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Petrov-Galerkin Finite Element Method for Convection-Diffusion-Reaction Problem

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ABSTRACT

In this paper, described and analyzed Petrov-Galerkin finite element (PGFE) method for solving of Convection-Diffusion-Reaction (CDR) problem, proved two properties of the bilinear form $a_{PG}(U,v)$, (continuity and ψ −elliptic). The stability and error estimate (semi-discrete and full-discrete) are proved and the approximate solutions are convergent with the error of $O(h^2), O(h+\tau)$, respectively. The numerical results of PGFE method show the convergence and accuracy of the method.

Keywords: *Petrov-Galerkin Finite Element, Standard Galerkin method, Finite Element method, Convection Diffusion Reaction problem, stability, Error analysis.*

1. INTRODUCTION:

The process of convection, diffusion and reaction is a basic process in the dynamic of chemical and biological liquids, heat, physical and mass transfer, etc. This is what attracts many specialists in these fields. Therefor, a number of athletes, engineers, and others have been interested in this method. Diffusion is the physical process by which a property is transferred by the random motion of a liquid molecules. While, Convection is a physical process by which some properties are transferred by the orderly motion of the flow. The behavior of fluids passing through a mass, forced heat, or vortex, transfer is described by a set of partial differential equations. The Finite Element (FE) method is one of the most powerful numerical approximation tools for solving the partial differential equation. This is because it can deal with a complex geometric shape and solid mathematics theory (mathematics analysis) that has evolved to analyze the performance of a FE method. The foundations of the FE method were established at the beginning of the twentieth century. But it was not widely used. Structural engineers were the first to use this method in the mid-century. Mathematicians were interested in analyzing and implementing the method in 1960. First presented in the publication by Clough, Turner and Martin [14] (1956). The FE method is a numerical procedure for solving the differential equations of physics and engineering problems. This idea of FE method is to turn a difficult problem into a simple problem. For some problems, the current mathematical tools will not be sufficient to get the exact solution. FE approximation is particularly powerful in two dimension and three dimension because the FE method can handle a domain Ω (geometrically complex) easily [11]. In application, typically the size of the diffusion is much smaller than the size of the convective term, we need to stabilization techniques are require in order to get physically sound numerical approximations.

Several treatments have been tried and developed for the standard FE method, including the streamline diffusion, artificial diffusivity method, upwind techniques, and weak Galerkin method [2][6]. PGFE method was suggested to eliminate the inaccuracies and oscillations obtained using the standard Galerkin (SG) method when applied to CDR problem in high Peclet number. This paper studies ones of the currently most popular FE stabilizations, the PGFE method. In 1978, Griffiths and Lorenz [5], PGFE method, using different test and trial functions, are applied to the solution of an unsymmetrical two point boundary value problem intended to simulate certain aspects of CDR problems. Analysis of the boundary layer behavior of the continuous problem, L-splines are introduced, and, by studying their behavior for coarse meshes, we are able to modify the original schemes to produce FE methods. And also for coarse meshes, when no nodes occur in the boundary layer, the accuracy at all nodal points is good. In 2010, E. Burman [4], suggested an analysis of the streamline upwind PG method of adapting using standard finite difference patterns to estimate time. For sufficiently smooth solution, we demonstrate uniform stability and near perfect convergence. If the solution, or data, is not smooth enough, a degradation of the stability and convergence estimates appears if Δt is small compared to δ (the stabilization parameter). In 2014, D. Broersen and R. Stevenson[1], a PG discretization is studied of an ultra-weak form of CDR equation in a mixed form. The genuinely ideal test space must be replaced by dropping it onto finite dimensional test search space to reach an actionable method. In this paper, will show and analyze the PGFE method of the linear CDR problem, space discretization. Analysis of two properties of the bilinear form $a_{PG}(U, v)$ and the error analysis in two cases (semi-discrete and fully-discrete PGFE method), discuss the numerical solution of the unsteady-state and steady-state PGFE method to solve the CDR problem, and compute L^2, L_∞ and H^1 error, organized this paper as follows. In section 2, present CDR problem. In section 3, show PG space and the condition to the stability parameter. In sections 4 and 5, show some essential definitions and solve CDR by PGFE method. In section 6, prove some important properties (continuity, ψ −elliptic and stability). In sections 7 and 8, the error analysis (semi-discrete and fully-discrete PGFE method). In section 9, show numerical results. In section 10, Discussion and Conclusion.

2. Time Dependent Modeling Problem :

Let Ω is a convex polygonal or polyhedral domain, where $\Omega \subset \mathbb{R}^2$ is a bounded open domain. A linear time dependent (CDR) equation is given by [7]:

Find $U: \mathfrak{D}_H = \Omega \times I \to \mathbb{R}$ such that

$$\begin{aligned} U_t - \epsilon \Delta U + \beta \cdot \nabla U + cU = f \quad & in \quad \mathfrak{D}_H \\ U = 0 \quad & on \ \Gamma \times I \\ U(x,0) = U_0(x) \quad & in \ \Omega \end{aligned} \tag{1}$$

with boundary Γ, $\beta(x,t)$ and $c(x,t)$ are given functions, $0 < \epsilon \ll 1$ is a constant diffusion coefficient, $I = (0,T]$ and $T > 0$ is a given final time.

Assumption 2.1 [10]:

We assumed that there is a positive constant μ_0;

$$(c - \frac{1}{2} \nabla \cdot \beta) = \mu \geq \mu_0 .$$

3. Petrov-Galerkin Space:

Let $\{\mathfrak{I}_h\}$ where $0 < h < 1$ denote a family of shape regular triangulation of Ω . We assume that the triangulation $\mathfrak{I}_h$ are quasi-uniform. The triangle of $\mathfrak{I}_h$ will be denoted by e_l (element).

Let ψ be trail space and ϕ a test space [4] where

$$\psi = H_0^1(\Omega) = \{v: v \in H^1(\Omega);\ v\,|_\Gamma = 0\}$$

$$\phi = \{w: w = v + \delta\beta \cdot \nabla v\,;\, v \in \psi\} \tag{2}$$

here δ indicates to a stability parameter which is constant in $\mathfrak{D}_H$.

It will be chosen as [13];

$$\delta = \begin{cases} \eta h & if \quad \epsilon < h \\ 0 & if \quad \epsilon \geq h \end{cases}$$

; $0 < \eta < \frac{1}{4}$ (small constant), and $\dim \psi = dim\phi$.

4. Petrov-Galerkin Method (PGM):

In Standard Galerkin (SG) method, a weak form of Equation (1) read as follows;

find $U \in \psi$ such that

$$(U_t, v) + (\epsilon \nabla U, \nabla v) + (\beta \cdot \nabla U, v) + (cU, v) = (f, v) \qquad , \forall U \in \psi \tag{3}$$

and $\qquad U(x, 0) = U_{0.}$

where

$(U, v) = \int Uv \, d\Omega,$

$L^2(\Omega) = \{U : \int_\Omega |U|^2 d\Omega < \infty\},$

$H^1(\Omega) = \{U \; and \; \nabla U \in L^2(\Omega)\},$

$H_0^1(\Omega) = \{U \in H^1(\Omega); U|_\Gamma = 0\},$

$L_\infty = \max_{U \in \Omega} |U|,$

and $(\cdot,\cdot)$ denote the inner product in $L^2(\Omega)$. In numerical simulations, ψ is replaced by FE space $\psi_{h,r}$ where $r \in \mathbb{N}$ be the degree of the local polynomial. The SGFE find $U_h \in \psi_{h,r} \ni v \in \psi_{h,r}$. Consider the case conforming FE method. The time continuous GFE method targets to find function $U_h \in \psi_{h,r}$ that fulfills a problem of form Equation (3) for all test functions from $\psi_{h,r}$ with an appropriate approximation of U_0 at the initial time.

$$\left(U_{t,h}, v\right) + (\epsilon \nabla U_h, \nabla v) + (\beta \cdot \nabla U_h, v) + (cU_h, v) = (f, v) \qquad , \forall v \in \psi_h \tag{4}$$

to derive the PGFE method, we replace v by w which is defined in (2);

$$(U_t, w) - (\epsilon \Delta U, w) + (\beta \cdot \nabla U, w) + (cU, w) = (f, w) \qquad , \forall w \in \phi \tag{5}$$

$$(U_t, v + \delta\beta \cdot \nabla v) - (\epsilon \Delta U, v + \delta\beta \cdot \nabla v) + (\beta \cdot \nabla U, v + \delta\beta \cdot \nabla v) + (cU, v + \delta\beta \cdot \nabla v) = (f, v + \delta\beta \cdot \nabla v) \; ; \forall v \in \psi \tag{6}$$

using the distribution of terms on both sides of Equation (6), the second term from the left hand side, we use the integration by parts, we have;

$$(-\Delta \epsilon U, v) = -\int_\Omega \epsilon \Delta U v \, d\Omega = \underbrace{-\epsilon \nabla U v|_\Gamma}_{=0} + \int_\Omega \epsilon \nabla U \nabla v d\Omega = (\epsilon \nabla U, \nabla v). \tag{7}$$

And [11](for linear element-shape function is linear)

$$-\int_\Omega \epsilon \Delta U \delta\beta \cdot \nabla v d\Omega = 0 \tag{8}$$

so that, the Equation (6) we obtain;

$$(U_t, v) + (U_t, \delta\beta \cdot \nabla v) + (\epsilon \nabla U, \nabla v) + (\beta \cdot \nabla U, v) + (\beta \cdot \nabla U, \delta\beta \cdot \nabla v) + (cU, v) + (cU, \delta\beta \cdot \nabla v) = (f, v) + (f, \delta\beta \cdot \nabla v) \quad ; \forall v \in \psi \tag{9}$$

$$(U_t, v + \delta\beta \cdot \nabla v) + a_{PG}(U, v) = (f, v + \delta\beta \cdot \nabla v) \tag{10}$$

where

$$a_{PG}(U, v) = (\epsilon \nabla U, \nabla v) + (\beta \cdot \nabla U + cU, v + \delta\beta \cdot \nabla v) \, . \tag{11}$$

