Hosoya Polynomials of Steiner Distance of the Sequential Join of Graphs

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الملخص

تضمن هذا البحث ايجاد متعددات حدود هوسويا لمسافة ستينر - n لكل من بيانات الجمع التتابعي J_3 و J_4 كما تم ايجاد متعددات حدود هوسويا لمسافة ستينر - **3** لبيان الجمع التتابعي J_m لـــ m من البيانات.

ABSTRACT

The Hosoya polynomials of Steiner n-distance of the sequential join of graphs J_3 and J_4 are obtained and the Hosoya polynomials of Steiner **3**-distance of the sequential join of m graphs J_m are also obtained.

Keywords: Steiner n-distance, Hosoya polynomial, Sequential Join.

1. Introduction

We follow the terminology of [**2,3**]. For a connected graph $G=(V,E)$ of order p, the ***Steiner distance*[5,6,7]** of a non-empty subset $S \subseteq V(G)$, denoted by $d_G(S)$, or simply $d(S)$, is defined to be the size of the smallest connected subgraph ***T(S)*** of ***G*** that contains ***S***; ***T(S)*** is a tree called a ***Steiner tree*** of ***S***. If **|*S*|=2**, then ***d(S)*** is the distance between the two vertices of ***S***. For $2 \le n \le p$ and **|*S*|=*n***, the Steiner distance of ***S*** is called ***Steiner n-distance of S*** in ***G***. The ***Steiner n-diameter*** of ***G***, denoted by $diam_n^* G$ or simply $\delta_n^*(G)$, is defined by:

$$diam_n^* G = \max\{d_G(S) : S \subseteq V(G), |S| = n\}.$$

Remark 1.1. It is clear that

(1) If $n > m$, then $diam_n^* G \ge diam_m^* G$.

(2) If $S' \subseteq S$, then $d_G(S') \le d_G(S)$.

The Steiner n-distance of a vertex $v \in V(G)$, denoted by $W_n^*(v,G)$, is the sum of the Steiner ***n***-distances of all ***n***-subsets containing ***v***. The sum of

Steiner $\boldsymbol{n}$-distances of all $\boldsymbol{n}$-subsets of $\boldsymbol{V(G)}$ is denoted by $\boldsymbol{d_n(G)}$ or $\boldsymbol{W_n^*(G)}$. It is clear that

$$\boldsymbol{W_n^*(G) = n^{-1} \sum_{v \in V(G)} W_n^*(v,G)} . \qquad ...(1.1)$$

The graph invariant $\boldsymbol{W_n^*(G)}$ ***is called Wiener index of the Steiner n-distance*** of the graph $\boldsymbol{G}$.

Definition 1.2[**1**] Let $\boldsymbol{C_n^*(G,k)}$ be the number of n-subsets of distinct vertices of $\boldsymbol{G}$ with Steiner $\boldsymbol{n}$-distance $\boldsymbol{k}$. The graph polynomial defined by

$$\boldsymbol{H_n^*(G;x) = \sum_{k=n-1}^{\delta_n^*} C_n^*(G,k)x^k} , \qquad ...(1.2)$$

where d_n^* is the Steiner $\boldsymbol{n}$-diameter of $\boldsymbol{G}$; is called the ***Hosoya polynomial of Steiner n-distance of G.***

It is clear that

$$\boldsymbol{W_n^*(G) = \sum_{k=n-1}^{\delta_n^*} kC_n^*(G,k)} \qquad ...(1.3)$$

For $1 \le \boldsymbol{n} \le \boldsymbol{p}$, let $\boldsymbol{C_n^*(u,G,k)}$ be the number of $\boldsymbol{n}$-subsets $\boldsymbol{S}$ of distinct vertices of $\boldsymbol{G}$ containing $\boldsymbol{u}$ with Steiner $\boldsymbol{n}$-distance $\boldsymbol{k}$. It is clear that

$\boldsymbol{C_1^*(u,G,}0\boldsymbol{)} = 1$.

