A Generalized Curvature of a Generalized Envelope

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الملخص

الهدف من هذا البحث هو دراسة بعض تطبيقات الانحناء المعمم [3] على الغلاف المعمم لعائلة من المستقيمات معطاة [7]،[8] باستخدام بعض مفاهيم التحليل غير القياسي الذي أوجده [5] .Robinson, A ووضعه .Nelson, E بأسلوب منطقي.

ABSTRACT

In this paper we study one of the applications of a generalized curvature [3] on the generalized envelope of a family of lines given in [7], [8], using some concepts of nonstandard analysis given by **Robinson**, **A**. [5] and axiomatized by **Nelson**, E..

Keywords: infinitesimals, monad, envelope, generalized curvature

1- Introduction:

The following definitions and notations are needed throughout this paper.

Every concept concerning sets or elements defined in classical mathematics is called **standard** [4].

Any set or formula which does not involve new predicates "standard, infinitesimals, limited, unlimited…etc" is called **internal,** otherwise it is called **external** [2], [4].

A real number x is called **unlimited** if and only if $|x| > r$ for all positive standard real numbers, otherwise it is called **limited** [2].

A real number x is called **infinitesimal** if and only if $|x| < r$ for all positive standard real numbers r [2].

Two real numbers x and y are said to be **infinitely close** if and only if $x - y$ is infinitesimal and denoted by $x \simeq y$ [2], [6].

If x is a limited number in $\mathbb{R}$, then it is infinitely close to a unique standard real number, this unique number is called the **standard part** of x or **shadow** of x denoted by $st(x)$ or ${}^{0}x$ [2], [4].

If x is a real limited number, then the set of all numbers, which are infinitely close to x, is called the **monad** of x and denoted by $\boldsymbol{m(x)}$ [2], [3].

A curve n is called **envelope** of a family of curves $\{g_a\}$ depending on a parameter a, if at each of its points, it is tangent to at least one curve of the family $\{g_a\}$, and if each of its segments is tangent to an infinite set of these curves [1].

The **projective homogenous plane** over $\mathbb{R}$, denoted by $\mathbb{P}^2_{\mathbb{R}}$ is the set:

$\mathbb{P}^2_{\mathbb{R}} = \mathbb{R}^2 \cup \{$one point at ∞ for each equivalence classes of parallel lines $\}$,we denoted it by (***PHP***) [1].

The **projective homogeneous coordinates** of a point $p(x,y) \in \mathbb{R}^2$ are (xa, ya, a), where a is any nonzero number, we denote it by (***PHC***). In this sense the projective homogeneous coordinates of any point is not unique. [1]

By a **parameterized differentiable curve**, we mean a differentiable map $\gamma : \mathbf{I} \to \mathbb{R}^3$ of an open interval $\mathbf{I}$ =($\boldsymbol{a}$,$\boldsymbol{b}$) of the real line $\mathbb{R}$ into $\mathbb{R}^3$ such that: $\gamma(t)=(x(t), y(t), z(t)) = x(t)e_1 + y(t)e_2 + z(t)e_3$, and x, y, and z are differentiable at t; it is also called **spherical curve** [2].

Definition 1.1 [7]

Let $A=\gamma(t)$ be a standard point on the curve γ, then the following cases occur for the point A with the existence of the order of derivatives of γ:

1- If $\gamma' \neq 0, \gamma'' \neq 0$ and $\gamma' \cdot \gamma'' \neq 0$ then the point is called **biregular** point.

2- If $\gamma' \neq 0$ then the point is called **regular** point.

3- If $\gamma' \neq 0$ and $\gamma' \cdot \gamma'' \neq 0$ then the point is called **only regular** point, and we say that the point is only regular point of order ***p-1*** if $\gamma' \neq 0$ and $\gamma' = \gamma'' = \mathbf{L} = \gamma^{(p-1)} = 0$, but $\gamma' \cdot \gamma^{(p)} \neq 0$. In this case we say that p is the order of the first vector derivative not **collinear** with γ'

4- If $\gamma' = 0$ then the point is called **singular** point. In general if

$\gamma' = \gamma'' = \mathbf{L} = \gamma^{(p-1)} = 0$ but $\gamma^{(p)} \neq 0$,then the point is called **singular point of order p**.