5. The Semi-Discrete and Fully-Discrete PGFE method :

For semi-discrete solution; find $U_h \in \psi_h \subset \psi$ such that

$$\left(U_{t,h}, v\right) + \left(U_{t,h}, \delta\beta \cdot \nabla v\right) + (\epsilon \nabla U_h, \nabla v) + (\beta \cdot \nabla U_h, v) + (\beta \cdot \nabla U_h, \delta\beta \cdot \nabla v) + (cU_h, v) + (cU_h, \delta\beta \cdot \nabla v) = (f, v) + (f, \delta\beta \cdot \nabla v) \quad , \forall v \in \psi_h \tag{12}$$

and, the fully-discrete solution at the time $t_n = n\tau$. The backward Euler PGFE method read as; find $U_h^n \in \psi_h \subset \psi$ such that

$$\left(\frac{U_h^n - U_h^{n-1}}{\tau}, v + \delta\beta\cdot\nabla v\right) + (\epsilon\nabla U_h^n, \nabla v) + (\beta\cdot\nabla U_h^n, v) + (\beta\cdot\nabla U_h^n, \delta\beta\cdot\nabla v) + (cU_h^n, v) + (cU_h^n, \delta\beta\cdot\nabla v) = (f^n, v) + (f^n, \delta\beta\cdot\nabla v) \quad , \forall v \in \psi_h. \tag{13}$$

If $\delta = 0$, we get SG method.

Lemma 5.1: For $U \in \psi$, satisfies

$\big((\beta\cdot\nabla + c)U, U\big) = ((c - \frac{1}{2}\nabla\cdot\beta)U, U).$

Proof:

Using integration by parts, we obtain;

$$\frac{1}{2}\int_\Omega \beta\cdot\nabla U U d\Omega = -\frac{1}{2}\int_\Omega \nabla\cdot(\beta U) U d\Omega \tag{14}$$

and using the product rule

$$= -\frac{1}{2}\int_\Omega (\nabla\cdot\beta) U U d\Omega - \frac{1}{2}\int_\Omega \beta\cdot\nabla U U d\Omega \tag{15}$$

it follows that

$$\int_\Omega \beta\cdot\nabla U U d\Omega = -\frac{1}{2}\int_\Omega (\nabla\cdot\beta) U U d\Omega \tag{16}$$

now

$$\int_\Omega (\beta\cdot\nabla + c) U U d\Omega = \int_\Omega \left(c - \frac{1}{2}\nabla\cdot\beta\right) U U d\Omega \tag{17}$$

the proof is complete ■

Definition 5.1:

Let $U \in \psi$ the solution of Equation (9) and let $U_h \in \psi_h$ the solution of equation (12),(13), the PG norm is defined by [7]

$$\|U_h\|_{PG} = (\epsilon\|\nabla U_h\|_{L^2(\Omega)}^2 + \mu\|U_h\|_{L^2(\Omega)}^2 + \delta\|\beta\cdot\nabla U_h\|_{L^2(\Omega)}^2)^{\frac{1}{2}}.$$

6. Properties of the bilinear form $a_{PG}(U, v)$.

Let ψ be a Hilbert Space with the scalar product $(\cdot,\cdot)_\psi$ and corresponding norm $\|\cdot\|_{PG}$ (the PG norm). Suppose that $a_{PG}(\cdot,\cdot): \psi\times\psi \to \mathbb{R}$ is a bilinear form. We shall prove two properties of the bilinear form, which are essential for establishing the error.

Lemma 6.1: (Continuity)

Let U the exact solution of Equation (9) and v be a test function belong to ψ then $a_{PG}(U, v)$ given by equation (11) is continuous.

Proof:

From Equation (11), we obtain;

$$|a_{PG}(U,v)| = |(\epsilon\nabla U, \nabla v) + (\beta\cdot\nabla U, v) + (cU, v) + (\beta\cdot\nabla U, \delta\beta\cdot\nabla v) + (cU, \delta\beta\cdot\nabla v)| = \sum_{i=1}^{4}\lambda_i . \tag{18}$$

To estimate λ_1 , by Cauchy-Schwartz inequality, we have

$$\lambda_1 = (\epsilon\nabla U, \nabla v) \le \left|\epsilon^{1/2}\right|_{L_\infty}\|\nabla U\|_{L^2(\Omega)}\left|\epsilon^{1/2}\right|_{L_\infty}\|\nabla v\|_{L^2(\Omega)}. \tag{19}$$

To estimate λ_2, by Lemma 5.1, Assumption 2.1 and Cauchy-Schwartz inequality, we obtain;

$$\lambda_2 = (\beta\cdot\nabla U + cU, v) = ((\underbrace{c - \frac{1}{2}\nabla\cdot\beta}_{\mu>0})U, v) \le \left|\mu^{\frac{1}{2}}\right|_{L_\infty}\|U\|_{L^2(\Omega)}\left|\mu^{\frac{1}{2}}\right|_{L_\infty}\|v\|_{L^2(\Omega)}. \tag{20}$$

To estimate λ_3, we have;

$$\lambda_3 = (\beta\cdot\nabla U, \delta\beta\cdot\nabla v) \le \left|\delta^{\frac{1}{2}}\right|_{L_\infty}\|\beta\cdot\nabla U\|_{L^2(\Omega)}\left|\delta^{\frac{1}{2}}\right|_{L_\infty}\|\beta\cdot\nabla v\|_{L^2(\Omega)}. \tag{21}$$

To estimate λ_4, we have;

$$\lambda_4 = (cU, \delta\beta\cdot\nabla v) \le |c|_{L_\infty}\|U\|_{L^2(\Omega)}|\delta|_{L_\infty}\|\beta\cdot\nabla v\|_{L^2(\Omega)}. \tag{22}$$

Substituting $\lambda_1 \cdots \lambda_4$ in Equation (18), we obtain;

$$|a_{PG}(U,v)| \leq \left|\epsilon^{\frac{1}{2}}\right|_{L_\infty} \|\nabla U\|_{L^2(\Omega)} \left|\epsilon^{\frac{1}{2}}\right|_{L_\infty} \|\nabla v\|_{L^2(\Omega)} + \left|\mu^{\frac{1}{2}}\right|_{L_\infty} \|U\|_{L^2(\Omega)} \left|\mu^{\frac{1}{2}}\right|_{L_\infty} \|v\|_{L^2(\Omega)} + \left|\delta^{\frac{1}{2}}\right|_{L_\infty} \|\beta\cdot\nabla U\|_{L^2(\Omega)} \left|\delta^{\frac{1}{2}}\right|_{L_\infty} \|\beta\cdot\nabla v\|_{L^2(\Omega)} + |c|_{L_\infty}\|U\|_{L^2(\Omega)}|\delta|_{L_\infty}\|\beta\cdot\nabla v\|_{L^2(\Omega)}$$

$$\leq [\,|\epsilon|_{L_\infty}\|\nabla U\|^2_{L^2(\Omega)} + \left(|\mu|_{L_\infty} + |c|_{L_\infty}\right)\|U\|^2_{L^2(\Omega)} + |\delta|_{L_\infty}\|\beta\cdot\nabla U\|^2_{L^2(\Omega)}]^{1/2} \times [\,|\epsilon|_{L_\infty}\|\nabla v\|^2_{L^2(\Omega)} + |\mu|_{L_\infty}\|v\|^2_{L^2(\Omega)} + 2|\delta|_{L_\infty}\|\beta\cdot\nabla v\|^2_{L^2(\Omega)}]^{1/2} \tag{23}$$

since $|c|_{L_\infty} \leq |\mu|_{L_\infty}$, then (23), we get;

$$\leq [|\epsilon|_{L_\infty}\|\nabla U\|^2_{L^2(\Omega)} + 2|\mu|_{L_\infty}\|U\|^2_{L^2(\Omega)} + |\delta|_{L_\infty}\|\beta\cdot\nabla U\|^2_{L^2(\Omega)}]^{1/2} \times [\,|\epsilon|_{L_\infty}\|\nabla v\|^2_{L^2(\Omega)} + |\mu|_{L_\infty}\|v\|^2_{L^2(\Omega)} + |\delta^2|_{L_\infty}\|\beta\cdot\nabla v\|^2_{L^2(\Omega)}]^{1/2}$$

$$\leq k\,\|U\|_{PG}\|v\|_{PG} \tag{24}$$

hence, we obtain;

$$|a_{PG}(U,v)| \leq k\,\|U\|_{PG}\|v\|_{PG} \tag{25}$$

where $k = \max\{|\epsilon|_{L_\infty}, |\mu|_{L_\infty}, |\delta|_{L_\infty}\}$.

the proof is complete ■

Lemma 6.2: ($\psi - elliptic$)

Let assumption 2.1 is satisfied. If the PG parameter is chosen;

$$0 < \delta \leq \frac{\mu}{2|c|^2_\infty} \tag{26}$$

Then the bilinear form $a_{PG}(\cdot,\cdot)$ associated with the PGFE method satisfied

$$a_{PG}(U,U) \geq \frac{1}{2}\|U\|^2_{PG} \tag{27}$$

Proof:

Let $v = U$ in Equation (11), we obtain;

$$a_{PG}(U,U) = (\epsilon\nabla U,\nabla U) + (\beta\cdot\nabla U,U) + (cU,U) + (\beta\cdot\nabla U + cU, \delta\beta\cdot\nabla U)$$

from assumption 2.1, we obtain;

$$= \epsilon(\nabla U,\nabla U) + (\underbrace{(c - \tfrac{1}{2}\nabla\cdot\beta)}_{\mu>0}U,U) + (\beta\cdot\nabla U, \delta\beta\cdot\nabla U) + (cU,\delta\beta\cdot\nabla U)$$

$$= \epsilon\,\|\nabla U\|^2_{L^2(\Omega)} + \mu\|U\|^2_{L^2(\Omega)} + \delta\|\beta\cdot\nabla U\|^2_{L^2(\Omega)} + (cU,\delta\beta\cdot\nabla U) \tag{28}$$

from definition 5.1 and [15];

$$\geq \|U\|^2_{PG} - |(cU,\delta\beta\cdot\nabla U)|. \tag{29}$$

The second term from (29), by Cauchy-Schwartz inequality, we have;

$$|(cU,\delta\beta\cdot\nabla U)| \leq \delta^{1/2}|c|_{L_\infty}\|U\|_{L^2(\Omega)}\delta^{1/2}\|\beta\cdot\nabla U\|_{L^2(\Omega)} \tag{30}$$

from the condition (26) , we get;

$$\leq \left(\frac{\sqrt{\mu}}{\sqrt{2}\,|c|_{L_\infty}}|c|_{L_\infty}\|U\|_{L^2(\Omega)}\,\delta^{\frac{1}{2}}\|\beta\cdot\nabla U\|_{L^2(\Omega)}\right. = \frac{\sqrt{\mu}}{\sqrt{2}}\|U\|_{L^2(\Omega)}\delta^{\frac{1}{2}}\|\beta\cdot\nabla U\|_{L^2(\Omega)} \tag{31}$$

by Young's-inequality;

$$\leq \frac{\mu}{2}\|U\|^2_{L^2(\Omega)} + \frac{1}{4}\delta\|\beta\cdot\nabla U\|^2_{L^2(\Omega)} \leq \frac{1}{2}\,\|U\|^2_{PG}\,. \tag{32}$$

From (29), we have;

$$a_{PG}(U,U) \geq \|U\|^2_{PG} - \frac{1}{2}\|U\|^2_{PG} = \frac{1}{2}\|U\|^2_{PG}\,. \tag{34}$$

The proof is complete ■

6.3: Stability for PGFE method.

In this section, we discuss the PG scheme's stability given in Equation (10), this means a suitable norm for a solution can be estimated with the problem data.

Theorem 6.3.1:
Let assumption 2.1 and lemma 6.2 be fulfilled. The solution of (10) satisfies, there exist a constant $\gamma > 0$ dependent on μ, δ such that

$$\|U_h\|^2_{L^2(\Omega)} + \|U_h\|^2_{L^2(0,T;PG)} \leq 2\gamma \|f\|^2_{L^2(0,T;\Omega)} + \frac{\mu}{2}\|U_h\|^2_{L^2(0,T;\Omega)} - 2\|U_{t,h}\|^2_{L^2(0,T;\Omega)}$$

Proof:
The Equation (10), let $v = U_h$, we obtain;

$$(U_{t,h}, U_h) + (U_{t,h}, \delta\beta \cdot \nabla U_h) + a_{PG}(U_h, U_h) = (f, U_h) + (f, \delta\beta \cdot \nabla U_h)$$
$$\sum_{i=1}^{3} S_i = \sum_{j=4}^{5} S_j \, . \tag{35}$$

To estimate $S_1 \cdots S_5$;

$$S_1 = (U_{t,h}, U_h) = \frac{1}{2}\frac{d}{dt}\|U_h\|^2_{L^2(\Omega)} \, . \tag{36}$$

By Cauchy and young's inequality, we obtain;

$$S_2 = (U_{t,h}, \delta\beta \cdot \nabla U_h) \leq \|U_{t,h}\|_{L^2(\Omega)} \|\delta\beta \cdot \nabla U_h\|_{L^2(\Omega)} \leq \|U_{t,h}\|^2_{L^2(\Omega)} + \frac{1}{4}\|\delta\beta \cdot \nabla U_h\|^2_{L^2(\Omega)} \, . \tag{37}$$

From lemma 6.2, we obtains;

$$S_3 = a_{PG}(U_h, U_h) \geq \frac{1}{2}\|U_h\|^2_{PG} \, . \tag{38}$$

By Cauchy-Schwartz inequality, we obtain;

$$S_4 = (f, U_h) \leq \frac{1}{\sqrt{\mu}}\|f\|_{L^2(\Omega)} \sqrt{\mu}\, \|U_h\|_{L^2(\Omega)} \tag{39}$$

by young's-inequality then (39), we have;

$$\leq \frac{1}{\mu}\|f\|^2_{L^2(\Omega)} + \frac{\mu}{4}\|U_h\|^2_{L^2(\Omega)} . \tag{40}$$

by Cauchy-Schwartz inequality, we obtain;

$$S_5 = (f, \delta\beta \cdot \nabla U_h)$$
$$\leq \|\delta^{1/2} f\|_{L^2(\Omega)} \left\|\delta^{\frac{1}{2}}(\beta \cdot \nabla U_h)\right\|_{L^2(\Omega)} \tag{41}$$

by Young's- inequality on (41) , we get;

$$\leq \delta\|f\|^2_{L^2(\Omega)} + \frac{1}{4}\delta\|\beta \cdot \nabla U_h\|^2_{L^2(\Omega)} . \tag{42}$$

Substituting $S_1 \cdots S_5$ in (35), we obtain;

$$\frac{1}{2}\frac{d}{dt}\|U_h\|^2_{L^2(\Omega)} + \frac{1}{2}\|U_h\|^2_{PG} \leq \frac{1}{\mu}\|f\|^2_{L^2(\Omega)} + \frac{\mu}{4}\|U_h\|^2_{L^2(\Omega)} + \delta\|f\|^2_{L^2(\Omega)} - \|U_{t,h}\|^2_{L^2(\Omega)} \tag{43}$$

multiply (43) by 2 and integral both side from (0 to t), we obtain;

$$\|U_h(t)\|^2_{L^2(\Omega)} - \|U_h(0)\|^2_{L^2(\Omega)} + \int_0^t \|U_h\|^2_{PG} dt \leq \int_0^t [2\gamma\|f\|^2_{L^2(\Omega)} + (\frac{\mu}{2}\|U_h\|^2_{L^2(\Omega)} - 2\|U_{t,h}\|^2_{L^2(\Omega)})dt \tag{44}$$

where $\gamma = \frac{1}{\mu} + \delta$, and $U^0 = 0$ (boundary), we obtain;

$$\|U_h(t)\|^2_{L^2(\Omega)} + \|U_h\|^2_{L^2(0,T;PG)} \leq 2\gamma\|f\|^2_{L^2(0,T;\Omega)} + \frac{\mu}{2}\|U_h\|^2_{L^2(0,T;\Omega)} - 2\|U_{t,h}\|^2_{L^2(0,T;\Omega)}$$

the proof is complete ■

7. The Error Analysis of Semi-Discrete PGFE method:

In this section, we derive error estimates for the semi-discrete PGFE method in the L^2-norm .

Theorem 7.1: Let U be the solution of equation (1), U_h be the solution of Equation (12) and $U \in L^2(H^1), U_t \in L^2(0,T;\, H^1)$ then there exist a constant C such that

$\|U - U_h\|_{L^2(\Omega)} \leq$

$$Ch^2[\|U\|_{L^2(H^1)} + \|U_t\|_{L^2(0,T;H^1)}] \qquad (45)$$

Proof:

Let IU the interpolate of U, and

$$E = U - U_h = \underbrace{(U - IU)}_{\theta} - \underbrace{(U_h - IU)}_{\xi}$$
$$= \theta - \xi \qquad (46$$

now

$$\|E\|_{L^2(\Omega)} = \|U - U_h\|_{L^2(\Omega)}$$

by triangle-inequality, we have;

$$\leq \|U - IU\|_{L^2(\Omega)} + \|U_h - IU\|_{L^2(\Omega)}$$
$$\leq \|\theta\|_{L^2(\Omega)} + \|\xi\|_{L^2(\Omega)} \qquad (47)$$

since [9]

$$\|\theta\|_{L^2(\Omega)} \leq Ch^2\|U\|_{L^2(H^1)} \,. \qquad (48)$$

Now

$$(U_t, w) + a_{PG}(U, w) = (f, w) \quad ; \forall w \in \phi \qquad (49)$$

$$(U_{t,h}, w) + a_{PG}(U_h, w) = (f, w) \qquad (50)$$

subtracting (49),(50), we obtain;

$$(U_t - U_{t,h}, w) + a_{PG}(U - U_h, w) = 0 \qquad (51)$$

and

$$((\theta - \xi)_t, w) + a_{PG}((\theta - \xi), w) = 0 \qquad (52)$$
$$(\theta_t, w) + a_{PG}(\theta, w) = (\xi_t, w) + a_{PG}(\xi, w) \qquad (53)$$

since

$$a_{PG}(U - IU, w) = a_{PG}(\theta, w) = 0 \qquad (54)$$

then (53), we have;

$$(\xi_t, w) + a_{PG}(\xi, w) = (\theta_t, w) \qquad (55)$$

let $w = \xi$ then (55), we obtain;

$$(\xi_t, \xi) + a_{PG}(\xi, \xi) = (\theta_t, \xi) \qquad (56)$$

L.H.S of (56) , from lemma 6.2, we obtain;

$$(\xi_t, \xi) = \frac{1}{2}\frac{d}{dt}\|\xi\|^2_{L^2(\Omega)}$$
$$a_{PG}(\xi, \xi) \geq \frac{1}{2}\|\xi\|^2_{PG} \qquad (57)$$

and by Cauchy-Schwartz, young's- inequality in R.H.S of (56), we obtain;

