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Modified Jackknifed Ridge Estimator in Bell Regression Model: Theory, Simulation and Applications

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ABSTRACT: Regression analyses look at how the answer variable and one or more explanatory factors are related. When the explanatory variables are linearly dependent, it becomes practically difficult to model this relationship in real-world applications. Several shrinkage estimators are suggested traditionally to avoid this problem. When there is linear dependency between the explanatory variables, one of the most widely used estimate techniques is the ridge estimator. In this paper, we suggested a jackknifed variant of the ridge estimator to reduce biasedness for the Bell regression model. The simulation results demonstrated that the proposed method is able to effectively outperform other alternatives estimators in terms of mean squared error (MSE). Further, the application results, which are based on two datasets: aircraft data and mine fracture data showed that the suggested estimator improves the ridge estimator's performance in terms of MSE.

Keywords: Bell regression model; Jackknife estimator; Multicollinearity; Ridge Regression; Simulation Study.

1. INTRODUCTION

Because it shows how a number of explanatory variables relate to the response variable of interest, statistical modelling is crucial in many scientific study fields. The response variable must have a normal distribution in a linear regression model, according to the assumption. This presumption might not be accurate in many practical instances, though. For instance, the response variable may be positively skewed in the field of medicine. Consequently, it might not be appropriate to use a linear regression model. The generalized linear model (GLM), a type of regression models that includes both continuous and discrete response variables, is becoming more and more common [1].

Because of the multicollinearity between the explanatory variables, $\mathbf{X}^T\mathbf{X}$ becomes singular or may amplify the variance of the maximum likelihood estimator (MLE). As a result, standard estimating techniques like MLE frequently deliver subpar results. To get around the multicollinearity in the linear regression model, various authors have suggested the ridge, Liu, Liu-type, and other estimators as alternatives to MLE [2-4]. These estimators have been extended to the GLMs [5-21].

By using a jackknife approach on these estimators, bias can be decreased. Using this method, experimental data can be processed to produce statistical estimators for unidentified parameters. The jackknife approach offers the benefit of giving an estimator with a minimal bias while maintaining the advantages of large samples [22].

The employment of the Jackknife technique with the ridge estimator is the primary goal of this research. The suggested estimator will effectively contribute to enhancing GLM's ridge estimator's performance.

2. Statistical Methodology

2.1 MLE estimator in Bell regression model

Assume that $(y_i, \mathbf{x}_i)$, $i = 1, 2, ..., n$ is independent observed data with the explanator variable vector $\mathbf{x}_i \in R^{p+1}$ and the response variable $y_i \in R$ which follows the Bell distribution. Then, the density function of y_i can be expressed as

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$$P(Y=y)=\frac{\theta^{y}e^{-e^{\theta}+1}B_{y}}{y!},\ y=0,1,2,..., \qquad (1)$$

where $\theta>0$ and $B_y=(1/e)\sum_{d=0}^{\infty}(d^{y}/d!)$ is the Bell numbers [23-25] with

$$E(y)=\theta e^{\theta}, \qquad (2)$$

$$Var(y)=\theta(1+\theta)e^{\theta}. \qquad (3)$$

Assuming $\psi=\theta e^{\theta}$ and $\theta=W_{\circ}(\psi)$ where $W_{\circ}(.)$ is the Lambert function [24]. Then,

$$P(Y=y)=\exp\left(1-e^{W_{\circ}(\psi)}\right)\frac{W_{\circ}(\psi)^{y}B_{y}}{y!},\ y=0,1,2,..., \qquad (4)$$