Theorem 1.2 [7]

Let γ be a standard curve of order C^n and **A** be a standard singular point of order *p-1* on γ; and let *B* and *C* be two points infinitely close to the point *A*, then the generalized curvature of γ at the point denoted by K_G and given by

$$K_G = \frac{(p!)^{\frac{q}{p}} \left| x^{(p)} y^{(q)} - x^{(q)} y^{(p)} \right|}{q! \left(x^{(p)2} + y^{(p)2} \right)^{\frac{q+p}{2p}}} = \frac{(p!)^{\frac{q}{p}} \left| g^{(p)} \times g^{(q)} \right|}{q! \left\| g^{(p)} \right\|^{\frac{q}{p}+1}},$$

where q is the order of the first vector derivative of g not collinear with $\gamma^{(p)}$.

Theorem 1.3 [7]

If $p_k(t) = r_k(t) = q_k(t) = 0$ for $1 \le k \le n$ (n standard) and $p_n(t)$, $r_n(t)$, $q_n(t)$ are not all zeros, then the ***PHC*** points of $\gamma(t)$ are of the form $(p_n(t), r_n(t), q_n(t))$ which does not depend on ε. Thus, we get the generalized nonclassical form of the envelope curve $\gamma(t)$ as follows:

$$(x(t), y(t)) = \left(\frac{X_\varepsilon(t)}{Z_\varepsilon(t)}, \frac{Y_\varepsilon(t)}{Z_\varepsilon(t)} \right)$$

$$= \left(\frac{v^{(n)}(t)w(t) - w^{(n)}(t)v(t)}{u^{(n)}(t)v(t) - v^{(n)}(t)u(t)}, \frac{w^{(n)}(t)u(t) - u^{(n)}(t)w(t)}{u^{(n)}(t)v(t) - v^{(n)}(t)u(t)} \right)$$

2- A Generalized Curvature of the Envelope of a Family of Lines

Throughout this section, we give a curvature formula for the envelope of a family of lines $L_t : u(t)x + v(t)y + w(t)z = 0$ represented by the components u, v, and w.

It is clear that every two infinitely closed points (points in the same monad) on the envelope curve of a family of lines determine two infinitely close lines in that monad.

That is, " $A(t_o), B(t_o) \in \gamma(t_o)$, where $B(t_o+\alpha)$ I $m(A(t_o))$ there exists a line $L_{t_o+\alpha} \in \{L_t\}$ such that $L_{t_o+\alpha} > L_{t_o}$ in $m(A(t_o))$, where $m(A(t_o))$ denotes the monad of the point A, where a is an infinitesimal number.

For finding curvature formula of the envelope of a family of lines, we follow the following algorithm.

1. Find the envelope curve using **Theorem 1.3** according to the case under consideration.
2. Find the singularity and collinearity order of the envelope curve.
3. Consider three infinitely closed points $A(t_o)$, $B(t_o+\alpha)$ and $C(t_o+\beta)$ on the envelope curve $\gamma(t)$ such that

   $A(t_o) \in L_{t_o}$, $B(t_o+\alpha) \in L_{t_o+\alpha}$ and $C(t_o+\beta) \in L_{t_o+\beta}$
4. Apply the generalized curvature formula given in **Theorem 1.2** at the points $A(t_o)$, $B(t_o+\alpha)$ and $C(t_o+\beta)$.

   Where a and b are infinitesimal numbers.

The following theorems will give a new formula of the generalized curvature of the envelope of a family of lines.

