Journal of Basrah Researches ((Sciences)) Vol.(45). No.2 (2019)

Available online at : http:// brsj.cepsbasra.edu . iq

ISSN -1817-2695

Received - -2019, Accepted 7- 10-2019

Imperfect Bifurcation of Cammasa-Holm Equation using

Wisam M. Sadeg, Mudhir A. Abdul Hussain

Department of Mathematics, College of Education for Pure Sciences, University of Basrah, Basrah, IRAQ. mud_abd@yahoo.com

Abstract

In this paper, we are studying imperfect bifurcation of the Camassa-Holm equation using the local method of Lyapunov-Schmidt.We showed that the bifurcation equation corresponding to the Camassa-Holm equation is given by the nonlinear system of two equations. The bifurcation diagram of the bifurcation equation showed that there are zero, two or four real solutions. We considered these solutions as solutions of a dynamical system, then we draw the phase portrait corresponding to the system.

keywords: Lyapunov-Schmidt Method, Bifurcation Theory, Nonlinear Systems.

1 Introduction

many of the nonlinear problems that appear in mathematics and physics can be written in the form of operator equation,

$$F(u, \lambda, \epsilon) = b, \quad u \in O \subset X,\ b \in Y\ , \lambda \in R^n. \quad (1)$$

where F is a smooth Fredholm map with zero index, ϵ is a small parameter indicate the perturbation of the equation

$$F(u, \lambda) = b, \quad u \in O \subset X\ , b \in Y\ , \lambda \in R^n. \quad (2)$$

X, Y are Banach spaces and O is an open subset of X. For these problems, the method of reduction to finite-dimensional equation,

$$\theta(\xi, \lambda, \varepsilon) = \beta, \ \ \xi \in \hat{M}, \ \ \beta \in \hat{N} \quad (3)$$

where theta is a map from $\hat{M}$ to $\hat{N}$, and $\hat{M}$, $\hat{N}$ are smooth finite-dimensional manifold we get them from reduction .

Vainberg [3], Loginov [6] and Sapronov [2, 7] are dealing with equation (1) into equation (3) by using local method of Lyapunov-Schmidt with the conditions that, equation (3) has all the topological and analytical properties of equation (1) (multiplicity, bifurcation diagram, etc). The Camassa-Holm equation

$$u_t - u_{tyy} + 3uu_y + 2ku_y = 2u_y u_{yy} + uu_{yyy} \quad ,$$

$$y \in R, \ \ t > 0$$

arises as a model for the unidirectional propagation of shallow water waves over an at bottom, u(y, t) representing the waters free surface in non-dimensional variables, and $k \in R$ being a parameter related to the critical shallow water speed. Equation above was first obtained [5] as an abstract bi-Hamiltonian equation with infinitely many conservation laws [8, 9]. An interesting property of the Camassa-Holm equation is that for a wide class of initial data the spatial derivative of the solution becomes unbounded in finite time while the solution remains uniformly bounded and this property is used to relate the Camassa-Holm equation to wave breaking in the hydrodynamical interpretation [11]. Many powerful methods have been presented for finding the traveling wave solutions, such as the Backlund transformation [5], tanh-coth method [10], bilinear method [4], symbolic computation method [12], and Lie group analysis method [1]. Furthermore, a great amount of works focused on various extensions and applications of the methods in order to simplify the calculation procedure. In this paper, we used the local method of Lyapunov- Schmidt to study the imperfect bifurcation of traveling wave solutions of Camassa-Holm equation

$$u_t - u_{tyy} + 3uu_y + (2k + \epsilon_1(y - ct))u_y + \epsilon_2 u_{yy} =$$

$$2u_y u_{yy} + uu_{yyy} \quad y \in R, \quad t > 0 \quad (4)$$

where ϵ_1 and ϵ_2 are small parameters indicate the perturbation parameters. For a traveling wave $u(y, t) = w(x)$, $x = y - ct$ we have

