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On Sm-Modules

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Abstract

Let R be a commutative ring with identity, and let M be a unitary R-module. We introduce a concept of sm-module as follows: M is called sm-module if and only if $\sqrt{ann_R N}$ is a semimaximal ideal of R, for each maximal submodule N of M.

In this paper, some properties and characterizations of sm-modules is given also, various basic results a bout sm-module are considered. Moreover, some relations between sm-modules and other types of modules are considered.

1. Introduction

Every ring considered in this paper will be assumed to be commutative with identity and every module is unitary. We introduce the following: An R-module M is called an sm-module if and only if $\sqrt{ann_R N}$ is a semimaximal ideal of R, for each maximal submodule N of M, where annRN={r:r∈R and rN=0}.

Our concern in this paper is to study sm-modules and to look for any relation between sm-modules and certain types of well-known modules. This paper consists of two sections. Our main concern in section one is to define and study sm-modules. We introduce some characterizations for this concept. Also, other basic results about this concept are given. In section two, we study the relation between sm-modules and max-modules, multiplication modules, bounded modules and with the other types of modules.

1- Basic Properties of sm-modules

In this section, we introduce the concept of sm-module and give some characterization and properties of this concept; we end this section by study the relationships between sm-modules and semisimple rings.

We start with the following definition.

1.1 Definition:

An R-module M is called sm-module if and only if $\sqrt{ann_R N}$ is a semimaximal ideal of R for each maximal submodule N of M. Specially, a ring R is called sm-ring if and only if R is sm-R-module.

Recall that an ideal I of a ring R is said to be semimaximal ideal if I is an intersection of finitely many maximal ideals of R, [1,Def.(1.2.1),p.16].

1.2 Examples and Remarks:

1. Z_6 as a Z-module is sm-module. In general Z_n as a Z-module is sm-module, where n is a positive integer and not prime number.

2. Let p be a prime number. Then the Z-module Z_p is not sm-module.
3. Z as a Z-module is not sm-module. Since pZ is maximal submodule for each p, p is prime number and $\sqrt{\mathrm{ann}_Z(pZ)} = \sqrt{0} = (0)$ for each p. Hence (0) is not semimaximal ideal of Z.
4. Every maximal submodule of an sm-module is an sm-module.

Proof: Let K be a maximal submodule of M. Then $\sqrt{\mathrm{ann}_R K}$ is semimaximal ideal of R (since M is sm-module). To show that K is sm-module. Let N be a maximal submodule of K. Since $N \subseteq K$, then $\mathrm{ann}_R K \subseteq \mathrm{ann}_R N$which implies that $\sqrt{\mathrm{ann}_R K} \subseteq \sqrt{\mathrm{ann}_R N}$, but $\sqrt{\mathrm{ann}_R K}$ is semimaximal ideal of R. Thus by [1, Prop.(1.2.11),p.20], $\sqrt{\mathrm{ann}_R N}$ is semimaximal ideal of R and hence K is an sm-module.

5. Let $M = \underset{p}{\oplus} Z_p$ as a Z-module. Then M is not sm-module.
6. Q as a Z-module is not sm-module.
7. The homomorphic image of sm-module is not sm-module.

   For example:Z_6 as a Z-module is an sm-module. Define $f: Z_6 \longrightarrow \frac{Z_6}{(\bar{2})}$, $f(n) = n + (\bar{2})$ fo all $n \in Z_6$. It is easily proved that f is homomorphism, but $\mathrm{ann}_Z \frac{Z_6}{(\bar{2})} \cong Z_2$, Z_2 is not sm-module by (2).
8. Let $M = Z \oplus Z_n$ be a Z-module, n is any positive integer is not sm-module.
9. Recall that an R-module M is said to be max-module if $\sqrt{\mathrm{ann}_R N}$ is a maximal ideal of R, for each non-zero submodule N of M, [2,Def.(2.1),p.4].

The following proposition shows that the class of sm-modules containing in the class of max-modules.

1.3 Proposition:

Every max-module is sm-module.