Define

$$\boldsymbol{H_n^*(u,G;x) = \sum_{k=n-1}^{\delta_n^*} C_n^*(u,G,k)x^k} . \qquad \ldots(1.4)$$

Obviously, for $2 \le \boldsymbol{n} \le \boldsymbol{p}$

$$\boldsymbol{H_n^*(G;x) = \frac{1}{n} \sum_{u \in V(G)} H_n^*(u,G;x)} . \qquad \ldots(1.5)$$

Ali and Saeed [**1**] were first who studied this distance-based graph polynomial for Steiner $\boldsymbol{n}$-distances, and established Hosoya polynomials of Steiner $\boldsymbol{n}$-distance for some special graphs and graphs having some kind of regularity, and for Gutman's compound graphs $\boldsymbol{G_1 \cdot G_2}$ and $\boldsymbol{G_1 : G_2}$ in terms of Hosoya polynomials of $\boldsymbol{G_1}$ and $\boldsymbol{G_2}$.

Definition 1.3[**2**] Let $\boldsymbol{G_1}$, $\boldsymbol{G_2}$, …, $\boldsymbol{G_m}$, $\boldsymbol{m}$ ³ $\mathbf{2}$, be vertex disjoint graphs. The sequential join of $\boldsymbol{G_1}$, $\boldsymbol{G_2}$, …, $\boldsymbol{G_m}$ is a graph denoted by

$\boldsymbol{J_m = G_1 + G_2 + ... + G_m}$,

and defined by

$$V(J_m) = \bigcup_{i=1}^{m} V(G_i),$$

$$E(J_m) = \left\{ \bigcup_{i=1}^{m} E(G_i) \right\} \bigcup \left\{ uv \,\middle|\, u \in V_i \text{ and } v \in V_{i+1},\ \text{for } i = 1, 2, \ldots, m-1 \right\}$$

in which $V_i = V(G_i)$, as depicted in the following figure.

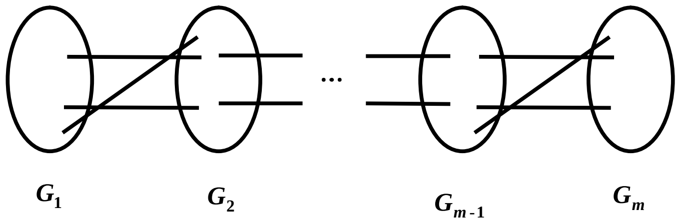

Fig. **1.1** J_m

It is clear that

$$p(J_m) = \sum_{i=1}^{m} p_i,\quad q(J_m) = q_m + \sum_{i=1}^{m-1} (q_i + p_i p_{i+1}),$$

in which

$p_i = p(G_i)$ and $q_i = q(G_i)$.

One can easily see that for $m \geq 3$, $\sum_{i=1}^{m} G_i$ is not commutative, that is for $m=3$ $G_1 + G_2 + G_3 \neq G_1 + G_3 + G_2$.

In [**8**], Saeed obtained the (ordinary) Hosoya polynomials of J_m, and in [**7**], Herish obtained the Steiner ***n***-diameter of the sequential join of ***m*** empty graphs and of ***m*** complete graphs. Also, the Hosoya polynomials of Steiner distance of the sequential join of ***m*** empty graphs and of ***m*** complete graphs were obtained. For $m \geq 3$ and $n \geq 2$, the Steiner ***n***-diameter of the sequential join of ***m*** complete graphs is given by[**7**]

$$diam_n^* J_m = \begin{cases} m + n - 3, & \text{if } 2 \leq n \leq p_1 + p_m \\ m + n - 3 - \alpha, & \text{if } p_1 + p_m + 1 \leq n \leq p, \end{cases}$$

where α is the smallest integer such that

$$p_1 + p_m + 1 \leq n \leq p_1 + p_m + \sum_{i=1}^{\alpha} r_i\ . \qquad \ldots(1.6)$$

It is obvious that Eq. **1.6** holds for the sequential join of ***m*** graphs J_m.

In this paper, a generalization of the results obtained in [**7**] is given. We obtained the Hosoya polynomials of Steiner ***n***-distance of J_3 and J_4; and the Hosoya polynomials of Steiner 3-distance of J_m, $m \geq 4$. We also obtained $H_n^*(J_3;x)$, for $n \geq 2$ and $H_3^*(J_m;x)$, for $m \geq 4$, where each of G_i, for $i = 1,2,\dots,m$ is a special graph.