$$-cw + cw'' + \frac{3}{2}w^2 + (2k + \epsilon_1(y - ct))w' + \epsilon_2 w''$$

$$= w'w'' + w'w'' + ww'''$$

after integration and taking constant integration to be zero equation (5) takes the form

$$cw'' + (\beta + \epsilon_1 x)w + \epsilon_2 w' + \frac{3}{2}w^2 - \frac{1}{2}(w')^2 - ww'' = 0 \quad (5)$$

$$\beta = 2k - c$$

Definition 1.1. *[13]*
Let E and F are Banach spaces, operater $A : E \to F$ which is linear and continuous and such that

1. *The kernel of A is finite-dimensional,*
2. *The Range of A is closed in F,*
3. *The Cokernel of A is finite-dimensional. is called Fredholm operator. The Fredholm index for operator A is the number dim(KerA)-dim(CokerA) .*

Definition 1.2. *The set of all $\lambda = \bar{\lambda}$ for the equation (1) which is it has a degenerate solution is called discriminate set Σ , $\bar{x} \in O$:*

$$F(\bar{x}, \bar{\lambda}, \epsilon) = 0\ , \quad Codim(Im\partial F/\partial x(\bar{x}, \bar{\lambda}, \epsilon)) > 0. \quad (6)$$

2 Lyapunov-Schmidt reduction

To study the problem (5) with boundary value problem, we will make it in the format operator equation, that is;

$$F(w, \lambda, \epsilon_1, \epsilon_2) = cw'' + (\beta + \epsilon_1 x)w + \epsilon_2 w' + \frac{3}{2}w^2 - \frac{1}{2}(w')^2 - ww'' \quad (7)$$

Where F : E $\to$ M is nonlinear Fredholm nonlinear map with zero index from Banach space E to Banach space M ,$E = C^2([0, 1], R)$ is the space of continuous functions which is have derivative from order at most tow,$M = C^0([0, 1], R)$ is the space of continuous functions and $w = w(x)$, x $\in$ [0, 1],$\lambda = (c, \beta)$. In this case, the bifurcation solutions for the equation (7) is equivalent to the bifurcation solutions of the operator equation

$$F(w, \lambda, \epsilon_1, \epsilon_2) = 0 \quad (8)$$

It is well known that by finite dimensional reduction theorem the solutions of problem (5) is equivalent to the solutions of finite dimensional system with $dim(kerFw(0, \lambda)) = 2$ variables and $dim(CokerFw(0, \lambda)) = 2$ equations, so first step in this work we shall find this system and then we analyze the results to find the bifurcation solutions of problem (5). Our purpose is to study the bifurcation solutions of problem (5) near the bifurcation solutions of the problem

$$cw'' + \beta w + \frac{3}{2}w^2 - \frac{1}{2}(w')^2 - ww'' = 0 \quad (9)$$

$$w(0) = w(1) = 0$$

The first step in this reduction we are determines the linearized equation which is corresponding to the equation (8), which is given by the following equation:

$$Ah = 0, \quad h \in E,$$

$$A = \frac{\partial f}{\partial w}(0, \lambda, 0, 0) = ch'' + \beta h, \quad x \in [0, 1], \quad (10)$$

$$h(0) = h(1) = 0$$

The solution of the linearized equation satisfied the boundary conditions is given by

$$e_p = c_p \sin(p\pi x), \ \ p = 1, 2, 3...$$

and the characteristic equation which is corresponding to above solution is

$$\beta - cp^2\pi^2 = 0.$$

This equation gives in the $c\beta$-plane characteristic lines lp. The characteristic lines l_p consist the points (c, β) in which the linearized equation above has non-zero solutions. The intersection of characteristic lines in the cβ-plane is a bifurcation point [8]. So the point (c, β) = (0,0) is a bifurcation point for equation (8). Localized parameters c, β as follows, $c = 0+\delta_1$, $\beta = 0+\delta_2$,where δ_1, δ_2 are small parameters which are lead to bifurcation along the modes