The link function is providing the relation of the mean and the natural parameter as $\mu_i=g^{-1}(\eta_i)=g^{-1}(\mathbf{x}_i^T\boldsymbol{\beta})$, where $\mathbf{x}_i^T=(1,x_{i2},x_{i3},...,x_{ip})$ and $\boldsymbol{\beta}=(\beta_0,\beta_1,...,\beta_p)^T$. The Bell regression model (BRM) can be modeled by assuming $\psi_i=\exp\left(\mathbf{x}_i^T\boldsymbol{\beta}\right)\exp\left(\exp\left(\mathbf{x}_i^T\boldsymbol{\beta}\right)\right)$ and $\log\psi_i=\mathbf{x}_i^T\boldsymbol{\beta}\exp\left(\mathbf{x}_i^T\boldsymbol{\beta}\right)$ as
$y_i\sim \text{Bell}(W_{\circ}(\psi_i))$. The log-likelihood is defined

$$\ell(\boldsymbol{\beta},\psi)=\sum_{i=1}^{n}y_i\log\left(\exp\left(\mathbf{x}_i^T\boldsymbol{\beta}\right)\exp\left(e^{\left(\mathbf{x}_i^T\boldsymbol{\beta}\right)}\right)\right)+\sum_{i=1}^{n}\left(1-e^{e^{\left(\mathbf{x}_i^T\boldsymbol{\beta}\right)}e^{e^{\left(\mathbf{x}_i^T\boldsymbol{\beta}\right)}}}\right)$$
$$+\log B_y-\log\left(\prod_{i=1}^{n}y_i!\right). \qquad (5)$$

Then, the estimated parameters using MLE are defined as

$$\hat{\boldsymbol{\beta}}_{\text{MLE}}=(\mathbf{X}^T\hat{\mathbf{B}}\mathbf{X})^{-1}\mathbf{X}^T\hat{\mathbf{B}}\hat{\mathbf{u}}, \qquad (6)$$

where $\hat{\mathbf{B}}=\text{diag}\left[(\partial\mu_i/\partial\eta_i)^2/V(y_i)\right]$ [26] and $\hat{\mathbf{u}}$ is a vector where i^{th} element equals to $\hat{u}_i=\log\hat{\psi}_i+[(y_i-\hat{\psi}_i)/\sqrt{\text{var}(\hat{\psi}_i)}]$. The MLE is as

$$\text{cov}(\hat{\boldsymbol{\beta}}_{\text{MLE}})=\left[-E\left(\frac{\partial^2\ell(\boldsymbol{\beta},\psi)}{\partial\boldsymbol{\beta}\,\partial\boldsymbol{\beta}^T}\right)\right]^{-1}=(\mathbf{X}^T\hat{\mathbf{B}}\mathbf{X})^{-1}. \qquad (7)$$

In the presence of multicollinearity, the near singularity of $\mathbf{X}^T\hat{\mathbf{B}}\mathbf{X}$ makes the estimation unstable and enlarges the variance [27].

2.2 Modified Jackknife Ridge Estimator in Bell regression model

Hoerl & Kennard (1970) proposed an estimator to handle linear dependency among the explanatory variables for the linear regression model (LRM). This estimator is popularly known as the ridge estimator and is given by:

$$\widehat{\boldsymbol{\beta}}_{\mathbf{k-LRM}}=(\mathbf{I}+k(\mathbf{X}^T\mathbf{X})^{-1})^{-1}(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\boldsymbol{y}, \qquad (8)$$

where k > 0 is the shrinkage parameter. It has been extended to different regression models to eliminate the threat of multicollinearity. Amin, Akram, & Majid (2021) recently defined the ridge estimator for the Bell regression model.Thus, the Bell Ridge estimator is given by:

$$\widehat{\boldsymbol{\beta}}_{\mathbf{k-BRM}}=\left(\mathbf{I}+k\left(\mathbf{X}^T\widehat{\hat{\mathbf{B}}}\mathbf{X}\right)^{-1}\right)^{-1}\left(\mathbf{X}^T\widehat{\hat{\mathbf{B}}}\mathbf{X}\right)^{-1}\mathbf{X}^T\widehat{\hat{\mathbf{B}}}\hat{\mathbf{u}} \qquad (9)$$

