On sp-Weakly Regular Rings

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الملخص

يقال للحلقة R أنها يمنى (يسرى) منتظمة بضعف من النمط -sπ إذا كان لكل $a \in R$ و عدد صحيح موجب n و $a^n \in a^n Ra^{2n} R$. هذا البحث هو استمرار لدراسة الحلقات المنتظمة بضعف من النمط -sπ في المصدر [8] R. D. Mahmood و A. M. Abdul-Jabbar كما تم توسيع الخواص الأساسية للحلقات المنتظمة بضعف من النمط -sπ، وتم أعطاء العلاقة بين الحلقات المنتظمة بضعف من النمط -sπ و الحلقات شبه المنتظمة النمط -π و الحلقات المنتظمة من النمط π- bi .

ABSTRACT

A ring R is said to be right(left) sπ-weakly regular if for each $a \in R$ and a positive integer n, $a^n \in a^n R a^{2n} R$ ($a^n \in R a^{2n} R a^n$). In this paper, we continue to study sπ-weakly regular rings due to R. D. Mahmood and A. M. Abdul-Jabbar [8]. We first consider properties and basic extensions of sπ-weakly regular rings, and we give the connection of sπ-weakly regular, semi π-regular and π-biregular rings.

1. Introduction

Throughout this paper rings are associative with identity. For a subset X of a ring R, the right annihilator of X in R is r(X) = { r ∈ R: xr = 0, for all x ∈ X}. J(R), Y(R) and N will stand respectively for the Jacobson radical, right singular ideal of R and the set of all nilpotent elements. A ring R is called right (left) s-weakly regular if for each a ∈ R, a ∈ aRa^2R (a ∈ Ra^2Ra). This concept was introduced by V. Gupta [4] and W. B. Vasantha Kandasamy [10]. As a generalization of this concept the authors in [8] defined sπ-weakly regular ring that is a ring such that for each a ∈ R and a positive integer n, $a^n \in a^nRa^{2n}R$ ($a^n \in Ra^{2n}Ra^n$). In the present work we develop further properties of sπ-weakly regular rings, and we give the connection of sπ-weakly regular rings with other rings. Recall that:

(1) R is called reduced if it has no nonzero nilpotent elements.
(2) According to Cohn [3], a ring R is called reversible if ab = 0 implies ba = 0 for a, b ∈ R. Obviously every reduced ring is reversible. It is easy to see that R is reversible if and only if right (left) annihilator in R is a two-sided ideal.

2. sp**WR-Rings**

In this section we continue to study sπWR-rings and we establish relation between such rings, semi π-regular and π-biregular rings.

Definition 2.1[8]:

R is called right(left) sπWR-ring if for each a ∈ R, there exists a positive integer n = n(a), depending on a such that $a^n \in a^nRa^{2n}R$ ($a^n \in Ra^{2n}Ra^n$). R is called sπWR-ring if it is both right and left sπWR-ring.

Recall that, a right ideal P of a ring R is said to be completely prime if for all a, b ∈ R such that ab ∈ P, then either a ∈ P or b ∈ P.

Proposition 2.2:

If R is a right sπWR-ring and $r(a^n) \subset r(a)$, for every a ∈ R and a positive integer n. Then every completely prime ideal of R is maximal.

Proof:

Let P be a completely prime ideal of R and J any ideal of R such that P ⊂ J. Then, there exists x ∈ J and x ∉ P. Since R is right sπWR-ring, there exist b, c ∈ R and a positive integer n such that $x^n = x^n b x^{2n} c$ and $x(1-b x^{2n}c) = 0$ (since $r(a^n) \subset r(a)$). So, $x(1-b x^{2n}c) = 0 \in P$. Since x ∉ P and P is completely prime ideal of R, so $(1- b x^{2n} c) \in P \subset J$, and hence

(1- b x^{2n} c) $\in$ J. Hence 1- b x^{2n} c + b x^{2n} c = 1 $\in$ J and therefore, J = R. Consequently P is a maximal ideal of R. ♦

Theorem 2.3:

If R is a right sπWR-ring and I is an ideal of R, then R/I is sπWR-ring.

Proof:

Let R be a right sπWR-ring. Then, there exist b,c $\in$ R such that a^n = a^n b a^{2n} c, for some positive integer n. Now,

$$(a + I)^n (b + I) (a + I)^{2n} (c + I) = (a^n + I) (b + I) (a^{2n} + I) (c + I)$$
$$= a^n b a^{2n} c + I$$
$$= a^n + I.$$

Therefore, R/I is a right sπWR-ring. ♦

Now, the following result is given in [7] and [1].