Proof: Let M be a max-module. Then for each non-zero submodule N of M, $\sqrt{\mathrm{ann}_R N}$ is maximal ideal of R. Hence by [1, Ex. and Rem.(1.2.2)(2),p.16], $\sqrt{\mathrm{ann}_R N}$ is semimaximal ideal of R for each non-zero submodule N of M. Therefore $\sqrt{\mathrm{ann}_R N}$ is semimaximal ideal of R for each maximal submodule N of M and hence M is sm-module.

Note that, the converse of proposition (1.3) is not true in general. For example, the Z-module $M=Z_2\oplus Z_{20}$ is sm-module, but is not max-module. Since $N_1=(\bar{0})\oplus(\bar{2})$ and $N_2=(\bar{0})\oplus(\bar{5})$ are maximal submodules of M. Then $\sqrt{\mathrm{ann}_Z N_1}=\sqrt{Z\cap 10Z}=\sqrt{10Z}=10Z$ is semimaximal ideal of R and $\sqrt{\mathrm{ann}_Z N_2}=\sqrt{Z\cap 4Z}=\sqrt{4Z}=2Z$ is semimaximal ideal of R, which implies M is sm-module. But $\sqrt{\mathrm{ann}_R N_1}$ is not maximal ideal of R. Thus M is not max-module.

The class of sm-modules is closed under direct sum as the following result shows.

1.4 Proposition:

Let M_1, M_2 be two sm-R-modules. Then $M_1\oplus M_2$ is also sm-R-modules.

Proof: Let $N=N_1\oplus N_2$ be a maximal submodule of M, where N_1, N_2 are maximal submodules of M_1 and M_2 respectively. Then $\sqrt{\mathrm{ann}_R N}=\sqrt{\mathrm{ann}_R(N_1\oplus N_2)}=\sqrt{\mathrm{ann}_R N_1\cap \mathrm{ann}_R N_2}=\sqrt{\mathrm{ann}_R N_1}\cap\sqrt{\mathrm{ann}_R N_2}$, but $\sqrt{\mathrm{ann}_R N_1}$ and $\sqrt{\mathrm{ann}_R N_2}$ are semimaximal ideals of R (since M_1 and M_2 are two sm-modules). Thus by [,Prop.(1.2.14),p.21], $\sqrt{\mathrm{ann}_R N}$ is semimaximal ideal of R and hence $M= M_1\oplus M_2$ is sm-module.

So, we have the following application of (1.4).

1.5 Corollary:

Let M_α be an sm-R-modules for all α. Then $\underset{\alpha\in\wedge}{\oplus} M_\alpha$ is an sm-module.

The following corollary is a special case of proposition (1.4).

1.6 Proposition:

Let M be an R-module. If M is sm-module, then M^2 is also sm-module.

Proof: It is clear that $M^2= M\oplus M$. So according to proposition (1.4), M^2 is an sm-module.

1.7 Remark:

It is not necessary that every direct summand of sm-module is sm-module, for example: $M=Z_2\oplus Z_{20}$ as a Z-module is sm-module, but the Z-module Z_2 is not sm-module.

Recall that an R-module M is said to be divisile if and only if rM=M for every non-element r in R, [3].

By using this concept, we have the following.

1.8 Proposition:

Let M be an R-module, if M is sm-module and every submodule N of M is divisible, then $\sqrt{\text{ann}_R N} = \sqrt{\text{ann}_R rN}$ for each maximal submodule N of M such that rN≠(0), r∈R.

Proof: It is abvious.

The following results are another characterizations of sm-module, but first we need to recall some definitions.

An R-module M is called semisimple if every submodule of M is a direct summand of M. And a ring R is said to be semisimple ring if and only if R is a semisimple R-module, [3].

A ring R is a Boolean ring in case each of its elements is an idempotent, [4].

Next, we have the following proposition.

1.9 Proposition:

Let M be an R-module. Then M is sm-module if and only if R/$\sqrt{\text{ann}_R N}$ is semisimple ring for each maximal submodule N of M.