Then,

$$\mathrm{Bias}(\widehat{\boldsymbol{\beta}}_{\mathrm{k-BRM}}) = -\mathrm{k}\mathbf{Q}\left(\mathbf{X}^T\widehat{\hat{\mathbf{B}}}\mathbf{X} + k\mathbf{I}\right)^{-1}\boldsymbol{\alpha} \quad (10)$$

$$\mathrm{Variance}(\widehat{\boldsymbol{\beta}}_{\mathrm{k-BRM}}) = \mathbf{Q}\left(\mathbf{X}^T\widehat{\hat{\mathbf{B}}}\mathbf{X} + k\mathbf{I}\right)^{-1}\mathbf{X}^T\hat{\mathbf{B}}\mathbf{X}\left(\mathbf{X}^T\widehat{\hat{\mathbf{B}}}\mathbf{X} + k\mathbf{I}\right)^{-1}\mathbf{Q}^T, \quad (11)$$

where $\mathbf{Q} = (q_1, q_2, ..., q_p)$ is the matrix of eigenvectors of the $\mathbf{X}^T\hat{\mathbf{B}}\mathbf{X}$ matrix, and $\alpha = \mathbf{Q}^T\boldsymbol{\beta}$. The mean square error (MSE) can be written as

$$\mathrm{MSE}(\widehat{\boldsymbol{\beta}}_{\mathrm{k-BRM}}) = \sum_{j=1}^{p}\frac{\lambda_j}{(\lambda_j+k)^2} + k^2\sum_{j=1}^{p}\frac{\alpha_j^2}{(\lambda_j+k)^2}. \quad (12)$$

Shrinkage estimators are biased estimators. In linear regression model, Singh, et al. [28] proposed the Jackknife procedure to alleviate the problem of bias in generalized ridge estimator. The theoretical and application of the jackknife estimator have been studied by several authors [22, 29-35].

The proposed estimator, Jackknifed ridge estimator (JR-BRM) in BRM can be expressed and derived. Let $\boldsymbol{\Lambda} = \mathrm{diag}(\lambda_1, \lambda_2, ..., \lambda_p)$ is the matrix of eigenvalues of the $\mathbf{X}^T\hat{\mathbf{B}}\mathbf{X}$ matrix, such that $\mathbf{Q}^T\mathbf{X}^T\hat{\mathbf{B}}\mathbf{X}\mathbf{Q} = \mathbf{M}^T\hat{\mathbf{B}}\mathbf{M} = \boldsymbol{\Lambda}$, where $\mathbf{M} = \mathbf{X}\mathbf{Q}$. Consequently, the MLE estimator of Eq. (6) can be re-written as [18]

$$\hat{\boldsymbol{\beta}}_{\mathrm{MLE}} = \mathbf{Q}\hat{\boldsymbol{\upsilon}}_{\mathrm{MLE}}, \quad (13)$$

where $\hat{\boldsymbol{\upsilon}}_{\mathrm{MLE}} = \boldsymbol{\Lambda}^{-1}\mathbf{M}^T\hat{\mathbf{B}}\hat{\mathbf{u}}$. As a result, the $\hat{\hat{\beta}}_{k-BRM}$ of Eq. (9) is re-written as

$$\widehat{\boldsymbol{\upsilon}}_{\mathrm{k-BRM}} = (\boldsymbol{\Lambda} + k\mathrm{I})^{-1}\mathbf{M}^T\hat{\mathbf{B}}\widehat{\boldsymbol{u}}. \quad (14)$$

Following the idea of Jackknife approach [36], let $\mathbf{u}_{(-i)}$, $\mathbf{m}_{(-i)}$, and $\mathbf{B}_{(-i)}$, respectively, are the ith row deleted from the vector $\mathbf{u}$, the ith row deleted from the matrix $\mathbf{M}$, and the ith row and column deleted from the matrix $\mathbf{B}$. Let $\widehat{\boldsymbol{u}}_{\mathrm{k-BRM}(-i)}$ be given by Eq. (9) with replacing $\mathbf{M}$, $\mathbf{B}$, and $\mathbf{u}$ by $\mathbf{M}_{(-i)}$, $\mathbf{B}_{(-i)}$, and $\mathbf{u}_{(-i)}$, thus ,

