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Bifurcation solutions of a fourth order Nonlinear Differential Equations system using" local method of Lyapunov –Schmidt"

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Abstract

In this paper, a system of two nonlinear differential equations of the fourth order has been studied by using local method of the Lyapunov –Schmidt. We showed that the bifurcation equation corresponding to this system is given by a nonlinear system of two algebraic equations. The discriminate set of nonlinear algebraic equations has been found and geometrically described.

Keywords: bifurcation; Bifurcation theory, Local method of Lyapunov-Schmidt.

المستخلص :

في هذا البحث قمنا بدراسة نظام من المعادلات التفاضلية اللاخطية من الرتبة الرابعة باستخدام طريقة ليابونوف-شمدت المحلية. اثبتنا ان معادلة التفرع المقابلة لهذا النظام تعطى على شكل نظام لاخطي من معادلتين جبريتين. كذلك وجدنا المعادلة المميزة لهذا النظام مع وصف هندسي لمخطط التفرع.

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1. Introduction

In mathematics, there is method that using to analysis nonlinear system when does not working the implicit function theorem. This method called the Lyapunov–Schmidt reduction. The method working on conversion the dimensional equation from the infinite to the finite in Banach spaces.

The method of Lyapunov–Schmidt (LS) is important in some applications such as, economics, engineering and also natural sciences [1,2]. In addition to that, the method (LS) is applying in chemical engineering, to simplification partial differential equation models until easy solving simulate numerically , and still retain all the parameters of the original model[3,4].

We must know that the form

$F: X \times R^n \longrightarrow Y$

$$F(x,\lambda) = 0, \quad x \in O \subset X, \quad \lambda \in R^n \tag{1}$$

Can be analyzed using LS,

Where F is a nonlinear Fredholm mapping[6,7], for index zero

$(\mathrm{index}(F) = \dim(\ker F) - \dim(\mathrm{coker}\ F))$, X and Y are real Banach spaces[5],O is an open subset in X .We can conversion (1) to the finite- dimensional equation,

$$\Theta(\zeta,\lambda) = \mathcal{B}, \ \zeta \in \mathcal{M} \subset O, \ \mathcal{B} \in N \subset Y, \tag{2}$$

in which $\mathcal{M}$ and N are smooth finite -dimensional manifolds.

Knowing that the equation(2) reserve all analytical and topological properties of the equation(1)[8].

the motion and oscillations of the wave of the elastic beam on the elastic foundations described by the following nonlinear *ODE*,

$$y'''' + \alpha\, y'' + \beta y + g(\hat{y}) = 0, \tag{3}$$

$$\hat{y} = (y, y', y'', y''', y'''').$$

where β are real parameters and $g(\hat{y})$ is the nonlinear part of *ODE*.

Equation (3) has been considered by Thompson and Stewart [9].

In [10] Abdul Hussain studied equation (3) when $\mathrm{g}(\hat{y}) = y^2 + y^3$ with the boundary conditions,

$$y(0) = y(1) = y''(0) = z''(1)$$

He detected the bifurcation solutions of equation (3) by using the local method of Lyapunov-Schmidt.

Abdul Hussain (2015) introduces a modify method for finding nonlinear Ritz approximation of Fredholm functional [11,12].

In this paper we considered the system of non-linear differential equations of the fourth order,

$$\frac{d^4u_1}{dx^4} + \lambda_1 \frac{d^2\mathrm{u}_1}{dx^2} + \mathrm{u}_1 + \mathrm{u}_1^2 + \mathrm{u}_2^3 = 0$$

$$\frac{d^4u_2}{dx^4} + \lambda_2 \frac{d^2u_2}{dx^2} + u_2 + u_2^2 + u_1^3 = 0$$

with boundary conditions :

$$\mathrm{u}_1(0) = \mathrm{u}_1(1) = \mathrm{u}_1''(0) = \mathrm{u}_1''(1) = 0$$

$$\mathrm{u}_2(0) = \mathrm{u}_2(1) = \mathrm{u}_2''(0) = \mathrm{u}_2''(1) = 0$$

where $\lambda = (\lambda_1, \lambda_2) \in R^2$.