Lemma 2.4:

Let R be a reduced ring. Then, for every a $\in$ R and every positive integer n,

(1) $r(a^n) = \ell(a^n)$.

(2) $a^n R \cap r(a^n) = 0$.

(3) $r(a) = r(a^n)$.

Theorem 2.5:

Let R be a reduced ring. Then, R is a right sπWR-ring if and only if $R / r(a^m)$ is sπWR-ring, for every a $\in$ R and a positive integer m.

Proof:

Assume that $R / r(a^m)$ is a right sπWR-ring, for every a $\in$ R and a positive integer m. Now,

$a^n + r(a^m) = (a^n + r(a^m)) (b + r(a^m)) (a^{2n} + r(a^m)) (c + r(a^m))$.

$= a^n b a^{2n} c + r(a^m)$. Then, $a^n - a^n b a^{2n} c \in r(a^m)$, this implies that $a^m (a^n - a^n b a^{2n} c) = 0$ and hence 1- b a^{2n} c $\in$ $r(a^{m+n}) = r(a^n)$ since R is reduced. So, a^n (1- b a^{2n} c) = 0 and $a^n = a^n b a^{2n} c$. Therefore, R is a right sπWR-ring.

Conversely, assume that R is a right sπWR-ring. Then, by Theorem 2.3, $R / r(a^m)$ is an sπWR-ring. ♦

Next we consider the Jacobson radical, the right singular ideal of a and the set of all nilpotent elements N of right sπWR-ring.

Theorem 2.6:

Let R be a right sπWR-ring. Then,

(1) J(R) = N

(2) If R is a reduced, then Y(R) is a nilideal of R.

Proof:

(1) Let $0 \neq x \in J(R)$. Then, there exist $b, c \in R$ and a positive integer n such that $x^n = x^n b x^{2n} c$. Then, $1 - b x^{2n} c$ is invertible. Therefore, there exists an invertible element u such that $(1 - b x^{2n} c) u = 1$. Multiply from the left by x^n, we obtain $(x^n - x^n b x^{2n} c) u = x^n$. Whence it follows that $x^n = 0$, so $x \in N$ and $J(R) \subseteq N$. Since $N \subseteq J(R)$. Thus, J(R) = N.

(2) Let a be a nonzero element in Y(R). Then, r(a) is an essential right ideal of R. Since R is right sπWR-ring, there exist $b, c \in R$ and a positive integer n such that $a^n = a^n b a^{2n} c$. Consider $r(a^n) \cap b a^{2n} R$, let $x \in r(a^n) \cap b a^{2n} R$. Then, $a^n x = 0$ and $x = b a^{2n} t$, for some $t \in R$. So, $a^n b a^{2n} t = 0$, then $a^n = 0$. Thus, $a^{2n} = 0$ and hence $b a^{2n} t = 0$, yields $x = 0$. Therefore, $r(a^n) \cap b a^{2n} R = 0$. Since $r(a^n)$ is a nonzero essential right ideal of R, then $b a^{2n} = 0$ and hence $a^{2n} = 0$. Thus, $a^n = 0$. So, Y(R) is a nilideal of R. ♦

Recall that, a ring R is called right(left) semi π-regular [1] if for every $a \in R$, there exists $b \in R$ and a positive integer n such that $a^n = a^n b$ and $r(a^n) = r(b)$ ($a^n = b a^n$ and $\ell(a^n) = \ell(b)$).

The following theorem gives the conditions of sπWR-ring to be semi π-regular.

Theorem 2.7:

If R is a reversible, sπWR-ring and $a^{2n} R = R a^{2n}$, for every $a \in R$ and a positive integer n, then R is semi π-regular.