$$\widehat{\boldsymbol{\upsilon}}_{\mathrm{k-BRM}(-i)} = \left(\mathbf{M}_{(-i)}^T\widehat{\hat{\mathbf{B}}}_{(-i)}\boldsymbol{M}_{(-i)} + k\mathbf{I}\right)^{-1}\mathbf{M}_{(-i)}^T\widehat{\hat{\mathbf{B}}}_{(-i)}\widehat{\boldsymbol{u}}_{(-i)}, \quad (15)$$

where $\left(\mathbf{M}_{(-i)}^T\widehat{\hat{\mathbf{B}}}_{(-i)}\boldsymbol{M}_{(-i)} + k\mathbf{I}\right)^{-1}$ is calculated according to Sherman-Morrison Woodbury theorem [37]. Consequently,

$$\widehat{\boldsymbol{\upsilon}}_{\mathrm{k-BRM}(-i)} = \widehat{\boldsymbol{\upsilon}}_{\mathrm{k-BRM}} - \frac{\left(\mathbf{M}_{(-i)}^T\widehat{\hat{\mathbf{B}}}_{(-i)}\boldsymbol{M}_{(-i)}+k\mathbf{I}\right)^{-1}\mathbf{m}_i^T(\widehat{\boldsymbol{u}}_i - \mathbf{m}_i^T\widehat{\boldsymbol{\upsilon}}_{\mathrm{k-BRM}})}{1-\mathbf{m}_i^T\left(\mathbf{M}_{(-i)}^T\widehat{\hat{\mathbf{B}}}_{(-i)}\boldsymbol{M}_{(-i)}+k\mathbf{I}\right)^{-1}m_i}, \quad (16)$$

Using the weighted pseudo values [36], which are calculated as

$$T_i = \widehat{\boldsymbol{\upsilon}}_{\mathrm{k-BRM}} + n\left(1 - \mathbf{m}_i^T\left(\mathbf{M}_{(-i)}^T\widehat{\hat{\mathbf{B}}}_{(-i)}\boldsymbol{M}_{(-i)} + k\mathbf{I}\right)^{-1}m_i\right)(\widehat{\boldsymbol{\upsilon}}_{\mathrm{k-BRM}} - \widehat{\boldsymbol{\upsilon}}_{\mathrm{k-BRM}(-i)}). \quad (17)$$

Therefore, the Bell Jackknife Ridge estimator (JR-BRM) is defined as

$$\widehat{\boldsymbol{\upsilon}}_{\mathrm{JR-BRM}} = \widehat{\boldsymbol{\upsilon}}_{\mathrm{k-BRM}} + \left(\mathbf{M}^T\widehat{\hat{\mathbf{B}}}^T\mathbf{M} + k\mathbf{I}\right)^{-1}\sum_{i=1}^{n}\mathbf{m}_i^T(\widehat{\boldsymbol{u}}_i - \mathbf{m}_i^T\widehat{\boldsymbol{\upsilon}}_{\mathrm{k-BRM}}). \quad (18)$$

Let $\mathbf{C} = \mathbf{M}^T \widehat{\mathbf{B}}^T \mathbf{M} + \mathrm{k}\mathbf{I}$, **then** $\widehat{\mathbf{\upsilon}}_{\mathrm{JR-BRM}}$ can be further simplified as

$$\widehat{\mathbf{\upsilon}}_{\mathrm{JR-BRM}} = (\mathbf{I} - kC^{-1})\widehat{\mathbf{\upsilon}}_{\mathrm{k-BRM}} = (\mathbf{I} - k^2C^{-2})\widehat{\mathbf{\upsilon}}_{MLE} \tag{19}$$