Theorem 2.1

Let $A= (t_o)$ be a regular point of the envelope curve γ of the family $L_t : u(t)X + v(t)Y + w(t)Z =0$ in ***PHC***, then the generalized curvature K_G of the envelope curve at a point A is given by

$$\frac{\left|\left(r'(t)q''(t)-r''(t)q'(t)\right)^2+\left(p''(t)q'(t)-p'(t)q''(t)\right)^2+\left(p'(t)r''(t)-p''(t)r'(t)\right)^2\right|^{\frac{1}{2}}}{2\left|p'(t)^2+q'(t)^2+r'(t)^2\right|^{\frac{3}{2}}}, \quad \text{...(2.1.1)}$$

where $p(t)$, $r(t)$ and $q(t)$ are as given in **Theorem 1.3** for $n=1$

Proof:

Let $A= \gamma(t_o)$ be a standard point on the envelope of the curve γ, and $B = \gamma(t_o+\alpha)$, $C = \gamma(t_o+\beta)$ be two points infinitely close to A. Let L_t , $L_{t+\alpha}$ and $L_{t+\beta}$ be three lines of the family $\{L_t\}$ having A ,B and C as contact points with the envelope curve, respectively.
Then;

L_t : $u(t)X + v(t)Y + w(t)Z =0$,

$L_{t+\alpha}$: $u(t+\varepsilon)X + v(t+\varepsilon)Y + w(t+\varepsilon)Z =0$,

$L_{t+\beta}$: $u(t+\varepsilon)X + v(t+\varepsilon)Y + w(t+\varepsilon)Z =0$.

Since, the point A is regular, then **Theorem 1.3** for $n=1$ is satisfied, and therefore $\gamma(t)= (p_1(t), r_1(t), q_1(t))$

Using the spherical case of the generalized curvature given in **Theorem 1.2** for a curve $\gamma = \left(x(t),y(t),z(t)\right)$, we get

$$K_G = \frac{\left|(y'z''-y''z')^2+(x''z'-x'z'')^2+(x'y''-x''y')^2\right|^{\frac{1}{2}}}{2\left|x'^2+y'^2+z'^2\right|^{\frac{3}{2}}} \quad \text{... (2.1.2)}$$

Now replacing each of x, y and z by $p_1(t)$, $r_1(t)$ and $q_1(t)$, respectively, we get the required result. ■

Theorem 2.2

Let $A= \gamma(t_o)$ be a singular point of the envelope curve γ of order $n-1$, and let m be the order of the first nonzero derivative which is not collinear with $\gamma^{(n)}(t)$, that is, $\gamma'(t) = \gamma''(t)=\cdots = \gamma^{(n-1)}(t) =0$, $\gamma^{(n)}(t) \neq 0$, and
$\gamma'(t).\gamma''(t)=\gamma'(t).\gamma'''(t)=\cdots=\gamma'(t).\gamma^{(m-1)}(t)=\cdots=\gamma^{(n-1)}.(t)\,\gamma^{(m-1)}(t)=0$,
$\gamma^{(n)}(t).\gamma^{(m)}(t) \neq 0$

Then, the generalized curvature K_G of the envelope curve γ at the points of the monad of $\boldsymbol{A}$ is given by

$$\frac{(\boldsymbol{n}!)^{\frac{m}{n}}\left|\left(\boldsymbol{r}^{(n)}\boldsymbol{q}^{(m)}-\boldsymbol{r}^{(m)}\boldsymbol{q}^{(n)}\right)^2+\left(\boldsymbol{p}^{(m)}\boldsymbol{q}^{(n)}-\boldsymbol{p}^{(n)}\boldsymbol{q}^{(m)}\right)^2+\left(\boldsymbol{p}^{(n)}\boldsymbol{r}^{(m)}-\boldsymbol{p}^{(m)}\boldsymbol{r}^{(n)}\right)^2\right|^{\frac{1}{2}}}{\boldsymbol{m}!\left|\boldsymbol{p}^{(n)^2}+\boldsymbol{q}^{(n)^2}+\boldsymbol{r}^{(n)^2}\right|^{\frac{m+n}{2n}}} \quad \ldots(2.2.1)$$