Theorem 1.1 [13] Suppose X and Y are real Banach spaces and $F(x,\lambda)$ is a C^1map defined in a neighborhood U of a point (x_0,λ_0) with range in Y such that $F(x_0,\lambda_0)=0$ and $F_x(x_0,\lambda_0)$ is a linear Fredholm operator. Then all solutions (x,λ) of $F(x,\lambda)=0$ near (x_0,λ_0) (with λ fixed) are in one-to-one correspondence with the solutions of a finite-dimensional system of N_1 real equations in a finite number N_0 of real variables. Furthermore, N_0=dim(ker$L$) and N_1=dim(coker$L$), $(L=F_x(x_0,\lambda_0))$.

2. Reduction to Bifurcation Equation.

Consider the following nonlinear system of ODEs

$$AU+\text{Ɲ}U=0 \tag{4}$$

By using the boundary conditions

$$U(0)=U(1)=U^{//}(0)=U^{//}(1)=0.$$

$$where \qquad A=\frac{d^4}{dx^4}+\frac{d^2}{dx^2}B+I, \qquad \mathrm{B}=\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}, \mathrm{I}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

and
$$\text{Ɲ}U=\begin{pmatrix}u_1^2 & u_2^3\\ u_2^2 & u_1^3\end{pmatrix}, \ \mathrm{U}=\begin{pmatrix}u_1\\ u_2\end{pmatrix}$$

we want to study the bifurcation regions of system (4) depending on local method of Lyapunov –Schmidt. To do that, firstly put equation (4) in the form of operator equation, *i.e.*

$$F(U,\lambda)=AU+\text{Ɲ}U \tag{5}$$

where $F: X \to Y$ represents the nonlinear Fredholm operator of zero index from Banach space X to Banach space Y, $X=C^4([0,1],R)$, $Y=C^0([0,1],R)$,

$x \in [0,1]$. Every solution of the equation (4) is a solution of the operator equation

$$F(U, \lambda) = 0 \tag{6}$$

Now, we will proof the reduction theorem (1.1) for equation (6).

Theorem 2.1 The bifurcation equation corresponding to the system (6) is given by nonlinear system of two equations,

$$\begin{pmatrix} \frac{3}{2}x_2^3 + \frac{8}{3\pi}x_1^2 + k_1 x_1 \\ \frac{3}{2}x_1^3 + k_2 x_2 \end{pmatrix} + o(|\tau|^2) + O(|\tau|^2)O(\gamma) = 0 \tag{7}$$

Where, $\tau = (x_1, x_2)$, $\quad k_1, k_2 \in R$.

In addition, the solutions of (6) are corresponding to the solutions of the nonlinear system.

Proof: Let

$F(U, \lambda) = 0$

Since, the Frēchet derivative for $F(U, \lambda)$, at $(0, \lambda)$ is

$$dF(0, \lambda)h = \frac{d^4}{dx^4}h + \frac{d^2}{dx^2}Bh + Ih\,,$$

Where $\lambda = (\lambda_1, \lambda_2)$, $\quad h = (h_1, h_2)^{\mathrm{T}}$,

After that, the linear equation corresponding to (6) is given by:

$Ah = 0 \;, \qquad h \in X,$

$$A = dF(0, \lambda) = \frac{d^4}{dx^4} + \frac{d^2}{dx^2}B + I,$$

hence the result of linear equation which satisfy

$U(0) = U(1) = U^{//}(0) = U^{//}(1) = 0.$ is getting by

$$h(x) = \begin{pmatrix} c_1 & \sin(p\pi x) \\ c_2 & \sin(q\pi x) \end{pmatrix}$$

substitute h(x) in the linear equation, we get on characteristic equation corresponding to the following form

$$\pi^4 p^4 - \lambda_1 \pi^2 p^2 + 1 = 0$$

$$\pi^4 q^4 - \lambda_2 \pi^2 q^2 + 1 = 0$$

these equations give in the space of parameters (λ_1, λ_2) characteristic planes ℓ_p. The characteristic planes ℓ_p consists the points (λ_1, λ_2) for which the linearized equation has non-zero solutions. The point of intersection of characteristic planes in the space of parameters (λ_1, λ_2) is a bifurcation point .

Hence, the bifurcation point of (6) is the point $(\lambda_1, \lambda_2)^T = (\frac{\pi^4+1}{\pi^2}, \frac{16\,\pi^4+1}{4\,\pi^2})^T$.

We can write the disturbance parameters λ_1, λ_2 as : .