The modified Jackknife ridge estimator has been proposed by Batah, et al. [38]. In a similar way, we propose a new estimator of $\widehat{\mathbf{\upsilon}}_{\mathrm{JR-BRM}}$. Following Algamal (2018), the proposed estimator, which is denoted by modified Jackknifed Bell ridge estimator (MJR-BRM), is obtained by replacing $\widehat{\mathbf{\upsilon}}_{MLE}$ in equation (19) with $\widehat{\mathbf{\upsilon}}_{k-BRM}$. Thus, the MJR-BRM estimator is defined as [18]:

$$\widehat{\mathbf{\upsilon}}_{\mathrm{MJR-BRM}} = (\mathbf{I} - k^2C^{-2})(\mathbf{I} - kC^{-1})\widehat{\mathbf{\upsilon}}_{MLE} \tag{20}$$

The bias, variance and MSE of $\widehat{\mathbf{\upsilon}}_{\mathrm{MJR-BRM}}$ is respectively defined as

$$\mathrm{Bias}(\widehat{\mathbf{\upsilon}}_{\mathrm{MJR-BRM}}) = -k\mathbf{H}C^{-1}\mathbf{\upsilon} \tag{21}$$

where $\mathbf{H} = (\mathbf{I} + kC^{-1} - k^2C^{-2})$

$$\mathrm{variance}(\widehat{\mathbf{\upsilon}}_{\mathrm{MJR-BRM}}) = \mathbf{F}\mathbf{\Lambda}^{-1}\mathbf{F}^T \tag{22}$$

where $\mathbf{F} = (\mathbf{I} - k^2C^{-2})(\mathbf{I} - kC^{-1})$, and

$$\mathrm{MSE}(\widehat{\mathbf{\upsilon}}_{\mathrm{MJR-BRM}}) = \mathbf{F}\mathbf{\Lambda}^{-1}\mathbf{F}^T + k^2\mathbf{H}C^{-1}\mathbf{\upsilon}\mathbf{\upsilon}^T C^{-1}\mathbf{H}^T \tag{23}$$

2.3 Theoretical comparison between $\widehat{\mathbf{\upsilon}}_{\mathrm{k-BRM}}$ and $\widehat{\mathbf{\upsilon}}_{\mathrm{MJR-BRM}}$

Lemma: "[39] Let $\mathbf{G}$ is a $p \times p$ positive definite matrix, $\mathbf{b}$ is a $p \times 1$ vector, and c is a positive constant. Then $c\mathbf{G} - \mathbf{b}\mathbf{b}^T$ is a nonnegative definite if and only if $\mathbf{b}^T \mathbf{G}^{-1}\mathbf{b} \le c$ is hold".

Theorem. "The proposed estimator $\widehat{\mathbf{\upsilon}}_{\mathrm{MJR-BRM}}$ is superior to estimator $\widehat{\mathbf{\upsilon}}_{\mathrm{k-BRM}}$ if and only if

$$k^2(\mathbf{H}C^{-1}\mathbf{\upsilon})^T\mathbf{Q}\left[(\mathbf{M}^T\widehat{\mathbf{W}}\mathbf{M} + k\mathbf{I})^{-1}\mathbf{M}^T\widehat{\mathbf{W}}\mathbf{M}(\mathbf{M}^T\widehat{\mathbf{W}}\mathbf{M} + k\mathbf{I})^{-1} - \mathbf{F}\mathbf{\Lambda}^{-1}\mathbf{F}^T + k^2C^{-1}\mathbf{\upsilon}\mathbf{\upsilon}^T C^{-1}\right]^{-1}(\mathbf{H}C^{-1}\mathbf{\upsilon})\mathbf{Q}^T < 1 \text{ "} \tag{24}$$

Proof.

The scalar MSE of the proposed is given by

$$\mathrm{MSE}(\widehat{\mathbf{\beta}}_{\mathrm{k-BRM}}) = \sum_{j=1}^{p} \frac{\lambda_j(\lambda_j + 2\lambda_j k)^2}{(\lambda_j + k)^6} + k^2 \sum_{j=1}^{p} \frac{(\lambda_j + 3\lambda_j k + k^2)^2 \alpha_j^2}{(\lambda_j + k)^6}. \tag{25}$$