$$(\theta_t, \xi) \leq \frac{\sigma}{2}C^2h^4\|U_t\|^2_{L^2(H^1)} + \frac{1}{2\sigma}\|\xi\|^2_{L^2(\Omega)}$$
$$\leq \frac{\sigma}{2}C^2h^4\|U_t\|^2_{L^2(H^1)} + \frac{1}{2\sigma}\|\xi\|^2_{PG} \qquad (58)$$

from (56), (57) and (58), we have;

$$\frac{1}{2}\frac{d}{dt}\|\xi\|^2_{L^2(\Omega)} + \left(\frac{1}{2} - \frac{1}{2\sigma}\right)\|\xi\|^2_{PG} \leq \frac{\sigma}{2}C^2h^4\|U_t\|^2_{L^2(H^1)} \qquad (59)$$

let $\sigma = 1$ in (59), we obtain;

$$\frac{d}{dt}\|\xi\|^2_{L^2(\Omega)} \leq C^2h^4\|U_t\|^2_{L^2(H^1)} \qquad (60)$$

integral both sides from (0 to t), we obtain;

$$\|\xi(t)\|^2_{L^2(\Omega)} - \|\xi^0\|^2_{L^2(\Omega)} \leq C^2h^4\|U_t\|^2_{L^2(0,T;H^1)} \qquad (61)$$

since $\xi^0 = 0$ because $(U_h^0 - IU_h^0) = 0$ (boundary), we have;

$$\|\xi(t)\|^2_{L^2(\Omega)} \leq C^2h^4\|U_t\|^2_{L^2(0,T;H^1)}. \qquad (62)$$

Now, from (47), (48) and (62), we obtain;

$$\|U - U_h\|_{L^2(\Omega)} \leq \|\theta\|_{L^2(\Omega)} + \|\xi\|_{L^2(\Omega)}$$
$$\leq Ch^2\|U\|_{L^2(\Omega)} + Ch^2\|U_t\|_{L^2(0,T;H^1)}$$

the proof is complete ■

8. The Error Analysis of Fully-Discrete PGFE method:

In this section, we derive error estimates for the fully-discrete PGFE method in the L^2-norm.

Theorem 8.1: Let U^n and U_h^n be the solution of (9) and (13) respectively, then there exists a constant $\mathcal{C}$ independent of h such that

$$\|U^n - U_h^n\|_{L^2(\Omega)}$$

$$\leq C\left[h\|U^n\|_{H^1(\Omega)} + h\|U_t\|_{L^2\left(0,T;H^1(\Omega)\right)} + \tau\sum_{n=1}^{N}\|U^n\|_{H^{r+1}(\Omega)}\right]. \quad (63)$$

Proof:

Let PU^n be L^2 −projection of U^n on to ϕ, and

$$E^n = U^n - U_h^n = \underbrace{(U^n - PU^n)}_{\theta^n} - \underbrace{(U_h^n - PU^n)}_{\xi^n} = \theta^n - \xi^n \quad (64)$$

by Triangle-inequality, we obtain;

$$\|U^n - U_h^n\|_{L^2(\Omega)} \leq \|U^n - PU^n\|_{L^2(\Omega)} + \|U_h^n - PU^n\|_{L^2(\Omega)} \leq \|\theta^n\|_{L^2(\Omega)} + \|\xi^n\|_{L^2(\Omega)}. \quad (65)$$

From [8]

$$\|\theta^n\|_{L^2(\Omega)} \leq Ch\|U^n\|_{H^1(\Omega)}. \quad (66)$$

Now from the equations;

$$\left(\frac{U^n - U^{n-1}}{\tau}, w\right) + a_{PG}(U^n, w) = (f^n, w) \quad , \forall w \in \phi \quad (67)$$

$$\left(\frac{U_h^n - U_h^{n-1}}{\tau}, w\right) + a_{PG}(U_h^n, w) = (f^n, w) \quad (68)$$

by subtracting (67) from (68), we obtain;

$$\frac{1}{\tau}\left((U^n - U_h^n) - (U^{n-1} - U_h^{n-1}), w\right) + a_{PG}\left((U^n - U_h^n), w\right) = 0 \quad (69)$$

$$\frac{1}{\tau}\left((\theta^n - \xi^n) - (\theta^{n-1} - \xi^{n-1}), w\right) + a_{PG}(\theta^n - \xi^n, w) = 0 \quad (70)$$

$$(\theta^n - \theta^{n-1}, w) + \tau a_{PG}(\theta^n, w) = (\xi^n - \xi^{n-1}, w) + \tau a_{PG}(\xi^n, w) \quad (71)$$

let $w = \xi^n$

$$(\xi^n, \xi^n) + \tau a_{PG}(\xi^n, \xi^n) = (\xi^{n-1}, \xi^n) + (\theta^n - \theta^{n-1}, \xi^n) + \tau a_{PG}(\theta^n, \xi^n) \quad (72)$$

from Lemma 6.2, we obtain;

$$\tau a_{PG}(\xi^n, \xi^n) \geq \frac{\tau}{2}\|\xi^n\|_{PG}^2 \quad (73)$$

by Cauchy-Schwartz and Young's –inequality, we have;

$$(\xi^{n-1}, \xi^n) \leq \|\xi^{n-1}\|_{L^2(\Omega)}\|\xi^n\|_{L^2(\Omega)} \leq \frac{1}{2}\|\xi^{n-1}\|_{L^2(\Omega)}^2 + \frac{1}{2}\|\xi^n\|_{L^2(\Omega)}^2 \quad (74)$$

and by Cauchy-Schwartz and Young's-inequality, we have;

$$(\theta^n - \theta^{n-1}, \xi^n) \leq \|\theta^n - \theta^{n-1}\|_{L^2(\Omega)}\|\xi^n\|_{L^2(\Omega)} \leq \frac{1}{2\sigma_1}\|\theta^n - \theta^{n-1}\|_{L^2(\Omega)}^2 + \frac{\sigma_1}{2}\|\xi^n\|_{L^2(\Omega)}^2 \leq \frac{1}{2\sigma_1}\|\theta^n - \theta^{n-1}\|_{L^2(\Omega)}^2 + \frac{\sigma_1}{2}\|\xi^n\|_{PG}^2 \quad (75)$$

from Lemma 6.1 and by Young's-inequality, we have;

$$\tau a_{PG}(\theta^n, \xi^n) \leq \tau C_1\|\theta^n\|_{PG}\|\xi^n\|_{PG} \leq \frac{(\tau C_1)^2}{2\sigma_2}\|\theta^n\|_{PG}^2 + \frac{\sigma_2}{2}\|\xi^n\|_{PG}^2 \quad (76)$$

since [7]

$$\|\theta^n\|_{PG}^2 \leq Ch^{r+\frac{1}{2}}\|U^n\|_{H^{r+1}(\Omega)}^2 \quad (77)$$

then

$$\tau a_{PG}(\theta^n, \xi^n) \leq \frac{(\tau C_1)^2}{2\sigma_2}Ch^{r+\frac{1}{2}}\|U^n\|_{H^{r+1}(\Omega)}^2 + \frac{\sigma_2}{2}\|\xi^n\|_{PG}^2. \quad (78)$$

Substituting (72)-(78) in (71), we obtain;

$$\|\xi^n\|_{L^2(\Omega)}^2 + \frac{\tau}{2}\|\xi^n\|_{PG}^2 \leq \frac{1}{2}\|\xi^{n-1}\|_{L^2(\Omega)}^2 + \frac{1}{2}\|\xi^n\|_{L^2(\Omega)}^2 + \frac{1}{2\sigma_1}\|\theta^n - \theta^{n-1}\|_{L^2(\Omega)}^2 + \frac{\sigma_1}{2}\|\xi^n\|_{PG}^2 + \frac{(\tau C_1)^2}{2\sigma_2}Ch^{r+\frac{1}{2}}\|U^n\|_{H^{r+1}(\Omega)}^2 + \frac{\sigma_2}{2}\|\xi^n\|_{PG}^2 \quad (79)$$

putting $\frac{\tau}{2} = \sigma_1 = \sigma_2 = constant$ and multiply by 2, we obtain;

$$\|\xi^n\|^2_{L^2(\Omega)} \le \|\xi^{n-1}\|^2_{L^2(\Omega)} + \frac{2}{\tau}\|\theta^n - \theta^{n-1}\|^2_{L^2(\Omega)} + 2\tau\, C_1^2 C h^{r+\frac{1}{2}}\|U^n\|^2_{H^{r+1}(\Omega)} \quad (80)$$

summing both sides for $n = 1\ to\ N$, we obtain;

$$\|\xi^N\|^2_{L^2(\Omega)} \le \|\xi^0\|^2_{L^2(\Omega)} + C_2\sum_{n=1}^{N}[\frac{1}{\tau}\|\theta^n - \theta^{n-1}\|^2_{L^2(\Omega)} + \tau\|U^n\|^2_{H^{r+1}(\Omega)}] \quad (81)$$

where $C_2 = 2 + 2C_1^2 C h^{r+\frac{1}{2}}$

since $\xi^0 = 0$ and there exist, $1 \le n^\alpha \le N$, such that

$$\left\|\xi^{n^\alpha}\right\|_{L^2(\Omega)} = \|\xi^n\|_{L^2(\Omega)} \quad (82)$$

then

$$\|\xi^n\|^2_{L^2(\Omega)} \le C_2[\frac{1}{\tau}\sum_{n=1}^{N}\|\theta^n - \theta^{n-1}\|^2_{L^2(\Omega)} + \tau\sum_{n=1}^{N}\|U^n\|^2_{H^{r+1}(\Omega)}] \quad (83)$$