2. Hosoya Polynomials of Steiner n-Distance of J_3 and J_4

In this section, we consider J_m, for ***m*=3** and ***m*=4**. Let ***S*** be any ***n***-subset of vertices of J_m. Let $B(G_i)$, for $i = 1,2,\dots,m$, be the number of all ***n***-subsets ***S*** such that $\langle S \rangle$ is connected in G_i. The following proposition determines the Hosoya polynomials of Steiner ***n***-distance of J_3.

Proposition 2.1. For $3 \leq n \leq p (= p_1 + p_2 + p_3)$,

$$H_n^*(J_3;x) = C_1 x^{n-1} + C_2 x^n,$$

where

$$C_1 = \binom{p}{n} - \binom{p_1+p_2}{n} - \binom{p_2}{n} + B(G_1) + B(G_2) + B(G_3),$$

$$C_2 = \binom{p_2}{n} + \binom{p_1+p_3}{n} - \left[B(G_1) + B(G_2) + B(G_3)\right],$$

and

$B(G_1), B(G_2)$ and $B(G_3)$ are as defined above.

Proof. It is clear that

$$diam_n^* J_3 = \begin{cases} n, & \text{if} \quad 3 \leq n \leq p_1 + p_3 \\ n-1, & \text{if} \quad \text{otherwise} \end{cases}.$$

Therefore,

$$H_n^*(J_3;x) = C_1 x^{n-1} + C_2 x^n$$

in which C_1 is the number of all ***n***-subsets of $V(J_3)$ with Steiner distance equals ***n*-1**, and C_2 is the number of all ***n***-subsets of $V(J_3)$ with Steiner distance equals ***n***.

Therefore,

$$C_2 = \sum_{i=1}^{3} \left\{ \binom{p_i}{n} - B(G_i) \right\} + \sum_{j=1}^{n-1} \binom{p_1}{j} \binom{p_3}{n-j}$$

$$= \binom{p_2}{n} + \binom{p_1+p_3}{n} - \left[B(G_1) + B(G_2) + B(G_3)\right].$$

Now, since

$$C_1 + C_2 = \binom{p}{n},$$

therefore

$$C_1 = \binom{p}{n} - C_2 = \binom{p}{n} - \binom{p_1 + p_3}{n} - \binom{p_2}{n} + B(G_1) + B(G_2) + B(G_3)$$

This completes the proof. ■

The following corollary computes the n-Wiener index of J_3.

Corollary 2.2. For $3 \le n \le p(= p_1 + p_2 + p_3)$,

$$W_n^*(J_3) = n\binom{p}{n} - C_1,$$

where C_1 is given in Proposition **2.1**. ■

Next, we shall find the Hosoya polynomials of Steiner n-distance of J_4.

Proposition 2.3. For $3 \le n \le p(= p_1 + p_2 + p_3 + p_4)$,

$$H_n^*(J_4; x) = C_1 x^{n-1} + C_2 x^n + C_3 x^{n+1},$$

where

$$\begin{aligned} C_1 = {} & \sum_{i=1}^{n-2} \sum_{j=1}^{n-1-i} \left[\binom{p_1}{i}\binom{p_2}{j}\binom{p_3}{n-i-j} + \binom{p_2}{i}\binom{p_3}{j}\binom{p_4}{n-i-j} \right] \\ & + \sum_{i=1}^{n-3} \sum_{j=1}^{n-2-i} \sum_{k=1}^{n-1-i-j} \binom{p_1}{i}\binom{p_2}{j}\binom{p_3}{k}\binom{p_4}{n-i-j-k} + \sum_{i=1}^{4} B(G_i) \\ & + \binom{p_1+p_2}{n} + \binom{p_2+p_3}{n} + \binom{p_3+p_4}{n} - \binom{p_1}{n} - 2\binom{p_2}{n} - 2\binom{p_3}{n} - \binom{p_4}{n}, \end{aligned}$$