The difference between $\mathrm{variance}(\widehat{\mathbf{\upsilon}}_{\mathrm{MJR-BRM}})$ and $\mathrm{variance}(\widehat{\mathbf{\upsilon}}_{\mathrm{k-BRM}})$ is

$$\mathrm{diag}\left\{\frac{\lambda_j}{(\lambda_j + k)^2} - \frac{\lambda_j(\lambda_j + 2\lambda_j k)^2}{(\lambda_j + k)^6}\right\}_{j=1}^{p} \tag{26}$$

Consequently,

$$(\mathbf{M}^T\widehat{\mathbf{W}}\mathbf{M} + k\mathbf{I})^{-1}\mathbf{M}^T\widehat{\mathbf{W}}\mathbf{M}(\mathbf{M}^T\widehat{\mathbf{W}}\mathbf{M} + k\mathbf{I})^{-1} - \mathbf{F}\mathbf{\Lambda}^{-1}\mathbf{F}^T$$

is positive definite since $(\lambda_j + k)^2 > \lambda_j + 2\lambda_j k$.

3. Simulation Study

In this part, we simulate explanatory factors that are correlated and a bell-shaped response variable y.

3.1 Simulation Design

The explanatory variables are obtained in line with the study of Kibria (2003) and Lukman et al. (2019) as follows:

$$x_{ij} = \sqrt{(1-\rho^2)}m_{ij} + \rho m_{i(j+1)}, i = 1, \dots, n; j = 1, \dots p \qquad (27)$$

where m_{ij} are independent standard normal pseudo-random numbers and ρ^2 denotes the correlation between the explanatory variables such that $\rho = 0.5, 0.7, 0.8, 0.9, 0.99$ and 0.999. We assumed that $y_i \sim bell(W_o(\mu_i))$, where

$$\log(\mu_i) = \eta_i = \beta_1 x_{i1} + \beta_2 x_{i2} + \dots + \beta_p x_{ip} \qquad (28)$$

The sample sizes are varied such that n=30, 50, 100 and 200 while p is taken to be 3, 8 and 12. The true values of the regression parameter β are chosen such that $\sum_{i=1}^{p} \hat{\beta}_i^2 = 1$ (Kibria & Lukman, 2020). The simulation study is conducted by adopting bellreg-package (Stan Development Team, 2020; Castellares, Ferrari, & Lemonte, 2018). The average the mean squared error (MSE) with 100s replications was employed as

$$MSE(\beta^*) = \frac{1}{1000} \sum_{j=1}^{1000} (\beta_{ij}^* - \beta_i)'(\beta_{ij}^* - \beta_i) \qquad (29)$$

where β_{ij}^* is the estimator and β_i is the parameter.

3.2 Simulation Results

Tables 3.1–3.2 reveal the MSE of the simulated outcomes under various simulation circumstances. Due to the existence of multicollinearity, MLE performance is not sufficient. For instance, the MSE for MLE is 68.562 at sample size 30, =0.9, and p=12, but the MSEs for k-BRM and MJR-BRM are 2.383 and 2.361, respectively. This is consistent with the literature's assertion that MLE is set back when the regressors are parallel. Additionally, we noticed that the MSE of each estimator rises as the level of multicollinearity does for a given sample size. For instance, when n=30 for p=8, the MSE for k-BRM are 1.218, 1.446, 2.022, 2.198, 2.618 and 3.103, respectively. Also, the MSE of each of the estimators decreases as the sample sizes and ρ=0.99 are as follows: 12.214 (n=30), 4.081 (n=50), 3.211 (n=100) and 1.882 (n=200). It is very obvious that the MSE rise as the number of explanatory variables (p) increase. The performances of ridge estimator and the modified jackknife ridge estimator are competitive especially when the level of multicollinearity is moderate- say ρ=0.5 and 0.7. However, the proposed estimator shows superiority when the level of multicollinearity becomes high.