Moreover, the Cartesian coordinate of the generalized curvature K_G of the envelope curve γ at the points of the monad of $\boldsymbol{A}$ is given by

$$\frac{(\boldsymbol{n}!)^{\frac{m}{n}}\left|\left(\frac{\boldsymbol{p}(\boldsymbol{t})}{\boldsymbol{q}(\boldsymbol{t})}\right)^{(n)}\cdot\left(\frac{\boldsymbol{r}(\boldsymbol{t})}{\boldsymbol{q}(\boldsymbol{t})}\right)^{(m)}-\left(\frac{\boldsymbol{p}(\boldsymbol{t})}{\boldsymbol{q}(\boldsymbol{t})}\right)^{(m)}\cdot\left(\frac{\boldsymbol{r}(\boldsymbol{t})}{\boldsymbol{q}(\boldsymbol{t})}\right)^{(n)}\right|}{\boldsymbol{m}!\left|\left(\frac{\boldsymbol{p}(\boldsymbol{t})}{\boldsymbol{q}(\boldsymbol{t})}\right)^{(n)^2}+\left(\frac{\boldsymbol{r}(\boldsymbol{t})}{\boldsymbol{q}(\boldsymbol{t})}\right)^{(n)^2}\right|^{\frac{m+n}{2n}}} \quad \ldots(2.2.2)$$

where n and m are positive integer numbers.

Proof:

First, applying the spherical case of the generalized curvature given in **Theorem 1.2** at $\boldsymbol{x}=\boldsymbol{p_1}(\boldsymbol{t})$, $\boldsymbol{y}=\boldsymbol{r_1}(\boldsymbol{t})$ and $\boldsymbol{z}=\boldsymbol{q_1}(\boldsymbol{t})$, we get the generalize curvature formula **(2.2.1)**. Since the point $(\boldsymbol{p_1}(\boldsymbol{t}), \boldsymbol{r_1}(\boldsymbol{t}), \boldsymbol{q_1}(\boldsymbol{t}))$ in ***PHC*** is equivalent to the point $(\boldsymbol{p_1}(\boldsymbol{t})/\boldsymbol{q_1}(\boldsymbol{t}),\boldsymbol{r_1}(\boldsymbol{t})/\boldsymbol{q_1}(\boldsymbol{t}),\boldsymbol{1})$, so again, applying the spherical case of generalized curvature, we get

$$K_G=\frac{(\boldsymbol{n}!)^{\frac{m}{n}}\left|\left(\boldsymbol{y}^{(n)}\boldsymbol{z}^{(m)}-\boldsymbol{y}^{(m)}\boldsymbol{z}^{(n)}\right)^2+\left(\boldsymbol{z}^{(n)}\boldsymbol{x}^{(m)}-\boldsymbol{z}^{(m)}\boldsymbol{x}^{(n)}\right)^2+\left(\boldsymbol{x}^{(n)}\boldsymbol{y}^{(m)}-\boldsymbol{x}^{(m)}\boldsymbol{y}^{(n)}\right)^2\right|^{\frac{1}{2}}}{\boldsymbol{m}!\left|\boldsymbol{x}^{(n)^2}+\boldsymbol{y}^{(n)^2}+\boldsymbol{z}^{(n)^2}\right|^{\frac{m+n}{2n}}} \ldots(2.2.3)$$

Thus, putting $\boldsymbol{x}=p_1(\boldsymbol{t})/q_1(\mathrm{t})$, $\boldsymbol{y}=\boldsymbol{r_1}(\boldsymbol{t})/\boldsymbol{q_1}(\boldsymbol{t})$ and $\boldsymbol{z}=\boldsymbol{1}$, in **(2.2.3),** we obtain the formula **(2.2.2)**. ■

Corollary 2.3

Let $\boldsymbol{A}=\gamma(\boldsymbol{t_o})$ be a singular point of the envelope curve γ satisfying the hypothesis of **Theorem 2.2**.