$$\begin{aligned} C_2 = {} & \binom{p}{n} - \sum_{i=1}^{n-2} \sum_{j=1}^{n-1-i} \left[\binom{p_1}{i}\binom{p_2}{j}\binom{p_3}{n-i-j} \right] \\ & - \sum_{i=1}^{n-3} \sum_{j=1}^{n-2-i} \sum_{k=1}^{n-1-i-j} \binom{p_1}{i}\binom{p_2}{j}\binom{p_3}{k}\binom{p_4}{n-i-j-k} - \sum_{i=1}^{4} B(G_i) \\ & - \binom{p_1+p_2}{n} - \binom{p_2+p_3}{n} - \binom{p_3+p_4}{n} - \binom{p_1+p_4}{n} \\ & + 2\binom{p_1}{n} + 2\binom{p_2}{n} + 2\binom{p_3}{n} + 2\binom{p_4}{n}, \end{aligned}$$

and

$$C_3 = \binom{p_1+p_4}{n} - \binom{p_1}{n} - \binom{p_4}{n}.$$

Proof. It is clear that $n-1 \le diam_n^* J_4 \le n+1$, therefore the Hosoya polynomials of Steiner n-distance of J_4 has the following form

$$H_n^*(J_4;x) = C_1 x^{n-1} + C_2 x^n + C_3 x^{n+1}.$$

To find C_1, C_2 and C_3, let S be any n-subset of vertices of J_4, then we have the following possibilities for the subset S.

(I) $d(S) = n-1$ if and only if S has any of the following subcases:

(1) S is a subset of V_i, for $i = 1,2,3,4$ and $\langle S \rangle$ is a connected subgraph of G_i. The number of these n-subsets is given by

$$B(G_1) + B(G_2) + B(G_3) + B(G_4).$$

(2) $S \subseteq V_k \cup V_{k+1}$ and $(S \cap V_k \ne \phi \wedge S \cap V_{k+1} \ne \phi)$, $k = 1,2,3$.

The number of these subsets S is given by

$$\sum_{i=1}^{n-1}\binom{p_1}{i}\binom{p_2}{n-i} + \sum_{i=1}^{n-1}\binom{p_2}{i}\binom{p_3}{n-i} + \sum_{i=1}^{n-1}\binom{p_3}{i}\binom{p_4}{n-i}$$

$$= \binom{p_1+p_2}{n} + \binom{p_2+p_3}{n} + \binom{p_3+p_4}{n} - \binom{p_1}{n} - 2\binom{p_2}{n} - 2\binom{p_3}{n} - \binom{p_4}{n},$$

(3) $(S \subseteq \bigcup_{i=1}^{3} V_i \wedge S \cap V_i \ne \phi)$ or $(S \subseteq \bigcup_{i=2}^{4} V_i \wedge S \cap V_i \ne \phi)$. The number of these n-subsets is given by

$$\sum_{i=1}^{n-2}\sum_{j=1}^{n-1-i}\left[\binom{p_1}{i}\binom{p_2}{j}\binom{p_3}{n-i-j} + \binom{p_2}{i}\binom{p_3}{j}\binom{p_4}{n-i-j}\right]$$

(4) $S \cap V_i \ne \phi$, $i = 1,2,3,4$. The number of these n-subsets is given by

$$\sum_{i=1}^{n-3}\sum_{j=1}^{n-2-i}\sum_{k=1}^{n-1-i-j}\binom{p_1}{i}\binom{p_2}{j}\binom{p_3}{k}\binom{p_4}{n-i-j-k}$$

From (**1**), (**2**), (**3**) and (**4**), we get C_1 as given in the statement of the proposition.

(II) $d(S) = n+1$ if and only if $S \subseteq V_1 \cup V_4$ and $(S \cap V_1 \ne \phi$ and $S \cap V_4 \ne \phi)$. The number of these S's is given by

$$\sum_{i=1}^{n-1}\binom{p_1}{i}\binom{p_4}{n-i} = \binom{p_1+p_4}{n} - \binom{p_1}{n} - \binom{p_4}{n}.$$

So, C_3 is as given.

Now, since $C_1 + C_2 + C_3 = \binom{p}{n}$,

therefore

$$C_2 = \binom{p}{n} - C_1 - C_3 .$$

This completes the proof. ■

Remark. The triple summation in C_1 is taken to be zero when n**=3**.

The following corollary computes $W_n^*(J_4)$.

Corollary 2.4. For $3 \le n \le p(= p_1 + p_2 + p_3 + p_4)$,

$$W_n^*(J_4) = n\binom{p}{n} - C_1 + C_3 ,$$

where C_1 and C_3 are given in Proposition **2.3.**■

Remark. For $m \ge 5$, the calculation of the coefficients of $H_n^*(J_m;x)$ is complicated.