take the first term in R.H.S

since

$$\theta^n - \theta^{n-1} = \int_{t_{n-1}}^{t_n}\theta_t dt$$

then

$$\|\theta^n - \theta^{n-1}\|_{L^2(\Omega)} \le \int_{t_{n-1}}^{t_n}\|\theta_t\|_{L^2(\Omega)}dt \quad (85)$$

and

$$\|\theta^n - \theta^{n-1}\|^2_{L^2(\Omega)} \le (\int_{t_{n-1}}^{t_n}\|\theta_t\|^2_{L^2(\Omega)}dt)^2 \le \tau^2[\int_{t_{n-1}}^{t_n}\|\theta_t\|_{L^2(\Omega)}\frac{dt}{\tau}]^2 \quad (86)$$

by Jensen's-inequality [3], we get;

$$\|\theta^n - \theta^{n-1}\|^2_{L^2(\Omega)} \le \tau^2\int_{t_{n-1}}^{t_n}\|\theta_t\|^2_{L^2(\Omega)}\frac{dt}{\tau} \le \tau\int_{t_{n-1}}^{t_n}\|\theta_t\|^2_{L^2(\Omega)}dt \quad (87)$$

and

$$\frac{1}{\tau}\sum_{n=1}^{N}\|\theta^n - \theta^{n-1}\|^2_{L^2(\Omega)} \le \int_0^T\|\theta_t\|^2_{L^2(\Omega)}dt \quad (88)$$

from (66) then (88), we get;

$$\le C_3h^2\int_0^T\|U_t\|^2_{H^1(\Omega)} = C_3h^2\|U_t\|^2_{L^2(0,T;H^1(\Omega))} \quad (89)$$

substituting (89) in (83), we obtain;

$$\|\xi^n\|^2_{L^2(\Omega)} \le C_2[C_3h^2\|U_t\|^2_{L^2(0,T;H^1(\Omega))} + \tau\sum_{n=1}^{N}\|U^n\|^2_{H^{r+1}}] \quad (90)$$

substituting (66) and (90) in (65), we obtain;

$$\|U^n - U_h^n\|_{L^2(\Omega)} \le C[h\|U^n\|_{H^1(\Omega)} + h\|U_t\|_{L^2(0,T;H^1(\Omega))} + \tau\sum_{n=1}^{N}\|U^n\|_{H^{r+1}(\Omega)}]$$

The proof is complete ■

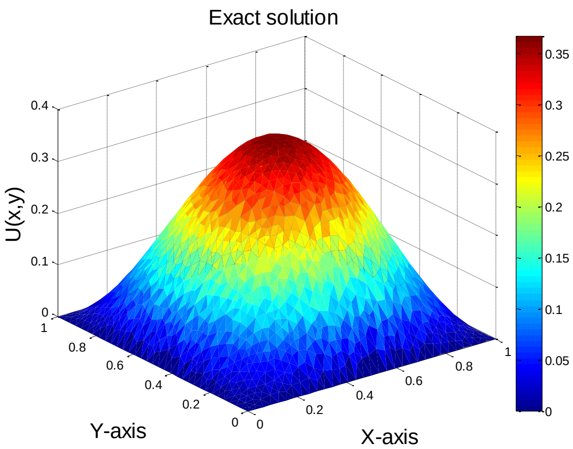
9. Numerical Results:

9.1: The Unsteady-State PGFE method:

Consider the problem(1) with the exact solution [16]

$$U(x,y,t) = \sin\pi x\sin\pi y\, e^{-t} \quad ,t > 0 \quad (91)$$

The initial condition is obtained from the exact solution with $I = (0, T]$ where $T = 1, \tau = 0.1$ $, \epsilon = 10^{-a},\ a = 0, 1, 2, 3, 4, 5,\ \beta = (10,10)^{T}$, $c = 1, h = 2^{-1}, 2^{-2}, 2^{-3}, 2^{-4}, 2^{-5}$.The exact solution of this problem is presented in Figure (1) and the approximate solution is representative in Figure (2)....(13), all Figures are in condition $h = 2^{-5}$.



Figure(1): The exact solution $U(x, y, t) = sin\pi x\ sin\pi y\ e^{-t}$ *for the CDR Problem in equation(1).*

Table(1): $L^2 - error, L_\infty - error\ and\ H^1 - error$ *for the CDR problem in(1) by unsteady-state SGFE method.*

ϵ	$L^2 - error$	$L_\infty - error$	$H^1 - error$
1	2.8103E-4	1.1000E-3	1.6000E-3
10^{-1}	3.8589E-4	1.7000E-3	8.9468E-4
10^{-2}	4.2689E-4	3.7000E-3	7.4727E-4
10^{-3}	6.0444E-4	5.4000E-3	7.3633E-4
10^{-4}	9.0000E-3	1.1660E-1	5.4000E-3
10^{-5}	3.9600E-2	5.1850E-1	1.4400E-2

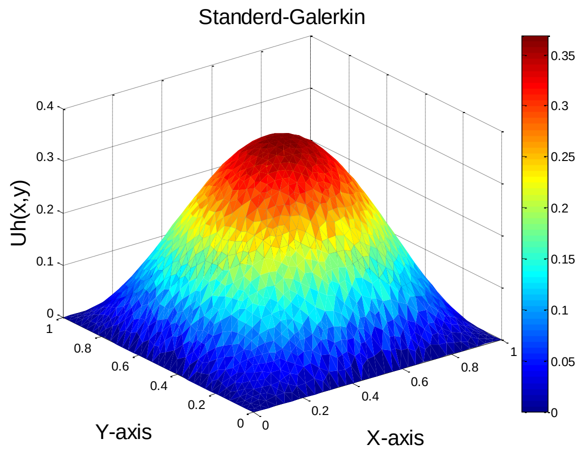


Figure (2): The approximate solution for the CDR problem (1) by SGFE method with $\epsilon = 1,\ \beta = (10,10)^{T},\ c = 1,\ h = 2^{-5}$ $, \tau = 0.1$

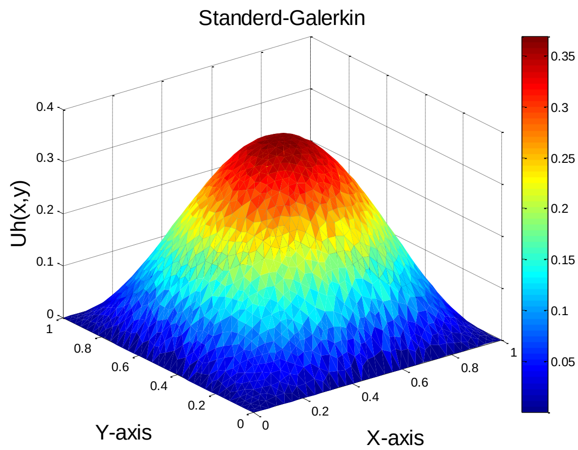
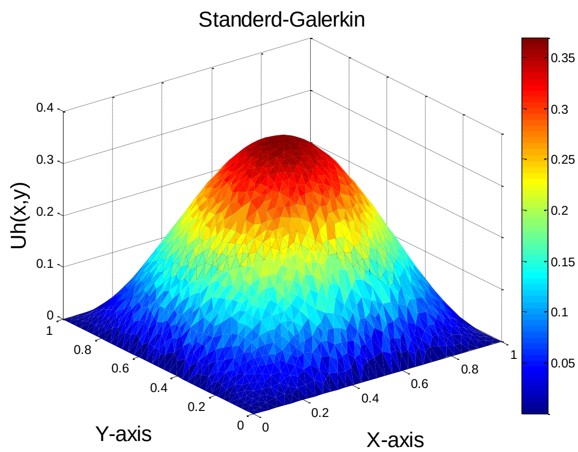
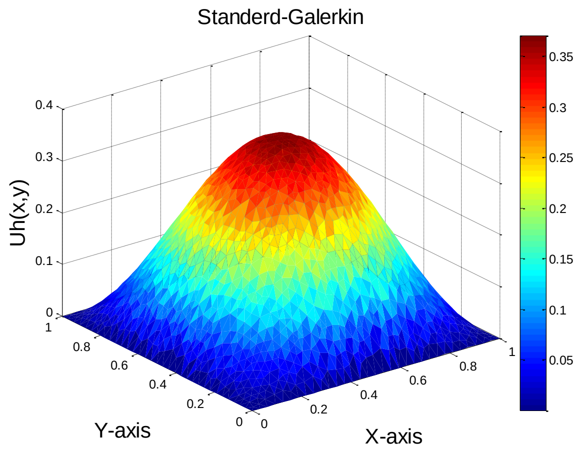


Figure (3): The approximate *solution for the CDR problem (1) by SGFE method with* $\epsilon = 10^{-1}, \beta = (10,10)^{T}, c = 1, h = 2^{-5}$ $, \tau = 0.1$



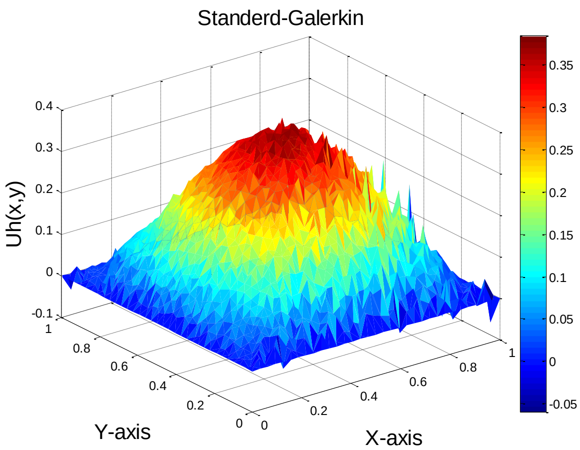
Figure(4): The approximate solution for the CDR problem (1) by SGFE method with $\epsilon = 10^{-2},\ \beta = (10,10)^{T}, c = 1, h = 2^{-5}$ $, \tau = 0.1$