Moreover, let the coefficient vector $(\boldsymbol{u}(\boldsymbol{t}), \boldsymbol{v}(\boldsymbol{t}), \boldsymbol{w}(\boldsymbol{t}))$ of the envelope curve has a singularity of order $\boldsymbol{n-1}$, then the generalized curvature K_G of the envelope curve γ at points in the monad of $\boldsymbol{A}$ is given by

$$\frac{(\boldsymbol{n}!)^{\frac{m}{n}}\left|(\boldsymbol{r}_n(\boldsymbol{t})\boldsymbol{q}_m(\boldsymbol{t})-\boldsymbol{r}_m(\boldsymbol{t})\boldsymbol{q}_n(\boldsymbol{t}))^2+(\boldsymbol{p}_m(\boldsymbol{t})\boldsymbol{q}_n(\boldsymbol{t})-\boldsymbol{p}_n(\boldsymbol{t})\boldsymbol{q}_m(\boldsymbol{t}))^2+(\boldsymbol{p}_n(\boldsymbol{t})\boldsymbol{r}_m(\boldsymbol{t})-\boldsymbol{p}_m(\boldsymbol{t})\boldsymbol{r}_n(\boldsymbol{t}))^2\right|^{\frac{1}{2}}}{\boldsymbol{m}!\left|\boldsymbol{p}_n^2(\boldsymbol{t})+\boldsymbol{q}_n^2(\boldsymbol{t})+\boldsymbol{r}_n^2(\boldsymbol{t})\right|^{\frac{m+n}{2n}}} \ldots(2.3.1)$$

and the cartesian coordinate curvature $K_G(\boldsymbol{t})$ of the envelope curve γ at $\boldsymbol{A}$ is given by

$$\frac{(n!)^{\frac{m}{n}}\left|\left(\frac{p_n(t)}{q_n(t)}\right)\left(\frac{r_m(t)}{q_m(t)}\right)-\left(\frac{p_m(t)}{q_m(t)}\right)\left(\frac{r_n(t)}{q_n(t)}\right)\right|}{m!\,\left|\left(\frac{p_n(t)}{q_n(t)}\right)^2+\left(\frac{r_n(t)}{q_n(t)}\right)^2\right|^{\frac{m+n}{2n}}} \qquad \ldots(2.3.2)$$

Proof:

By **Theorem 2.2** we have

$$K_G(t)=\frac{(n!)^{\frac{m}{n}}\left|\left(r^{(n)}q^{(m)}-r^{(m)}q^{(n)}\right)^2+\left(p^{(m)}q^{(n)}-p^{(n)}q^{(m)}\right)^2+\left(p^{(n)}r^{(m)}-p^{(m)}r^{(n)}\right)^2\right|^{\frac{1}{2}}}{m!\,\left|p^{(n)2}+q^{(n)2}+r^{(n)2}\right|^{\frac{m+n}{2n}}}$$

Since the coefficient vector $(u(t), v(t), w(t))$ of the envelope curve has a singularity of order $n\text{-}1$, so we get

$$u'(t)=v'(t)=w'(t)=\cdots=u^{(n-1)}(t)=v^{(n-1)}(t)=w^{(n-1)}(t)=0,$$

and $\quad (u^{(n)}(t),v^{(n)}(t),w^{(n)}(t)) \neq \vec{0}$

Therefore,

$$\left.\begin{aligned} p^{(n)}(t) &= v^{(n)}(t)w(t)-w^{(n)}(t)v(t)=p_n(t)\\ r^{(n)}(t) &= w^{(n)}(t)u(t)-u^{(n)}(t)w(t)=r_n(t)\\ q^{(n)}(t) &= u^{(n)}(t)v(t)-v^{(n)}(t)u(t)=q_n(t)\end{aligned}\right\} \qquad \ldots(2.3.3)$$

Hence, the result of the first part is proved.