Proof:

Let R be sπWR-ring. Then, for every $a \in R$, there exist $b, c \in R$ and a positive integer n such that $a^n = a^n b a^{2n} c$. Since $b a^{2n} \in R a^{2n} = a^{2n} R$. So, $a^n = a^{3n} h c$, for some $h \in R$. This implies that $a^n = a^n a^{2n} t$ (set t = hc). If we set $y = a^{2n} t$, then $a^n = a^n y$. It only remains to show that $r(a^n) = r(y)$. Let $x \in r(a^n)$, then $a^n x = 0$ and $x a^n a^n t = 0$, so $x a^{2n} t = 0$ and $x y = 0$. Hence $x \in \ell(y) = r(y)$ and $r(a^n) \subseteq r(y)$. On the other hand, let $z \in r(y)$, then $y z = 0$ and $a^{2n} r z = 0$, this implies that $a^n a^{2n} t z = 0$, so $a^n z = 0$, thus $z \in r(a^n)$ and $r(y) \subseteq r(a^n)$. Therefore, $r(a^n) = r(y)$ and R is a right semi π-regular ring in the same method we can easily show that R is a left semi π-regular. ♦

A ring R is called π-biregular [9] if for any $a \in R$, $R\, a^n\, R$ is generated by a central idempotent, for some positive integer n.

We begin by stating following lemma, which will be used in proof of our main result.

Lemma 2.8:

A ring R is π-biregular if and only if for every $a \in R$, $R\, a^n\, R \oplus r(a^n) = R$, for some positive integer n.

Proof:

See [9]. ♦

Proposition 2.9:

Let R be a reduced ring. Then, R is an sπWR-ring if and only if R is a π-biregular ring.

Proof:

Assume that R is π-biregular ring. Then, by Lemma 2.8, $R = r(a^n) \oplus R\, a^n\, R$. Since R is reduced, then $r(a^n) = r(a^{2n})$ for every positive integer n and hence $R = r(a^{2n}) \oplus R\, a^n\, R$. In particular, $1 = b + t_1\, a^{2n}\, t_2$, for some t_1, $t_2 \in R$ and $b \in r(a^n)$. So, $a^n = a^n\, t_1\, a^{2n}\, t_2$. Therefore, R is sπWR-ring.

Conversely, assume that R is sπWR-ring, then for every $a \in R$, there exist $t_1, t_2 \in R$ such that $a^n = a^n\, t_1\, a^{2n}\, t_2$, for some positive integer n. This implies that $a^n = a^n\, t_1\, a^n\, a^n\, t_2$. So, $a^n = a^n\, t_1\, a^n\, t$ ($t = a^n\, t_2$), for some $t \in R$.Thus, $a^n\, R = a^n\, R\, a^n\, R$. Also, by [8, Theorem 2.7], $R = r(a^n) \oplus R\, a^{2n}\, R$. Thus, $R = r(a^n) \oplus R\, a^n\, R$. So, by Lemma 2.8, R is π-biregular ring. ♦

3. Commutative sp-Weakly Regular Rings

In this section, all our rings are commutative. We discuss some properties of sπ-weakly regular rings.

We now introduce the following lemma, which may be used frequently in the sequel.

Lemma 3.1:

A ring R is local if and only if for any elements $r, s \in R$ such that $r + s = 1$ implies that either r or s is a unit.

Proof:

See [6]. ♦

Theorem 3.2:

Let R be a local, sπ-weakly regular ring. Then, every element in R is either a unit or nilpotent.

Proof:

Assume that R is a local ring and a $\in$ R. Since R is sπ-weakly regular, there exist x, y $\in$ R such that $a^n = a^n$ x a^{2n} y, for some positive integer n. So, a^n (1- x a^{2n} y) = 0 and hence a^n (1- a^{2n} x y) = 0. If a^n = 0, then a is nilpotent. If 1- a^{2n} x y $\neq$ 0, and $a^n \neq 0$, then 1- a^{2n} x y is a zero divisor, that is 1- a^{2n} x y is non unit. Since (1- a^{2n} x y) + a^{2n} x y = 1 by Lemma 3.1, a^{2n} x y is a unit. This implies that a is a unit. If 1- a^{2n} x y = 0, then a is a unit. Which completes the proof. ♦

Proposition 3.3:

Let R be sπ-weakly regular ring. Then, each element of R is either a unit or zero divisor.

Proof:

Let a be a non zero divisor in R and R is sπ-weakly regular, there exist elements x, y $\in$ R and a positive integer n such that $a^n = a^n$ x a^{2n} y. So, a^n (1- x a^{2n} y) = 0 and hence a^n (1- a^{2n} x y) = 0. Since a is a non zero divisor, then a^n is nonzero divisor. Therefore, 1- a^{2n} x y = 0, which implies that a^{2n} (x y) = 1. Hence a^{2n} x y = a (a^{2n-1} (x y)) = 1. Thus, a is a unit. ♦

Theorem 3.4:

If P is a prime ideal of a ring R and R / P is sπ-weakly regular ring. Then, P is a maximal ideal.