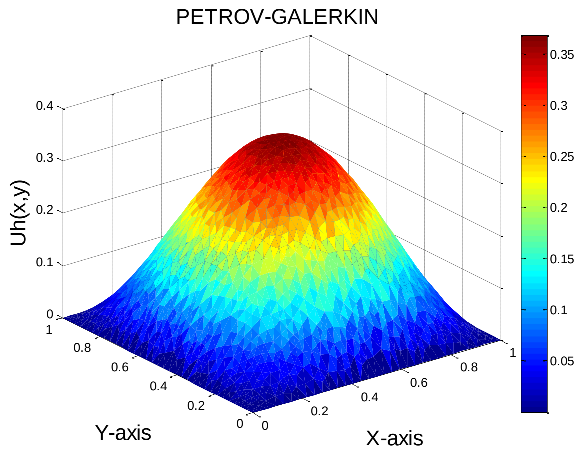
Figure(5): The approximate solution for the CDR problem (1) by SGFE method with $\epsilon = 10^{-3}, \beta = (10,10)^T, c = 1, h = 2^{-5}$ $, \tau = 0.1$

Table(2): $L^2 - error, L_\infty - error$ *and* $H^1 - error$ *for the CDR problem in (1) by unsteady-state PGFE method.*

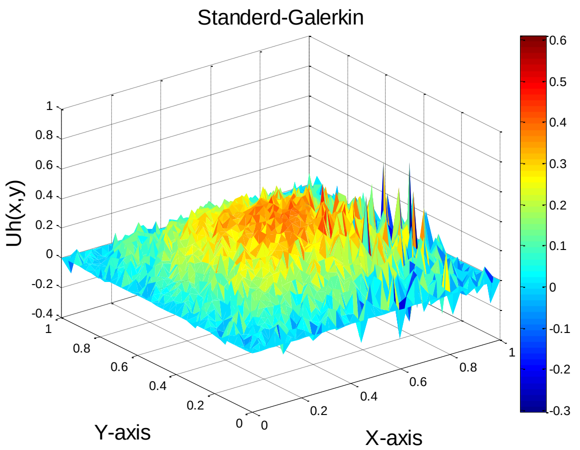
ϵ	$L^2 - error$	$L_\infty - error$	$H^1 - error$
1	$1.2819E-4$	$7.0891E-4$	$1.5000E-3$
10^{-1}	$2.7086E-4$	$1.3000E-3$	$9.2420E-4$
10^{-2}	$3.4210E-4$	$2.8000E-3$	$8.5172E-4$
10^{-3}	$3.9325E-4$	$3.9000E-3$	$8.2575E-4$
10^{-4}	$4.0948E-4$	$4.1000E-3$	$8.2199E-4$
10^{-5}	$4.1153E-4$	$4.2000E-3$	$8.2156E-4$



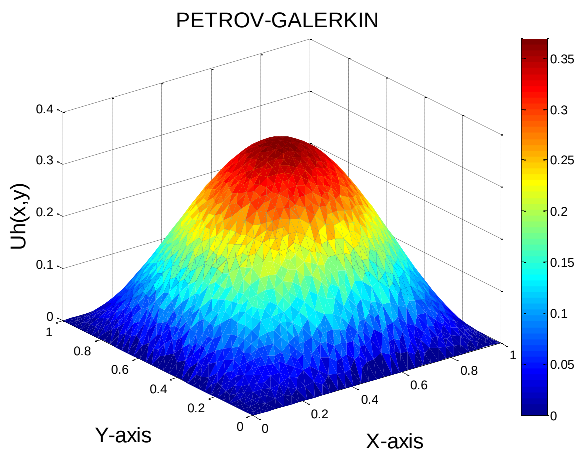
Figure(6): The approximate solution for the CDR problem (1) by SGFE method with $\epsilon = 10^{-4}, \beta = (10,10)^T, c = 1, h = 2^{-5}$ $, \tau = 0.1$



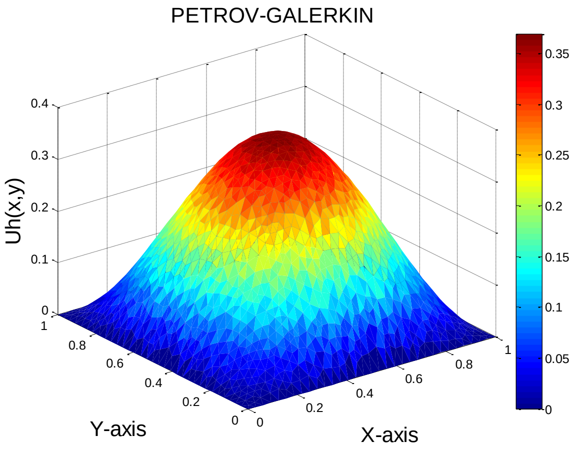
Figure(8): The approximate solution for the CDR problem (1) by PGFE method with $\epsilon = 1, \beta = (10,10)^T, c = 1, h = 2^{-5},$ $\tau = 0.1$



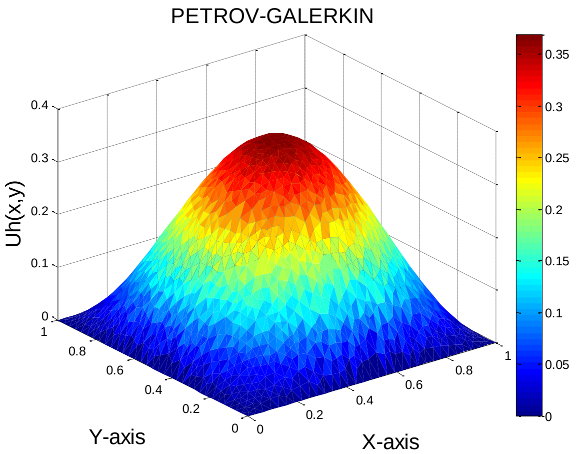
Figure(7); The approximate solution for the CDR problem (1) by SGFE method with $\epsilon = 10^{-5}, \ \beta = (10,10)^T, \ c = 1, \ h = 2^{-5}$ $, \tau = 0.1$



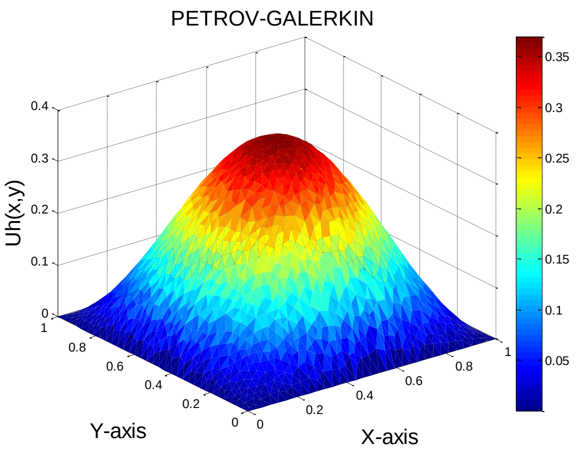
Figure(9): The approximate solution for the CDR problem (1) by PGFE method with $\epsilon = 10^{-1}, \beta = (10,10)^T, c = 1, h = 2^{-5}$ $, \tau = 0.1$



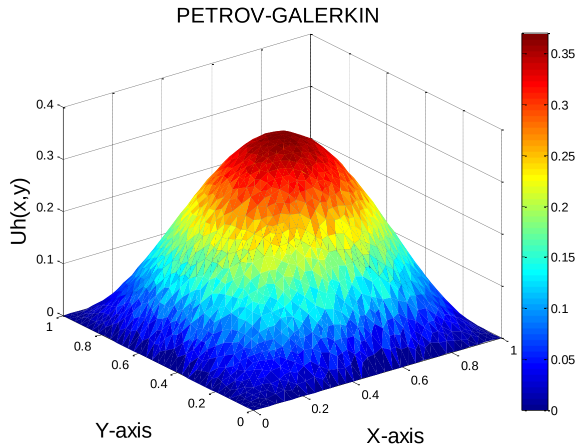
Figure(10): The approximate solution for the CDR problem (1) by PGFE method with $\epsilon = 10^{-2}, \beta = (10,10)^T, c = 1, h = 2^{-5}$ $, \tau = 0.1$



Figure(11): The approximate solution for the CDR problem (1) by PGFE method with $\epsilon = 10^{-3}, \beta = (10,10)^T, c = 1, h = 2^{-5}$, $\tau = 0.1$



Figure(12): The approximate solution for the CDR problem(1) by PGFE method with $\epsilon = 10^{-4}, \beta = (10,10)^T, c = 1, h = 2^{-5}$, $\tau = 0.1$



Figure(13): The approximate solution for the CDR problem (1) by PGFE method with $\epsilon = 10^{-5}, \beta = (10,10)^T, c = 1, h = 2^{-5}$ $, \tau = 0.1$

9.2: The Steady-State PGFE method.

Consider the steady-state for CDR problem;

$$-\epsilon \Delta U + \beta \cdot \nabla U + cU = f \;\; in \;\; \Omega = [0,1]^2 \tag{92}$$

with boundary condition $U = 0 \; in \; \Gamma$ (the boundary of Ω), where ϵ is a diffusion parameter, β is known convection filed, c is a reaction parameter, f is a load vector.