To prove the second part put $x=\mathbf{p_n(t)/q_n(t)}$, $y=\mathbf{r_n(t)/q_n(t)}$ and $z=\mathbf{1}$ and then apply the spherical curvature formula **(2.2.3)** to obtain the formula **(2.3.2)**. ■

Corollary 2.4

Let $A=\gamma(t_o)$ be a singular point of the envelope curve γ satisfying the hypothesis of **Theorem 2.2**. Let $\gamma(t)=(p(t), r(t), q(t))$ be such that $q(t)$ has a nonzero constant value, then the generalized curvature K_G of the envelope curve γ at points of the monad of A is given by

$$\frac{(n!)^{\frac{m}{n}}\left|\left(p^{(n)}(t)r^{(m)}(t)-p^{(m)}(t)r^{(n)}(t)\right)^2\right|}{m!\,\left|p^2(t)+r^2(t)\right|^{\frac{m+n}{2n}}}\cdot q^{\frac{m-n}{n}} \qquad \ldots(2.4.1)$$

Proof:

Without loss of generality we use the cartesian coordinate form **(2.2.2)** of **Theorem 2.2** to obtain

$$K_G(t)=\frac{(n!)^{\frac{m}{n}}\left|\left(\frac{p(t)}{q(t)}\right)^{(n)}\left(\frac{r(t)}{q(t)}\right)^{(m)}-\left(\frac{p(t)}{q(t)}\right)^{(m)}\left(\frac{r(t)}{q(t)}\right)^{(n)}\right|}{m!\left|\left(\frac{p(t)}{q(t)}\right)^{(n)^2}+\left(\frac{r(t)}{q(t)}\right)^{(n)^2}\right|^{\frac{m+n}{2n}}} \quad \ldots (2.4.2)$$

Since the value of $q(t)$ is constant, we get

$$K_G(t)=\frac{(n!)^{\frac{m}{n}}\left(\frac{1}{q}\right)^2\left|p(t)^{(n)}r(t)^{(m)}-p(t)^{(m)}r(t)^{(n)}\right|}{m!\left(\frac{1}{q}\right)^{\frac{m+n}{n}}\left|p(t)^{(n)^2}+r(t)^{(n)^2}\right|^{\frac{m+n}{2n}}}$$

$$=\frac{(n!)^{\frac{m}{n}}\left|\left(p^{(n)}(t)r^{(m)}(t)-p^{(m)}(t)r^{(n)}(t)\right)^2\right|}{m!\left|p^2(t)+r^2(t)\right|^{\frac{m+n}{2n}}}\cdot q^{\frac{m-n}{n}} \quad . \blacksquare$$

Remark 2.5

If $q(t)=0$ then, by using either equation **(2.2.1)** or the equation **(2.3.1)**, we can find a spherical generalized curvature K_G , but it does not represent a real curvature of the envelope curve . We shall call such value of curvature **Ideal Curvature** of a curve γ at points of the monad of $A=\gamma(t_o)$.

Example 2.6

Consider the family of lines $2x - 3ty + t^3 = 0$

By applying the algorithm given at the beginning of this section, we get

$u = 2$ $\quad u' = 0$ $\quad u'' = 0$ $\quad u''' = 0$
$v = -3t$ $\quad v' = -3$ $\quad v'' = 0$ $\quad v''' = 0$
$w = 2t^3$ $\quad w' = 6t^2$ $\quad w'' = 12t$ $\quad w''' = 12$

Now we determine the singularity and collinearity

$\gamma(0)=(0,0)$ $\quad \gamma'(0)=(0,0)$ $\quad \gamma''(0)=(0,2)$ $\quad \gamma'''(0)=(12,0)$

Thus γ has a first singularity order (that is $n=2$) and the order of collinearity is equal to 3. The envelope curve $\gamma(t)$ is given by