$$e_1(x) = c_1 \sin(\pi x), \quad e_2(x) = c_2 \sin(2\pi x) \qquad (11)$$

where $||e_1||_H = ||e_2||_H = 1,\ and$

$$c_1 = c_2 = \sqrt{2}.\ Let\ N = ker(A) = span\{e_1, e_2\},$$

then the space E can be written by decomposed in direct sum of two subspaces, N and $N^{\perp}$,

$$E = N \oplus N^{\perp}, \quad N^{\perp} = v \in E : v \perp N$$

Similarly, the space M can be written by decomposed in direct sum of two subspaces, N and $\tilde{N}^{\perp}$,

$$M = N \oplus \tilde{N}^{\perp}, \quad \tilde{N}^{\perp} = v \in M : v \perp N$$

. then There exist two projections P : E $\rightarrow N$ and I - P : E $\rightarrow \perp$ N such that $Pw = u$, $(I - P)w = v$ and hence every vector $w \in E$ can be written in the form $w = u + v$, $u = \sum_{i=1}^{2} y_i e_i \in N$ $v \in N^{\perp}$, $y_i =< w, e_i >$. Also, there exist projections Q: $M \rightarrow N$ and I - Q:$M \rightarrow \tilde{N}^{\perp}$ such that

$$QF(w, \lambda, \epsilon_1, \epsilon_2) = F_1(w, \lambda, \epsilon_1, \epsilon_2),$$

$$(I - Q)F(w, \lambda, \epsilon_1, \epsilon_2) = F_2(w, \lambda, \epsilon_1, \epsilon_2),$$

and hence

$$F(w, \lambda, \epsilon_1, \epsilon_2) = F_1(w, \lambda, \epsilon_1, \epsilon_2) + F2(w, \lambda, \epsilon_1, \epsilon_2),$$

$$F_1(w, \lambda, \epsilon_1, \epsilon_2) = \sum_{i=1}^{2} v_i(w, \lambda, \epsilon_1, \epsilon_2)e_i \in N$$

$$F_2(w, \lambda, \epsilon_1, \epsilon_2) \in \hat{N}$$

,

$$v_i(w, \lambda, \epsilon_1, \epsilon_2) =< F(w, \lambda, \epsilon_1, \epsilon_2), e_i >$$

. and we can write in the form

$$QF(w, \lambda, \epsilon_1, \epsilon_2) = 0,$$

$$(I - Q)F(w, \lambda, \epsilon_1, \epsilon_2) = 0$$

or

$$QF(u + v, \lambda, \epsilon_1, \epsilon_2) = 0,$$

$$(I - Q)F(u + v, \lambda, \epsilon_1, \epsilon_2) = 0$$

By implicit function theorem, there exists a smooth map $\Phi : N \rightarrow N^{\perp}$(depending on λ), such that,$\phi(w, \lambda, \epsilon_1, \epsilon_2) = v$ and

$$(I - Q)F(u + \phi(w, \lambda, \epsilon_1, \epsilon_2), \lambda, \epsilon_1, \epsilon_2) = 0$$

To find the solutions of the equation $F(w, \lambda, \epsilon_1, \epsilon_2) = 0$ in the neighbourhood of the point $w = 0$ it is sufficient to find the solutions of the equation

$$QF(u + \phi(w, \lambda, \epsilon_1, \epsilon_2), \lambda, \epsilon_1, \epsilon_2) = 0 \qquad (12)$$

Equation (12) is called the bifurcation equation of equation (7) and then we have bifurcation equation in form

$$\theta(y,\lambda,\epsilon_1,\epsilon_2)=0, \qquad y=(y_1,y_2), \qquad \lambda=(c,\beta)$$

where

$$\theta(y,\lambda,\epsilon_1,\epsilon_2)=F_1(u+\phi(w,\lambda,\epsilon_1,\epsilon_2),\lambda,\epsilon_1,\epsilon_2)$$