The numbers $B(G_1)$, $B(G_2)$ and $B(G_3)$ are given in Proposition **2.1** can be counted for some specific graphs G_1, G_2 and G_3 as in the following examples.

Example 2.5. Let N_{p_1}, N_{p_2} and N_{p_3} be empty graphs of orders p_1, p_2 and p_3 respectively, then

$$B(N_{p_1}) = B(N_{p_2}) = B(N_{p_3}) = 0.$$

Example 2.6. Let K_{p_1}, K_{p_2} and K_{p_3} be complete graphs of orders p_1, p_2 and p_3 respectively, then

$$B(K_{p_i}) = \binom{p_i}{n}, \text{ for } i = 1,2,3 .$$

Example 2.7. Let P_{α_1}, P_{α_2} and P_{α_3} be path graphs of orders α_1, α_2 and α_3 respectively, then

$$B(P_{\alpha_i}) = \alpha_i - n + 1, \text{ for } i = 1,2,3 .$$

Example 2.8. Let C_{α_1}, C_{α_2} and C_{α_3} be cycle graphs of orders α_1, α_2 and α_3 respectively, then

$$B(C_{\alpha_i}) = \alpha_i, \text{ for } i = 1,2,3 .$$

Example 2.9. Let W_{α_1}, W_{α_2} and W_{α_3} be wheel graphs of orders α_1, α_2 and α_3 respectively, then

$$B(W_{\alpha_i}) = \binom{\alpha_i - 1}{n-1} + \alpha_i - 1, \text{ for } i = 1,2,3.$$

Example 2.10. Let K_{α_i,β_i}, for $i = 1,2,3$, be complete bipartite graphs of partite sets of size α_i β_i, then

$$B(K_{\alpha_i,\beta_i}) = \binom{\alpha_i + \beta_i}{n} - \binom{\alpha_i}{n} - \binom{\beta_i}{n}, \text{ for } i = 1,2,3.$$

3. Hosoya Polynomials of Steiner 3-Distance of J_m $(m \geq 5)$

In this section, we consider $J_m = G_1 + G_2 + \ldots + G_m$, for $m \geq 5$. The following theorem determines Hosoya polynomials of Steiner **3**-distance of J_m.

Theorem 3.1. For $m \geq 5$,

$$H_3^*(J_m;x) = (A + Bx)x^2 + \tfrac{1}{2}\sum_{j=i+1}^{m}\sum_{i=1}^{m-1} p_i p_j (p_i + p_j - 2)x^{j-i+1}$$
$$+ \sum_{j=i+2}^{m}\sum_{i=1}^{m-2} p_i p_j \left(\sum_{r=1}^{j-1} p_r\right) x^{j-i},$$

where

$$A = \sum_{i=1}^{m}\left[\sum_{v \in V_i}\binom{\deg v}{2} - 2\gamma_i\right], \quad B = \sum_{i=1}^{m}\binom{p_i}{3} - A,$$

in which γ_i, for $i = 1,2,\ldots,m$ is the number of non-identical triangles K_3 as a subgraph in G_i.

Proof. Let S be any **3**-subset of vertices of J_m, then we have three main cases for the subset S.

(I) If $S \subseteq V_i$, for $i = 1,2,\ldots,m$, then

(a) $d(S) = 2$, when $\langle S \rangle$ is a connected subgraph in G_i, and by Lemma **3.4.4.** of [7], the number of such **3**-subsets S is given by

$$A = \sum_{i=1}^{m}\left[\sum_{v \in V_i}\binom{\deg v}{2} - 2\gamma_i\right].$$

(b) $d(S) = 3$, when $\langle S \rangle$ is a disconnected subgraph in G_i, and the number of such **3**-subsets S is given by

$$B = \sum_{i=1}^{m} \binom{p_i}{3} - A.$$

Case(**I**) produces the polynomial

$$F_1(x) = (A + Bx)x^2.$$

(II) Either two vertices of S are in V_i and one vertex of S in V_j, $i < j$, or one vertex of S in V_i, and two vertices of S in V_j, for $1 \le i < j \le m$. For each such cases of S,

$$d(S) = j - i + 1,$$

and the number of ways of choosing such S is given by

$$\sum_{j=i+1}^{m} \sum_{i=1}^{m-1} \left[\binom{p_i}{2} p_j + \binom{p_j}{2} p_i \right],$$

and, this produces the polynomial

$$F_2(x) = \frac{1}{2} \sum_{j=i+1}^{m} \sum_{i=1}^{m-1} [p_j p_i (p_i - 1) + p_i p_j (p_j - 1)] x^{j-i+1}$$

$$= \frac{1}{2} \sum_{j=i+1}^{m} \sum_{i=1}^{m-1} p_i p_j (p_i + p_j - 2) x^{j-i+1}$$