$(X\varepsilon(t), Y\varepsilon(t), Z\varepsilon(t))$
$=(v'(t)w(t)-w'(t)v(t), w'(t)u(t)-u'(t)w(t), u'(t)v(t)-v'(t)u(t))$
$=(6t^3, 6t^2, 12)$

Since the value of **q(t)** is constant, so using **Corollary 2.4**, we get,

$$K_G=\frac{(2!)^{\frac{3}{2}}\left|\left(p^{(2)}(t)r^{(3)}(t)-p^{(3)}(t)r^{(2)}(t)\right)^2\right|}{3!\left|p^2(t)+r^2(t)\right|^{\frac{3+2}{2\times 2}}}\cdot q^{\frac{3-2}{2}}=\frac{1}{\sqrt{6}}\cdot\sqrt{12}=\sqrt{2}$$

Note that if we use the cartesian coordinate, we find that $\gamma(t)$ is equal to

$$(x(t),y(t))=\left(\frac{X_{\varepsilon}(t)}{Z_{\varepsilon}(t)},\ \frac{Y_{\varepsilon}(t)}{Z_{\varepsilon}(t)}\right)=\left(\frac{v'(t)w(t)-w'(t)v(t)}{u'(t)v(t)-v'(t)u(t)},\ \frac{w'(t)u(t)-u'(t)w(t)}{u'(t)v(t)-v'(t)u(t)}\right)$$
$$=(1/2)(t^3,\ t^2)$$

Here γ also has a first singularity order (that is $n=2$) and the order of collinearity is equal to 3. Thus by using the usual two dimensional forms of the generalized curvature, we get, (see **Figure 2.3**)

$$K_G=\frac{(2!)^{\frac{3}{2}}\left|\left(x^{(2)}(t)y^{(3)}(t)-x^{(3)}(t)y^{(2)}(t)\right)^{2}\right|}{3!\ \left|\left(x^{(2)}(t)\right)^{2}+\left(y^{(2)}(t)\right)^{2}\right|^{\frac{3+2}{2\times 2}}}=\left.\frac{(2!)^{\frac{3}{2}}\left|3t\cdot 0-3\cdot 1\right|}{3!\ \left|(3t)^{2}+(1)^{2}\right|^{\frac{3+2}{2\times 2}}}\right|_{t=0}=\sqrt{2}$$

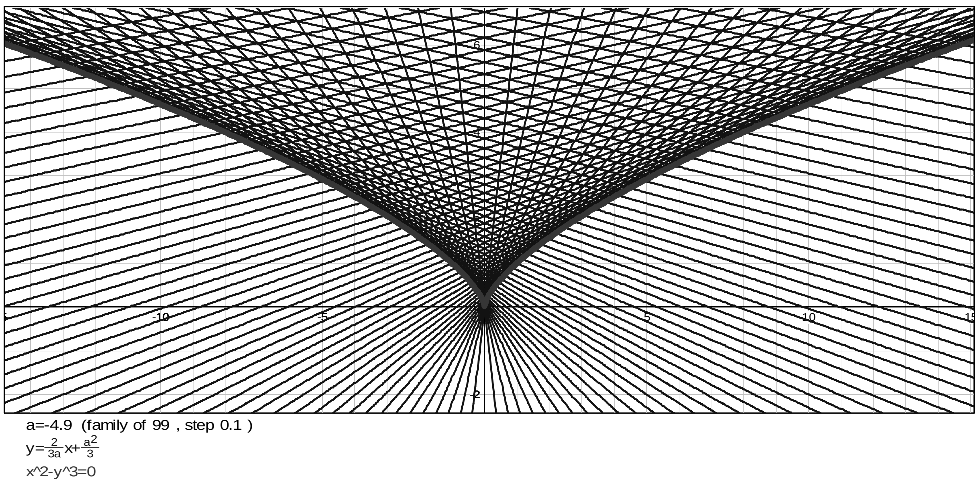

Figure 2.3

Remark: The graph of the equation of the above example is plotted with specific software **Omnigraph V3.1b-2005**.

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