_o, <.,.>, [.,.], (.,.)), and consider $\mathrm{End}_k(w_o)$ embedded diagonally in $\mathrm{End}_k(w)$. By the final step of the proof of theorem 1 in [1] applied to <.,.>, [.,.] and (.,.) there exists F in $\mathrm{End}_k(w)$ such that $A=\mathrm{End}_C(w) = \mathrm{End}_D(w) = \mathrm{End}_H(w)$, where *C* and *D* (resp. *H*) denotes the ¯ – subalgebra and **£** – subalgebra (resp. *O* – subalgebra) of $\mathrm{End}_K(w)$ generated by *F*. Let us denote by E the ¯– **£** – **™**– subalgebra of $\mathrm{End}_k(w)$ generated by *F*. Since *C, D, H Í E*, it follows that $\mathrm{End}_E(w) \subseteq \mathrm{End}_C(w) = \mathrm{End}_D(w) = \mathrm{End}_H(w) = A$. On the other hand, *A* is a ¯– **£** – **™**– subalgebra of $\mathrm{End}_k(w)$ whose elements commute with *F*, therefore $\mathrm{End}_A(w)$ is a ¯– **£** – **™**– subalgebra of $\mathrm{End}_k(w)$ containing *F*, and so $E \subseteq \mathrm{End}_A(w)$. From this, $A \subseteq \mathrm{End}_E(w)$.

Remark 2. The process of representing the algebras of commuting involutions can be explained through the following diagram :

No. of involution	**Underliny vector – space of represented algebra**	**Construction of Nondegenerate form with respect to involution**	**Generator of represented algebra with involution**	**The condition on the element of the field *k*, where** $S, \overline{\Sigma}, \ldots \hat{I}\ k$
N=1	$w_o = U_1 \text{ Å } U_2$	< . , . >	T ∈ $End_k(w_o)$ and T ∈ M_{2x2}(¢)	$S\overline{\Sigma} = 1$
N=2	$w_o = U_1 \text{Å} U_2 \text{Å} U_3 \text{Å} U_4$	< . , . > , [. , .]	T ∈ $End_k(w_o)$ and T ∈ M_{4x4}(¢)	$S\overline{\Sigma} = S'\overline{\Sigma}' = 1$
N=3	$w_o = U_1 \text{Å} \ldots\ldots \text{Å} U_6$	<. , .> , [. , .] , (. , .)	T ∈ $End_k(w_o)$ and T ∈ M_{6x6}(¢)	$S\overline{\Sigma} = S'\overline{\Sigma}' = S''\overline{\Sigma}'' = 1$
N=4	$w_o = U_1 \text{Å} \ldots\ldots \text{Å} U_8$	<.,.> , [.,.] , (.,.) , {.,.}	T ∈ $End_k(w_o)$ and T ∈ M_{8x8}(¢)	$S\overline{\Sigma} = S'\overline{\Sigma}' = S''\overline{\Sigma}'' = S'''\overline{\Sigma}''' = 1$
N=5	$w_o = U_1 \text{Å} \ldots\ldots \text{Å} U_{10}$	<.,.>, [. , .], (. , .), {. , .}, ((. , .))	T ∈ $End_k(w_o)$ and T∈M_{10x10}(¢)	$S\overline{\Sigma} = S'\overline{\Sigma}' = S''\overline{\Sigma}'' = S'''\overline{\Sigma}''' = S''''\overline{\Sigma}'''' = 1$
. . . .	. . . .	. . . .	. . . .	. . . .

REFERENCES

[1] M. Cabrera and Amir A. Mohammed, "A representation theorem for algebras with commuting involutions", (2000), Linear algebra and its Applications 306 (25-31).

[2] M. Cabrera, A. Moreno, A. Rodriguez, E. I. zel'manov, "Jordan Polynomials can be analytically recognized," (1996), Studia Math, 137-147.

[3] N. Jacobson, "Lectures in abstract algebra II, Linear algebra", (1953), Grad, Texts in Mathematics, Vol., 31, Springer, New York.

[4] S. Lang, "Algebra, third edition", (1971), Addison-Wesley, New York.

[5] A. Moreno, "Extending the norm from special Jordan triple systems to their associative envelopes", Banach Algebras '97, Proc. 13th Internat. Conf. on Banach Algebras, July 20-August 3, 1997, Walter de gruyter, Berlin, 1998, pp. 363-375.

[6] H. G. Quebbemann, "A representation theorem for algebras with involution", linear algebra Appl., (1987), 193-195.