Table 3.1: Average MSE when p=3.

n	Estimator	r					
		0.50	0.70	0.80	0.90	0.99	0.999
30	MLE	1.864	1.980	3.027	4.598	12.214	15.092
	k-BRM	0.022	0.066	0.857	1.068	1.220	3.110
	MJR-BRM	**0.022**	**0.066**	**0.855**	**1.068**	**1.209**	**3.062**
50	MLE	1.214	1.616	1.857	2.576	4.081	6.166
	k-BRM	0.017	0.049	0.677	1.841	1.215	2.083
	MJR-BRM	**0.016**	**0.035**	**0.424**	**1.820**	**1.209**	**2.034**
100	MLE	0.908	1.213	1.346	2.272	3.211	4.760
	k-BRM	0.014	0.029	0.567	0.696	0.777	1.766
	MJR-BRM	**0.013**	**0.022**	**0.128**	**0.491**	**0.512**	**1.752**
150	MLE	0.046	0.664	1.063	1.483	1.882	2.187
	k-BRM	0.011	0.011	0.011	0.032	0.033	0.056
	MJR-BRM	**0.010**	**0.011**	**0.011**	**0.033**	**0.033**	**0.056**

Table 3.2: Average MSE when p=8.

n	Estimator	r					
		0.50	0.70	0.80	0.90	0.99	0.999
30	MLE	2.649	6.028	8.146	9.698	27.927	172.581
	k-BRM	1.218	1.446	2.022	2.198	2.618	3.103
	MJR-BRM	**1.186**	**1.377**	**1.820**	**1.856**	**2.480**	**2.780**
50	MLE	1.816	2.826	3.479	4.244	5.248	19.780

	k-BRM	1.175	1.191	1.217	1.721	1.766	2.102
	MJR-BRM	**1.170**	**1.173**	**1.201**	**1.350**	**1.532**	**1.858**
100	MLE	1.702	1.919	1.919	2.242	4.178	14.154
	k-BRM	1.111	1.132	1.138	1.142	1.169	1.188
	MJR-BRM	**1.107**	**1.131**	**1.136**	**1.140**	**1.147**	**1.173**
150	MLE	1.167	1.337	1.543	1.849	2.196	8.772
	k-BRM	0.344	0.345	0.346	1.019	1.061	1.091
	MJR-BRM	**0.143**	**0.178**	**0.297**	**1.026**	**1.055**	**1.073**

Table 3.3: Average MSE when p=12.

n	Estimator	r					
		0.50	0.70	0.80	0.90	0.99	0.999
30	MLE	4.285	7.478	16.594	68.562	159.347	327.885
	k-BRM	1.428	2.058	2.121	2.383	2.850	3.594
	MJR-BRM	**1.414**	**1.804**	**1.981**	**2.361**	**2.723**	**3.025**
50	MLE	2.589	4.668	8.682	14.811	35.965	166.081
	k-BRM	1.288	1.624	1.859	2.023	2.204	2.416
	MJR-BRM	**1.228**	**1.532**	**1.749**	**2.011**	**2.132**	**2.235**
100	MLE	2.241	3.802	5.487	5.945	7.263	20.594
	k-BRM	1.141	1.380	1.387	1.394	1.450	1.656
	MJR-BRM	**1.183**	**1.366**	**1.377**	**1.392**	**1.512**	**1.670**
150	MLE	1.846	3.773	4.744	5.084	4.801	10.437
	k-BRM	1.139	1.160	1.294	1.353	1.365	1.379
	MJR-BRM	**1.084**	**1.092**	**1.206**	**1.328**	**1.335**	**1.356**

4. Numerical Result

In this section, we will adopt two real-life data to evaluate the performance of the existing estimators and the proposed.