Proof:

Let a $\in$ R. Then, a + P $\in$ R / P, since R / P is sπ-weakly regular, there exist b + P, c + P $\in$ R / P and a positive integer n such that

$a^n + P = (a^n + P) (b + P) (a^{2n} + P)(c + P)$

$a^n + P = a^n b a^{2n} c + P$. So, $a^n - a^n b a^{2n} c \in P$ and hence $a^n (1- b a^{2n} c) \in P$. Since $a^n \notin P$, then (1- b a^{2n} c) $\in$ P, gives

$1 + P = b c a^{2n-1} a + P$

= (b c a^{2n-1} + P) (a + P) , this shows that a + P has an inverse. Thus, R/P is a division ring. Whence P is a maximal ideal. ♦

Corollary 3.5:

If R is an sπ-weakly regular ring, then every prime ideal of R is maximal.

Proof:

Let P be a prime ideal of R. Since R is sπ-weakly regular ring, then by Theorem 2.3, R/P is sπ-weakly regular. Thus, by Theorem 3.4, P is a maximal. ♦

Lemma 3.6:

If I is a primary ideal, then $\sqrt{I}$ is prime.

Proof:

See [5]. ♦

Lemma 3.7:

Let I be an ideal of R. If $\sqrt{I}$ is a maximal ideal of R, then I is primary ideal.

Proof:

See [2]. ♦

Theorem 3.8:

If R is sπ-weakly regular ring and I is an ideal of R, then I is primary if and only if $\sqrt{I}$ is prime.

Proof:

If I is primary, then by Lemma 3.6, $\sqrt{I}$ is prime.

Conversely, suppose that $\sqrt{I}$ is prime. Since R is sπ-weakly regular ring, then by Corollary 3.5, $\sqrt{I}$ is maximal. Therefore, by Lemma 3.7, I is primary. ♦

The following result is the relationship between sπ-weakly regular ring with it is ideals by adding the condition that every ideal of R is completely semi-prime.

Theorem 3.9:

If every ideal of R is completely semi-prime, then R is sπ-weakly regular if and only if for each ideal I of R, I = $\sqrt{I}$ holds.

Proof:

Let R be sπ-weakly regular. It is obvious that $I \subseteq \sqrt{I}$. Now, let $b \in \sqrt{I}$, then $b^n \in I$ and hence $b^{2n} \in I$, for some positive integer n. Now, $b^n = b^n t_1 b^{2n} t_2$, for some $t_1, t_2 \in R$. Since $b^n \in I$ and every ideal of R is completely semi-prime, then $b \in I$. Therefore, $\sqrt{I} = I$.

Conversely, assume that I = $\sqrt{I}$, for each ideal I of R. If we set $I = a^n R a^{2n} R$, for some positive integer n. Therefore, $a^n R a^{2n} R = \sqrt{a^n R a^{2n} R}$. Now, $a^{3n} \in a^n R a^{2n} R$, then $a^n \in \sqrt{a^n R a^{2n} R} = a^n R a^{2n} R$. So, $a^n \in a^n R a^{2n} R$. Whence R is sπ-weakly regular. ♦

REFERENCES

[1] Al-Kouri M. R. M. (1996), On π-regular rings, M. Sc. Thesis, Mosul University.

[2] Burton, D. M. (1967), Introductuin to Modern Abstract Algebra, Addison-Wesley Publishing Company, Inc.

[3] Cohn P. M. (1999), Reversible rings, Bull. London Math. Soc., 31, Pp.641-648.

[4] Gupta, V. (1984), A generalization of strongly regular rings, Acta. Math. Hung., vol. 43 (1-2),Pp. 57-61.

[5] Hungerford, T. W.(1974), Algebra, Holt Rinehart and Winston, Inc.

[6] Lambek J. (1966), Lectures on rings and modules, Blaisdell Publishing Co., London.

[7] Mahmood, A. S. (1990), On von Neumann regular rings, M. Sc. Thesis, Mosul University.

[8] Mahmood R. D. and Abdul-Jabbar A. M., sπ-weakly regular rings, Accepted In Raf. J. Of Comp. Sc. And Math.

[9] Naoum, F. S. (2004), On π-regular rings, Ph. D. Thesis, Mosul University.

[10] Vasantha Kandasamy, W. B. (1993), s-weakly regular group rings, Arch. Math. (BRNO), 39-41.