The exact solution is [12]

$$U(x,y) = xy(1-x)(1-y) \tag{93}$$

with

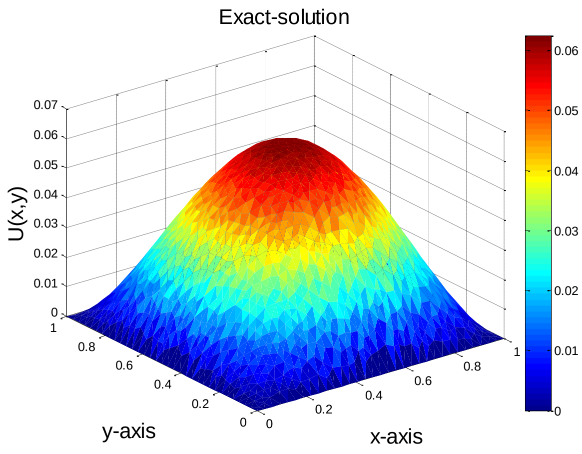
$\epsilon = \; 10^{-a}, a = 0, 1, 2, 3, 4, 5$

$\beta = (10,10)^T$,

$c = 1$,

$h = 2^{-1}, 2^{-2}, \; 2^{-3}, \; 2^{-4}, \; 2^{-5}$.

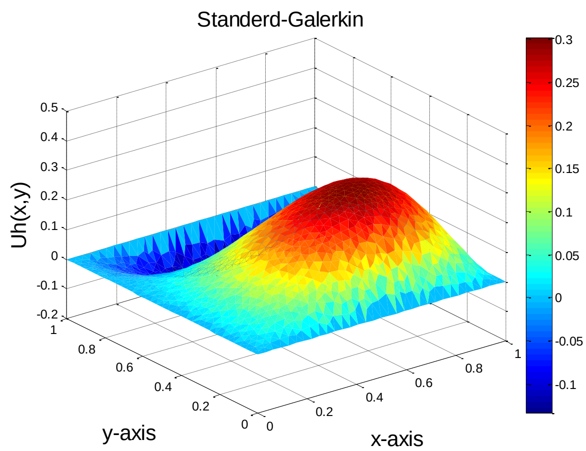
The exact solution of this problem is presented in Figure (14) and the approximate solution is representative in Figure (15) to Figure (26), all Figures are in condition $h = 2^{-5}$.



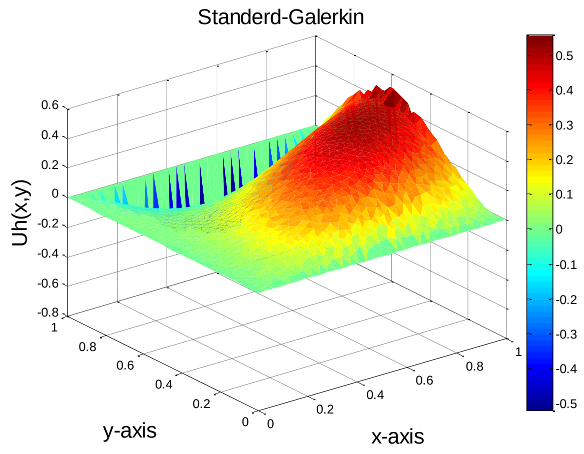
Figure(14): The exact solution $U(x,y) = xy(1-x)(1-y)$ *for the CDR Problem in (92) .*

Table (3): $L^2 - error, L_\infty - error$ *and* $H^1 - error$ *for the CDR problem in (92) using steady-state SGFE method*

ϵ	$L^2 - error$	$L_\infty - error$	$H^1 - error$
1	0.0670	0.2517	0.8220
10^{-1}	0.1338	0.5479	1.0293
10^{-2}	0.1582	1.3319	1.0627
10^{-3}	0.2548	2.2825	1.0475
10^{-4}	0.5797	4.0044	0.8692
10^{-5}	0.8258	4.9507	0.3930



Fiqure (15): The approximate solution for the CDR problem (92) by SGFE method with $\epsilon = 1, \beta = (10,10)^T, c = 1, h = 2^{-5}$.



Fiqure (16): The approximate solution for the CDR problem(92) by SGFE method with $\epsilon = 10^{-1}, \beta = (10,10)^T, c = 1, h = 2^{-5}$

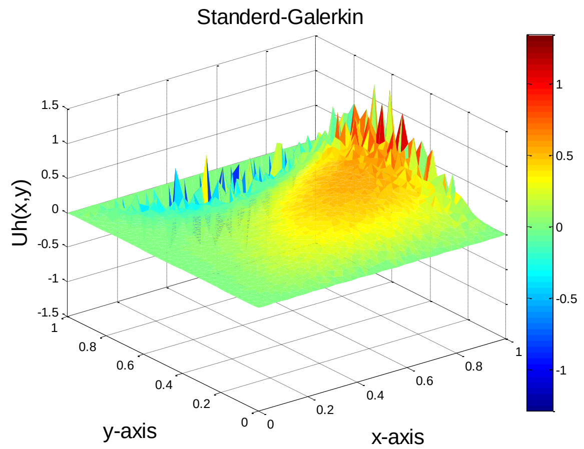


Figure (17): The approximate solution for the CDR problem (92) by SGFE method with $\epsilon = 10^{-2}, \beta = (10,10)^T, c = 1, h = 2^{-5}$

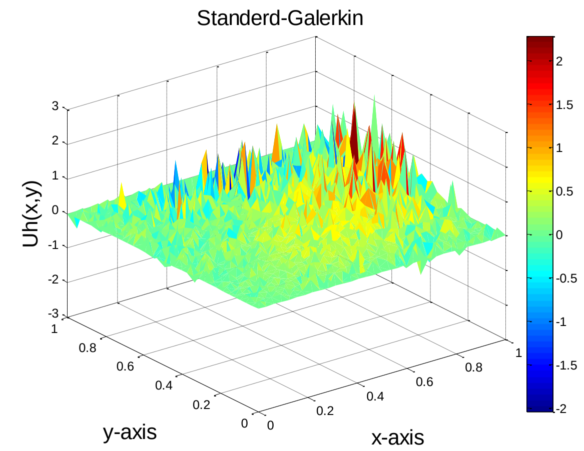


Figure (18): The approximate solution for the CDR problem (92) by SGFE method with $\epsilon = 10^{-3}, \beta = (10,10)^T, c = 0, h = 2^{-5}$

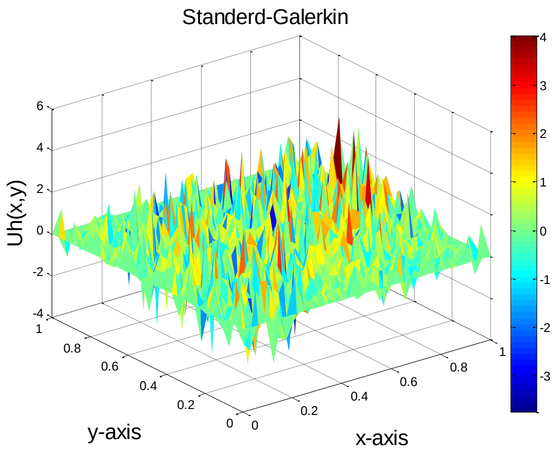
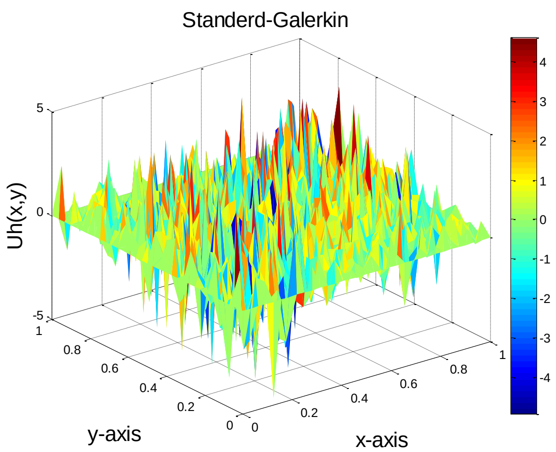


Figure (19): The approximate solution for the CDR problem (92) by SGFE method with $\epsilon = 10^{-4}, \beta = (10,10)^T, c = 1, h = 2^{-5}$



Figure(20): The approximate solution for the CDR problem (92) by SGFE method with $\epsilon = 10^{-5}, \beta = (10,10), c = 1, h = 2^{-5}$.

Table (4): $L^2 - error$, $L_\infty - error$ *and* $H^1 - error$ *for the CDR problem in (92) using steady-state PGFE method.*

ϵ	L^2-error	L_∞-error	H^1 -error
1	$1.3915E - 5$	$7.9170E - 5$	$8.2928E - 4$
10^{-1}	$2.3033E - 5$	$1.4894E - 4$	$4.0818E - 4$
10^{-2}	$3.3485E - 5$	$2.4274E - 4$	$3.0550E - 4$
10^{-3}	$5.6300E - 5$	$5.3939E - 4$	$2.2835E - 4$
10^{-4}	$9.2239E - 5$	$6.7236E - 4$	$1.3151E - 4$
10^{-5}	$1.0411E - 4$	$7.9251E - 4$	$4.7698E - 5$

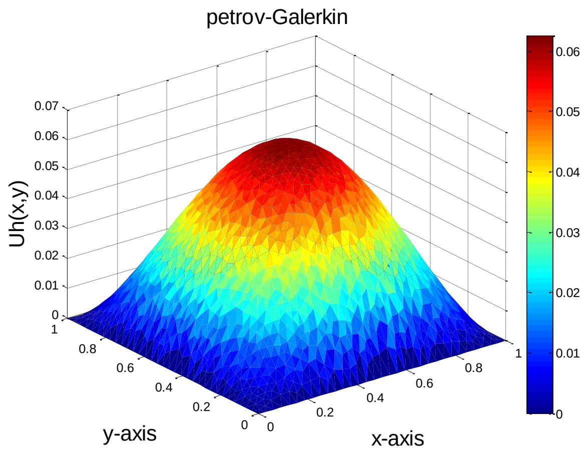


Figure (21): The approximate solution for the CDR problem (92) by PGFE method with $\epsilon = 1, \beta = (10,10)^T, c = 1, h = 2^{-5}$