4.1 Aircraft Data

This dataset is originally assumed to follow the Poisson regression model [40] .among others. The condition number is 219.3654 [15]. The variance in the number of damaged areas on the aircraft, however, is greater than double the mean (2.0569). This makes it clear that the data show over-dispersion. The regression estimates and MSE for each of the adopted estimators used in this investigation are shown in Table 4.1. The Hoerl et al. (1975) biassing parameter k was chosen to be the biassing parameter for the Bell ridge and Modified Jackknife Bell ridge methods.

$$\hat{k} = \frac{p}{\sum_{j=1}^{p} \hat{v}_j^2} \quad (30)$$

Then,

$$MSE(\widehat{\boldsymbol{\beta}}_{MLE}) = \sum_{j=1}^{p} \frac{1}{\lambda_j} \quad (31)$$

where λ_j is the eigenvalue of $\mathbf{X}^{\mathrm{T}}\widehat{\mathbf{W}}\mathbf{X}$.

$$MSE(\widehat{\boldsymbol{\beta}}_{k-BRM}) = \sum_{j=1}^{p} \frac{\lambda_j}{(\lambda_j+k)^2} + k^2 \sum_{j=1}^{p} \frac{\hat{v}_j^2}{(\lambda_j+k)^2} \quad (32)$$

where $\hat{v}_j^2$ is the jth squared of the MLE.

$$MSE(\widehat{\boldsymbol{\beta}}_{\mathrm{MJR-BRM}}) = \sum_{j=1}^{p} \frac{\lambda_j\left(\lambda_j^2+2\lambda_j k\right)^2}{(\lambda_j+k)^6} + k^2 \sum_{j=1}^{p} \frac{\left(\lambda_j^2+3\lambda_j k+k^2\right)^2 \hat{v}_j^2}{(\lambda_j+k)^6} \quad (33)$$

Table 4.1: Bell regression estimates.

		1	RM
ɔt	:5	ı9	806
	ı1	;3	737
	;0	;5	692
	.7	·6	152

·7 9 745

The results in Table 4.2 revealed that the proposed estimator generally dominate the Bell ridge and the maximum likelihood estimator. This support the literature that the jackknife ridge estimator often reduce the bias of the ridge estimator and in turn lower its mean squared error.

4.2 Mine fracture data

The dataset was originally adopted by Myer et al. (2021) to model count data. However, we obtained the ratio of the variance of the distribution to the mean and observed that the Poisson regression model is not a good fit because the variance is about 135 more than the mean (135.614). It is evident that there is over-dispersion which make the bell regression more suitable for this model. This conclusion is well supported in the study of Amin et al. (2021). The model is described as follows: $y_i \sim bell\left(W_o\left(\mu_i\right)\right),$ such that

$$\log(\mu_i) = \eta_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \cdots + \beta_4 x_{i4} \tag{34}$$

The eigenvalues are computed to be 2.400553e+06, 1.848779e+05, 2.440728e+04, 2.076920e+03, and 5.242369e-01. The CI for the dataset under study is 2139.89 which signal severe multicollinearity. This justifies the reason for considering other method of estimation than the method of maximum likelihood. The regression coefficients and their corresponding mean squared error for each of the estimators are displayed in Table 4.2.

Table 4.2: Bell regression estimates.

Coef.	MLE	k-BRM	MJR-BRM
Intercept	-3.8576	-0.0267	-0.0001
x1	-0.0013	-0.0009	-0.0009
x2	0.0643	0.0196	0.0192
x3	-0.0009	-0.0061	-0.0061
x4	-0.0279	-0.0225	-0.0225
MSE	1.9081	0.0014	0.0012

From Table 4.2, it is evident that the maximum likelihood performance suffers setback due to the presence of multicollinearity. The ridge and the proposed method performance are better. However, the Modified Jackknife Bell ridge exhibit the lowest mean squared error and this made it the most preferable among the estimators under study.

5. Conclusions

Bell regression model has been devised as alternative to Poisson regression model when over-dispersion in count data is exitance. The parameter estimate of the Bell regression model is carried out by maximum likelihood using the frequentist approach, and the Fisher information is generated. As a result, they suggested the Bell ridge regression technique. The literature makes it clear that jackknifing the ridge estimator will result in a more reliable estimate. As a result, we created a novel parameter estimate technique for this study called the Modified Jackknife Bell Ridge estimator. Through the use of two empirical applications and simulation research, the technique proposed in this work is demonstrated. In conclusion, modelling count data with over-dispersion is better suited to the Bell regression model. When the model has multicollinearity, the Modified Jackknife Bell Ridge estimator is also advised.

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