$\lambda_1 = \frac{\pi^4+1}{\pi^2} + \delta_1$, $\lambda_2 = \frac{16\,\pi^4+1}{4\,\pi^2} + \delta_2$, where δ_1, δ_2 are small disturbance responsible for modes bifurcation,

$e_1(x) = (c_1 \sin(\pi x), 0)^T$, $e_2(x) = (0, c_2 \sin(2\pi x))^T$ where,

$c_1 = c_2 = \sqrt{2}$, $\int_0^1 \sin(n\pi x)\sin(m\pi x)\,dx = \begin{cases} 1 \text{ } if \ \ n = m \\ 0 \text{ } if \ n \neq m \end{cases}$

suppose $\aleph = \ker(A) = \text{span}\{e_1, e_2\}$, then the space X can be written in direct sum of two subspaces $\aleph$ and the orthogonal complement of $\aleph$

$X = \aleph \oplus \aleph^{\perp}$, $\aleph^{\perp} = \left\{ z \in X : \int_0^1 z e_k \, dx = 0 , k = 0,1 \right\}$

In the same way, the space Y can be written in direct sum of two subspaces $\aleph$, and the orthogonal complement to $\aleph$. $Y=\aleph \oplus \widetilde{\aleph}^{\perp}$,

$\widetilde{\aleph}^{\perp} = \{v \in Y : \int_0^{\pi} ve_k \, dx = 0 \,, \; wherever, \;\; k = 1{,}2\}$

There are two projections $\mathcal{P}: X \to \aleph$ also, $(\mathrm{I} - \mathcal{P}): X \to \widehat{\aleph}^{\perp}$

$\mathcal{P}$ U = W and (I $-$ $\mathcal{P}$) $U = Z$ where U= $\binom{u_1}{u_2}$, W = $\binom{w_1}{w_2}$, $Z = \binom{z_1}{z_2}$

All vector $U \in X$ are written in the unique relation: $U = W + Z$

U = $\sum_{i=1}^{2} x_i e_i \in \ker(A)$, $z \perp \ker(G)$, $x_i = \langle U, e_i \rangle$

similarly, there exist two projections $Q: Y \to \aleph^{\perp}$ and $(I - Q): Y \to \widetilde{\aleph}^{\perp}$

such that

$QF(U, \lambda) = F_1(U, \lambda),$

$(I - Q)F(U, \lambda) = F_2(U, \lambda),$

And hence

$F(U, \lambda) = F_1(U, \lambda) + F_2(U, \lambda)$

$= QF(U, \lambda) + (I - Q)F(U, \lambda),$

$$F_1(U, \lambda) = QF(U, \lambda) = \sum_{i=1}^{2} v_i \,(U, \lambda) e_i \in \aleph,$$

$F_2(U, \lambda) = (I - Q)F(U, \lambda) \in \widetilde{\aleph}^{\perp},$ $\qquad v_i(u, \lambda) = \langle F(U, \lambda), e_i \rangle$

hence (5) becomes

$QF(U,\lambda) = 0,$

$(I - Q)F(U,\lambda) = 0$

or

$QF(W + W,\lambda) = 0$

$(I - Q)F(W + Z,\lambda) = 0$

using the implicit function theorem, we get a smooth map $\phi : \aleph \to \aleph^{\perp}$ such that $\phi\ (W,\lambda) = Z$ and

$(\mathrm{I} - Q)\ F\ (W + \phi(W,\lambda),\lambda) = 0$

To find the solution for $F(U,\lambda)$ =0 near the point U=0 it is sufficient to find the solution of the equation

Q F(W + $\phi(W,\lambda),\lambda$) = 0 (8)

This equation is called bifurcation equation for equation (6). this means, the bifurcation equation is $\theta(\tau,\lambda) = 0$, $\tau = (x_1, x_2)$, $\lambda = (\lambda_1 , \lambda_2)$

Where

$\theta(\tau,\lambda) = F_1(W + \phi(W,\lambda),\lambda)$

(8) can be rewritten as:

F(W + Z, λ) = A(W + Z) + N (W + Z)

Simple calculations show that N (W + Z) = N W + terms consists elements of v

Hence

F(W + Z, λ) = A W + N W + …

Where the dots denote the terms that consists of the element z. hence,

$\theta(\tau,\lambda) = Q$ F(W + Z, λ)

$= \sum_{i=1}^{2}\langle A\, W + N\, W\,, e_i\rangle + \cdots = 0$ (9)