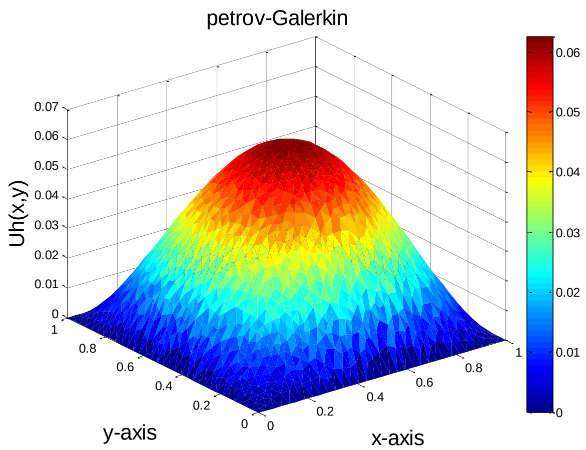


Figure (22): The approximate solution for the CDR problem (92) by PGFE method with $\epsilon = 10^{-1}, \beta = (10,10)^T, c = 1, h = 2^{-5}$

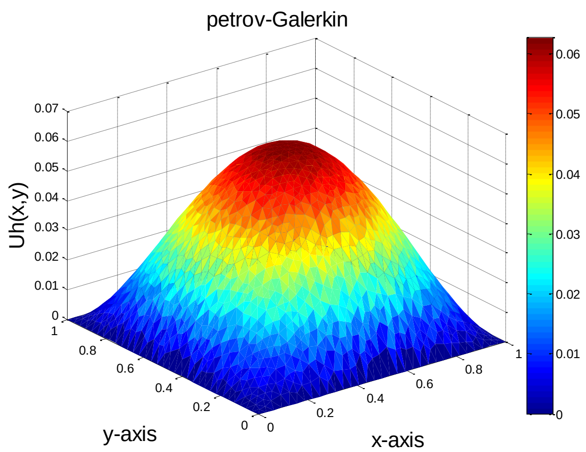


Figure (23): The approximate solution for the CDR problem (92) by PGFE method with $\epsilon = 10^{-2}, \beta = (10,10)^T, c = 1, h = 2^{-5}$.

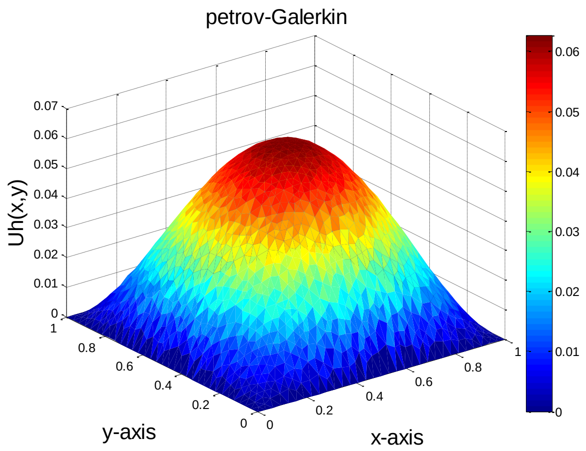


Figure (24): The approximate solution for the CDR problem (92) by PGFE method with $\epsilon = 10^{-3}, \beta = (10,10)^T, c = 1, h = 2^{-5}$

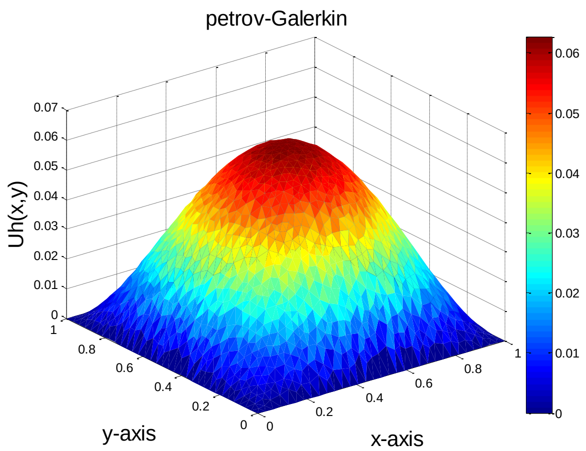


Figure (25):The approximate solution for the CDR problem (92) by PGFE method with $\epsilon = 10^{-4}, \beta = (10,10)^T, c = 1, h = 2^{-5}$

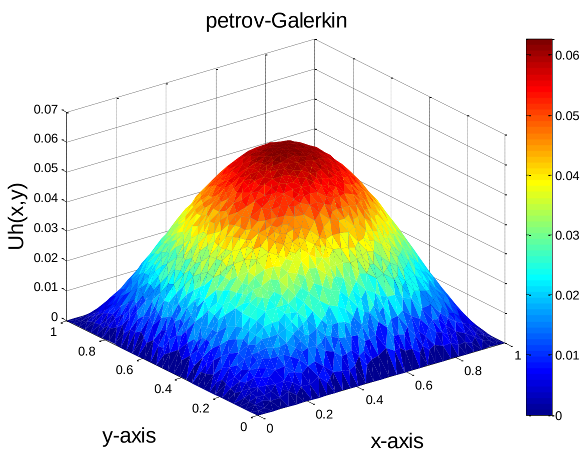


Figure (26): The approximate solution for the CDR problem (92) by PGFE method with $\epsilon = 10^{-5}, \beta = (10,10\,)^T, c = 1, h = 2^{-5}$.

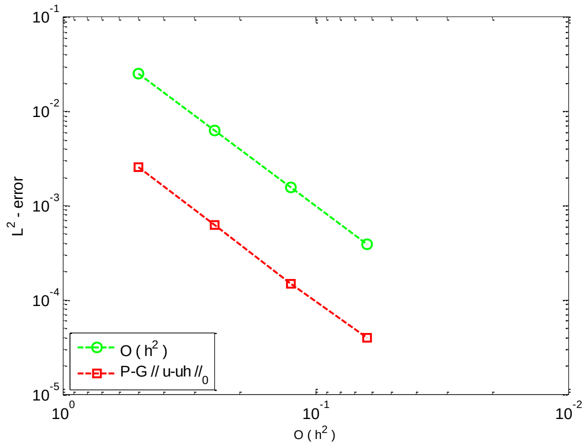
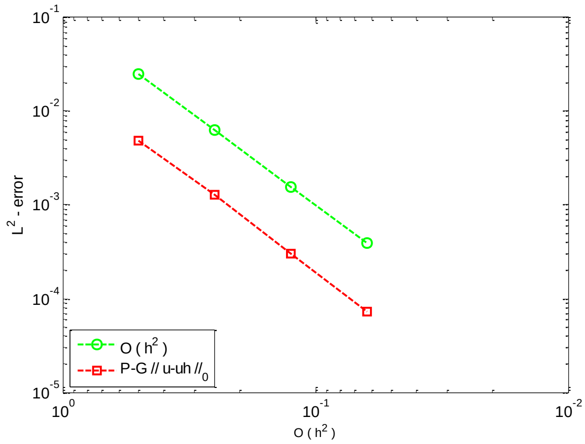


Figure (27):Convergence rate for the CDR problem in (92) by PGFE method with $\epsilon = 1$



Figure(28):Convergence rate for the CDR problem in (92) by PGFE method with $\epsilon = 0.1$

10: Discussion and Conclusion:

A fully-discrete and semi-discrete PG scheme for the time dependent CDR equation is studied in this paper. Error analysis and numerical experiments are presented to demonstrate the computational effectiveness of PG method, two examples were given to compare the performance of PG method with unsteady-state and steady-state. Numerical tests indicate that PG method is stable. The exact solutions and numerical results from unsteady-state and steady-state PGFE method are consistent, see Figures (1),(8) …(13), and (14), (21) …. (26). The stability of the PGFE method depends on the stability parameter in stability term $\delta\beta \cdot \nabla v$. The convergence in the PGFE method, the semi-discrete where time stays continuous and the fully-discrete with the error of $O(h^2), O(h+\tau)$, respectively. when comparing tables (1) and (2) for the unsteady-state and also between tables (3) and (4) for the steady-state, a significant improvement and regularity was observed in the numerical results of the PGFE method compared to the numerical results for SG method. This is very clear from $L^2-error, L_\infty-error\ and\ H^1-error$ in tables, and also Figures (8)..(13), (21)..(26), where the error in PGFE method is much less than the SG method. Note convergence rate of PGFE method, as in Figures (27) and (28). In SG method, when the diffusion coefficient (ϵ) is small with respect to the flow (convection) coefficient (β), where (ϵ) takes values from 1 to 10^{-5} , notice oscillations in the solutions and this is what we see in the Figures from (15)…(20), these oscillations in solutions were eliminated by PG method and this is evident in Figure from (8)…(13), and (21)…(26).

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طريقة بتروف ـ كالركن للعناصر المحددة لمسألة الانتشار والحمل والتفاعل (رد الفعل) الحراري

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المستخلص:

في هذه الورقة, تم وصف وتحليل طريقة بتروف كالركن للعناصر المحددة لمسألة الحمل والانتشار والتفاعل (رد الفعل) الحراري, أثبتت خاصيتين للشكل الخطي $a(U,v)$ (ألاستمرارية و الاهليجية). تم أثبات الاستقرارية وتقدير ألخطا (شبة منفصل ومنفصل تماما), واثبتنا أن الحل التقريبي متقارب بنسبة خطأ $O(h+\tau), O(h^2)$ على ألتوالي. تؤكد النتائج العددية لطريقة بتروف- كالركن التقارب والدقة للطريقة.

الكلمات المفتاحية:

بتروف كالركن للعناصر المحددة, طريقة كالركن القياسية, طريقة العناصر المحددة, مسألة الحمل والانتشار والتفاعل (رد الفعل) الحراري, الاستقرارية, تحليل ألخطا.