Proof: If M is sm-module, then $\sqrt{\text{ann}_R N}$ is semimaximal ideal of R. Thus by [1,Prop.(1.2.5),p.17], $\frac{R}{\sqrt{\text{ann}_R N}}$ is semisimple ring.

Conversely, if $\frac{R}{\sqrt{\text{ann}_R N}}$ is semisimple ring, then by [1, Prop.(1.2.5),p.17], $\sqrt{\text{ann}_R N}$ is semimaximal ideal of R. Thus M is sm-module by def. (1.1).

Now, we deduce the following corollaries.

1.10 Corollary:

Every module M over semisimple ring R is sm-module.

Proof: The result follows directly from [1, Rem.(1.1.34)(3),p.9] and prop.(1.9).

It is known that if R is a Boolean ring, then every proper ideal of R is semimaximal ideal, [1,Cor.(1.2.7),p.18].

1.11 Corollary:

Every module M over a Boolean ring is sm-module.

Proof: It is clear that $\sqrt{\text{ann}_R N}$ is a proper ideal of R for each maximal submodule N of M. Then the result follows from [1,Cor.(1.2.7),p.18] and prop.(1.9).

2- Modules Related to sm-modules

In this section, we study the relationships between sm-modules and multiplication, bounded, uniform, projective Z-regular and prime modules.

We note that if M is sm-module, then it is not necessary that R is sm-ring, for example: The Z-module Z_4 is sm-module, but Z is not sm-ring. Moreover if R is sm-ring and M is an R-module, then M is not necessarily sm-module, for example: Consider the Z_6-module Z_2, Z_6 is sm-ring, but Z_2 is not sm-module.

Recall that an R-module M is called faithful R-module if $\text{ann}_R M=0$, [4].

An R-module M is said to be multiplication module if for every submodule N of M, there exists an ideal I of R such that N=IM, [5].

However, in the class of the faithful multiplication module, they are equivalent as the following result shows.

2.1 Proposition:

If M is a faithful multiplication R-module, then M is sm-module if and only if R is sm-ring.

Proof: If M is sm-module. To show that R is sm-module, let I be a maximal ideal of R. Then IM is maximal submodule of M. Hence N=IM is a maximal submodule of M. Thus $\sqrt{\text{ann}_R N}$ is a semimaximal ideal of R because M is sm-module. On the other hand, since M is faithful multiplication R-module, then $\text{ann}_R N= \text{ann}_R I$, so $\sqrt{\text{ann}_R N} = \sqrt{\text{ann}_R I}$. Thus $\sqrt{\text{ann}_R I}$ is a semimaximal ideal and R is a sm-ring.

Conversely, if R is sm-ring. To show that M is sm-module, let N be a maximal submodule of M. Since M is a multiplication R-module, N=IM for some ideal I of R. But M is faithful,

$ann_R N = ann_R IM = ann_R I$ and so $\sqrt{ann_R N} = \sqrt{ann_R IM} = \sqrt{ann_R I}$ which is a semimaximal ideal of R. Therefore M is an sm-module.

Recall that an R-submodule N of M is called essential in M if for each non-zero R-submodule L of M, $N \cap L \neq 0$, [5].

Now, we can give the following result:

2.2 Proposition:

Let M be an R-module and let $0 \neq x \in M$ such that:

1. Rx is an essential submodule of M.
2. $\sqrt{ann_R(x)}$ is a semimaximal ideal of R, and
3. $\sqrt{ann_R M} = \sqrt{ann_R(x)}$.

Then M is an sm-module.