Simplified (9) :

$\langle A\, W + N\, W\,, e_1\rangle\, e_1 + \langle A\, W + N\, W\,, e_2\rangle\, e_2 + \ldots = 0$ (10)

From the inner product we have

$\langle A\, W + N\, W\,, e_1\rangle = \langle A\, w_1 +, e_1\rangle + \langle N\, w_1, e_1\rangle$

As a result we get :

$\langle A\, W + N\, W\,, e_1\rangle = \frac{3}{2}x_2^3 + \frac{8}{3\pi}x_1^2 + k_1x_1$

$\langle A\, W + N\, W\,, e_2\rangle = \frac{3}{2}x_1^3 + k_2x_2$

Where

$k_1 = e_1 = \alpha_1(\lambda)e_1$ and $k_2 = e_2 = \alpha_2(\lambda)e_2$.

The latter with (10) implies the assertion

3. Analysis equation of bifurcation

Equation (7) has the same analytical and topological characteristics properties of equation (5). Let

$$g(\tau,\tilde{\lambda}) = \begin{pmatrix} \frac{3}{2}x_2^3 + \frac{8}{3\pi}x_1^2 + k_1x_1 \\ \frac{3}{2}x_1^3 + k_2x_2 \end{pmatrix} + o(|\tau|^2) + O(|\tau|^2)O(\gamma)$$

then the map $g(\tau,\tilde{\lambda})$ is contact equivalent to the map

$$\phi_1(\tau,\tilde{\lambda}) = \begin{pmatrix} \frac{3}{2}x_2^3 + \frac{8}{3\pi}x_1^2 + k_1x_1 \\ \frac{3}{2}x_1^3 + k_2x_2 \end{pmatrix} \qquad (11)$$

wheree $k_1, k_2 \in R$

This means that, to determine the Discriminant set of the map $g(\tau,\tilde{\lambda})$ it is sufficient to determine the Discriminant set of the map $\phi_1(\tau,\tilde{\lambda})$ so for this purpose we must find the parameters equation in the form of

$h(\gamma) = 0$, $\gamma = (k_1, k_2) \in R^2$ such that the set of all $\gamma = (k_1, k_2)$ in which equation $\phi_1(\tau,\tilde{\lambda}) = 0$ has solutions achieve $h(\gamma) = 0$, for where $h : R^2 \longrightarrow R$ is a map. To found $h(\gamma)$ we will solve the below equation,

$$\phi_1(\tau,\tilde{\lambda}) = 0 \qquad (12)$$

After that, we substitute x_1 , x_2 in the below equation :

$$\det\left(\frac{\partial \phi_1(\tau,\gamma)}{\partial x_i}\right) = 0 \quad , \; i = 1, 2, \dots \qquad (13)$$

so that, we getting on Discriminate set of (12), we draw k_1k_2-plane by the aid of Maple 17.

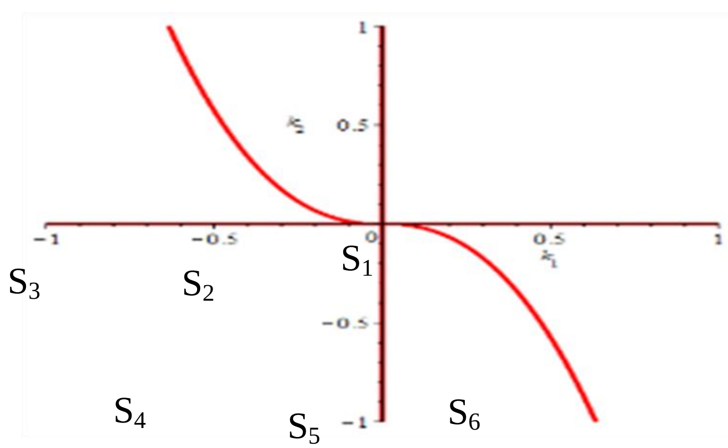

Fig. 1 The Discriminant set of equation $\phi(\tau,\tilde{\lambda}) = 0$

The Discriminate set decomposes the solution set into disjoint regions; every region has a fixed number of solutions. In the regions S_1 ,S_4 we have three regular solutions, where 1 , -1 , -1represent topological indices, but, the regions S_3 , S_6 , we have only one regular solution with topological index -1, in the regions S_2 , S_5 , we have three regular solution with topological indices: -1,1,1.

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