(III) One vertex of S in V_i, one vertex in V_j, $j \ge i + 2$, and the third vertex in V_r, $i < r < j$. For such case

$$d(S) = j - i,$$

and the number of all possibilities of such S is

$$\sum_{j=i+2}^{m} \sum_{i=1}^{m-2} p_i p_j \left(\sum_{r=i+1}^{j-1} p_r \right),$$

and this produces the polynomial

$$F_3(x) = \sum_{j=i+2}^{m} \sum_{i=1}^{m-2} p_i p_j \left(\sum_{r=i+1}^{j-1} p_r \right) x^{j-i}.$$

Now adding the polynomials $F_1(x), F_2(x)$ and $F_3(x)$ obtained in **(I)**, **(II)** and **(III)**, we get the required result. ■

The numbers A and B are given in Theorem **3.1** can be counted when G_i, for $i = 1, 2, ..., m$, has a special form, as in the following examples.

Example 3.2. Let N_{p_i}, for $i = 1, 2, ..., m$ be empty graphs of orders p_i, then

$A = 0$ and $B = \sum_{i=1}^{m} \binom{p_i}{3}$.

Example 3.3. Let K_{p_i}, for $i = 1,2,\ldots,m$ be complete graphs of orders p_i, then

$$A = \sum_{i=1}^{m} \binom{p_i}{3} \text{ and } B = 0.$$

Example 3.4. Let P_{α_i}, for $i = 1,2,\ldots,m$ be path graphs of orders α_i, then

$$A = \sum_{i=1}^{m} \left[\alpha_i - 2\right] = p - 2m \text{ and } B = \sum_{i=1}^{m} \binom{\alpha_i}{3} - p + 2m.$$

Example 3.5. Let C_{α_i}, for $i = 1,2,\ldots,m$ be cycle graphs of orders α_i, then

$$A = \sum_{i=1}^{m} \alpha_i = p \text{ and } B = \sum_{i=1}^{m} \binom{\alpha_i}{3} - p.$$

Example 3.6. Let W_{α_i} for $i = 1,2,\ldots,m$ be wheel graphs of orders α_i, then

$$A = \sum_{i=1}^{m} \left[\sum_{v \in V_i} \binom{\deg v}{2} - 2\gamma_i \right] = \sum_{i=1}^{m} \left[(\alpha_i - 1)\binom{3}{2} + \binom{\alpha_i - 1}{2} - 2(\alpha_i - 1) \right]$$

$$= \sum_{i=1}^{m} \binom{\alpha_i}{2},$$

and

$$B = \sum_{i=1}^{m} \binom{\alpha_i}{3} - \sum_{i=1}^{m} \binom{\alpha_i}{2} = \tfrac{1}{6} \sum_{i=1}^{m} \alpha_i(\alpha_i - 1)(\alpha_i - 5).$$

Example 3.7. Let K_{α_i,β_i}, for $i = 1,2,\ldots,m$, be complete bipartite graphs of partite sets of size α_i β_i, then it is known that K_{α_i,β_i} contains no odd cycles, and so $\gamma_i = 0$, for $i = 1,2,\ldots,m$.

Hence,

$$A = \sum_{i=1}^{m} \left[\alpha_i \binom{\beta_i}{2} + \beta_i \binom{\alpha_i}{2} \right] = \tfrac{1}{2} \sum_{i=1}^{m} \alpha_i\beta_i(\alpha_i + \beta_i - 2),$$

and

$$B = \sum_{i=1}^{m} \left[\binom{\alpha_i + \beta_i}{3} - \tfrac{1}{2}\alpha_i\beta_i(\alpha_i + \beta_i - 2) \right].$$

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