Proof: Let N be a maximal submodule of M. Since Rx is an essential submodule of M, there exists $0 \neq t \in R$ such that $0 \neq tx \in N$ and hence $(tx) \subseteq N$. This implies that $ann_R N \subseteq ann_R(tx)$ and so $\sqrt{ann_R N} \subseteq \sqrt{ann_R(tx)}$. But $N \subseteq M$, then $\sqrt{ann_R M} \subseteq \sqrt{ann_R N}$ and hence $\sqrt{ann_R(x)} \subseteq \sqrt{ann_R N}$ by (condition 3). Thus,

$$\sqrt{ann_R(x)} \subseteq \sqrt{ann_R N} \subseteq \sqrt{ann_R(tx)} \quad \ldots(1)$$

Let $r \in \sqrt{ann_R(tx)}$, then $r^n tx = 0$ for some $n \in Z^+$ and $r^n t \in ann_R(x)$. But $tx \neq 0$; that is $t \notin annR(x)$ and by (condition 2) $\sqrt{ann_R(x)}$ is semimaximal ideal of R, so $r \in \sqrt{ann_R(x)}$. Thus

$$\sqrt{ann_R(tx)} \subseteq \sqrt{ann_R(x)} \quad \ldots(2)$$

Thus by (1) and (2), $\sqrt{ann_R(tx)} = \sqrt{ann_R(x)}$ and so $\sqrt{ann_R N} = \sqrt{ann_R(x)}$. Therefore (by condition 2) $\sqrt{ann_R N}$ is a semimaximal ideal of R and M is an sm-module by def. (1.1).

An R-module M is called uniform if every non-zero R-submodule of M is essential [5].

So, we have that following application of (2.2).

2.3 Corollary:

Let M be a uniform R-module such that $\sqrt{ann_R(x)}$ is semimaximal ideal of R and $\sqrt{ann_R M} = \sqrt{ann_R(x)}$ for some $x\neq 0$. Then M is sm-module.

An R-module M is said to be bounded module if there exists an element $x\in M$ such that $ann_R M = ann_R(x)$ [6].

By using this concept, we have the following.

2.4 Proposition:

If M is a bounded R-module such that $\sqrt{ann_R(x)}$ is a semimaximal ideal of R for some $0\neq x\in M$, then M is an sm-module.

Proof: We have M is bounded, then $ann_R M = ann_R(x)$ for some $0\neq x\in M$ and so $\sqrt{ann_R M} = \sqrt{ann_R(x)}$ for some $0\neq x\in M$. Therfore by coro.(2.3), M is sm-module.

The following results are another consequences of proposition (2.4), but first we need to recall the definition of projective module.

An R-module M is called projective if for every R-module epimorphism $h:A\longrightarrow B$ and $f\in Hom_R(M,B)$, there exists $g\in Hom_R(M,A)$ such that $h\circ g=f$ [3,p.217].

2.5 Corollary:

Let R be an integral domain. Then every projective R-module M such that $\sqrt{ann_R(x)}$ is semimaximal ideal of R for some $0\neq x\in M$ is an sm-module.

Proof: According to [6,Coro.(1.1.12),p.10] and proposition (2.4).

2.6 Corollary:

Let M be a cyclic R-module such that $\sqrt{ann_R(x)}$ is semimaimal ideal of R for some $0\neq x\in M$. Then M is sm-module.

Proof: The result is directly by [6, Coro.(1.1.3),p.7] and proposition (2.4).

Recall that an R-module M is called regular if given any element m in M, there exists $f\in M^*$ such that $m=f(m)m$ where $M^*=Hom_R(M,R)$, [3].

The J-radical J(N) of a submodule N of an R-module M is defined as the intersection of all maximal submodules containing N; that is $J(N)=\cap\{P\subseteq M: P \text{ is maximal and } N\subseteq P\}$, [7].

By using these concepts, we have the following.

2.7 Proposition:

Let M be a regular multiplication R-module and N be a submodule of M. Then the following statements are equivalent:

1. M is an sm-module.

2. $\sqrt{\text{ann}_R(J(N))}$ is semimaximal ideal of R.

3. $R/\sqrt{\text{ann}_R(J(N))}$ is semisimple ring.

Proof: (1) $\Rightarrow$ (2): Let M be an sm-module. Then $\sqrt{\text{ann}_R N}$ is semimaximal ideal of R for each maximal submodule N of M. Thus by [7,Prop.(2.3),p.4], J(K)=K for every submodule K of M which implies that J(N)=N for every maximal submodule N of M and hence $\sqrt{\text{ann}_R(J(N))}$ is semimaximal ideal of R.

(2) $\Rightarrow$ (3) It is abvious according to [,Prop.(1.2.5),p.17].

(3) $\Rightarrow$ (1) It follows directly by proposition (1.).

Next, the following definitions are needed.

An R-module M is said to be a prime module if $\text{ann}_R M=\text{ann}_R N$ for every non-zero submodule N of M, [8].

An R-module M is called quasi-maximal module if and only if $\sqrt{\text{ann}_R M}$ is semimaximal ideal of R, [9].

However, in the class of prime module the two concepts sm-module and quasi-maximal module are equivalent as the following result shows.

2.8 Proposition:

Let M be a prime R-module. Then M is sm-module if and only if M is quasi-maximal module.

Proof: If M is sm-module, then $\sqrt{\text{ann}_R N}$ is semimaximal ideal of R for each maximal submodule N of M. Hence $\text{ann}_R M = \text{ann}_R N$ (since M is prime module). Therefore $\sqrt{\text{ann}_R M}=\sqrt{\text{ann}_R N}$, which implies that $\sqrt{\text{ann}_R M}$ is semimaximal ideal of R and hence M is quasi-maximal module.

Conversely, let M be a quasi-maximal module. Then $\sqrt{ann_R M}$ is semimaximal ideal of R. Now, for every non-zero submodule N of M, $ann_R M = ann_R N$ because M is prime module. Then $\sqrt{ann_R M} = \sqrt{ann_R N}$ for every non-zero submodule N of M, but $\sqrt{ann_R M}$ is semimaximal ideal of R, which implies that $\sqrt{ann_R N}$ is semimaximal ideal of R for every non-zero submodule N of M and hence $\sqrt{ann_R N}$ is semimaximal ideal of R for each maximal submodule N of M, which completes the proof.

The condition M is prime can not be dropped from proposition (2.8) as the following examples shows.

2.9 Example:

Consider $M=Z_2 \oplus Z_3$ as a Z-module. M is not prime Z-module.

M is quasi-maximal module. Since $\sqrt{ann_Z M} = \sqrt{ann_Z (Z_2 \oplus Z_3)} = \sqrt{6Z} = 6Z$ is semimaximal ideal of Z. But M is not sm-module. Since $(\bar{0})$ is the only maximal submodule of M and $\sqrt{ann_Z (\bar{0})} = \sqrt{Z} = Z$ is not semimaximal ideal.

The following results are another consequences of proposition (2.8), but first we need to recall some definitions

An R-module M is said to be searial R-module if the R-submodules of M are linearly with respect to inclusion, [8].

An R-module M is called Z-regular if for all $m \in M$, there exists $f \in Hom_R(M,R)=M^*$ such that $f(m)m=m$, [6].

An R-module M is called semiprime if and only if $ann_R N$ is a semiprime ideal of R for each non-zero R-submodule N of M, [10].

Hence, we have the following consequences of (2.8).

2.10 Corollary:

Let M be a Z-regular serial R-module. Then M is sm-module if and only if M is quasi-maximal module.

Proof: From [10,Prop.(4.2.2),p.71], [10,Prop.(4.2.1),p.70], we get M is prime module and hence by proposition (2.8) we get the result.

2.11 Corollary:

Let M be a uniform semiprime R-module. Then M is quasi-maximal module if and only if $\frac{R}{\sqrt{\operatorname{ann}_R N}}$ is semisimple ring for each maximal submodule N of M.

Proof: If M is quasi-maximal module. Then the result follows from [10,Prop.(4.2.3),p.72], proposition (2.8) and proposition (1.9).

Conversely, If $\frac{R}{\sqrt{\operatorname{ann}_R N}}$ is semimaximal ring for each maximal submodule N of M. Thus by proposition (1.9), we get M is sm-module and hence the result follows according to [10,Prop.(4.2.3),p.72] and proposition (2.8).

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