. Equation (7) can be written in the form,

$$F(u+v,\lambda,\epsilon_1,\epsilon_2)=A(u+v)+B(u+v).$$

$$=Au+\epsilon_1 xu+\epsilon_2 u'+\frac{3}{2}u^2-\frac{1}{2}(u')^2-uu''+...$$

where

$$B(u+v)=\epsilon_1 x(u+v)+\epsilon_2(u+v)'+\frac{3}{2}(u+v)^2$$

$$-\frac{1}{2}((u+v)')^2-(u+v)(u+v)''+...$$

and hence

$$\theta(y,\lambda,\epsilon_1,\epsilon_2)=F(u+v,\lambda,\epsilon_1,\epsilon_2)=$$

$$\sum_{i=1}^{2}<Au+\epsilon_1 xu+\epsilon_2 u'+\frac{3}{2}u^2-$$

$$\frac{1}{2}(u')^2-uu'',e_i>e_i+...=0 \qquad (13)$$

where $(<.,.>_H$ is the scalar product in Hilbert space $L^2([0,1],R))$. Equation (13) implies that

$$\sum_{i=1}^{2}<Au+\epsilon_1 xu+\epsilon_2 u'+\frac{3}{2}u^2-\frac{1}{2}(u')^2-uu'',e_i>e_i$$

$$+...=0e_1+0e_2$$

after some calculations of equation for above equation we have the following result

$$(A_1y_1^2+A_2y_2^2+A_3y_1-A_4y_2)e_1$$

$$+(B_1y_1y_2+B_2y_1+B_3y_2)e2=0e_1+0e_2$$

where

$$\langle f,g\rangle=\int_0^1 f(x)*g(x)dx$$

Theorem 2.1. *The bifurcation equation*

$$\theta(y,\lambda,\epsilon_1,\epsilon_2)=F_1(u+\phi(w,\lambda,\epsilon_1,\epsilon_2),\lambda,\epsilon_1,\epsilon_2)$$

corresponding to the equation (8) have the following form ,

$$\theta(y,\tilde{\lambda})=\begin{pmatrix} A_1y_1^2+A_2y_2^2+A_3y_1-A_4y_2 \\ B_1y_1y_2+B_2y_1+B_3y_2 \end{pmatrix}$$

$$+o(|y|^2)+O(|y|^2)O(\delta)=0$$

where O is capital order , o is smaller order . And $y=(y_1,y_2)$, $\tilde{\lambda}=(A_3,A_4,B_2,B_4)\in R^4$, $\delta=(\delta_1,\delta_2)$.
The discriminate $\sum$ set of the equation $\theta(y,\tilde{\lambda})$ is locally equivalent in the neighborhood of the point zero to the discriminate set of the following equation ,

$$\theta_1(\tilde{y},\gamma)=\begin{pmatrix} \tilde{y}_1^2+\tilde{y}_2^2+\lambda_1\tilde{y}_1-\lambda_2\tilde{y}_2 \\ 2\tilde{y}_1\tilde{y}_2+\lambda_3\tilde{y}_1+\lambda_4\tilde{y}_2 \end{pmatrix}=0$$

this means, to study the discriminate set of equation $\theta(y,\tilde{\lambda})$ it is suitable to study the discriminate set of equation $\theta_1(y,\tilde{\lambda})$. By changing variables,

$$x_1=\tilde{y}_1+\frac{\lambda_1}{2}, \qquad x_2=\tilde{y}_1+\frac{\lambda_2}{2},$$

we have the above equation is equivalent to the following equation

$$\theta_1(x,\hat{\lambda}) = \begin{pmatrix} x_1^2 + x_2^2 - \beta_1 \\ 2x_1x_2 + \tilde{\lambda}_1 x_1 + \tilde{\lambda}_2 x_2 + \beta_2 \end{pmatrix} = 0$$

$where\ x = (x_1, x_2)\ ,\ \hat{\lambda} = (\hat{\lambda}1, \hat{\lambda}2, \beta 1, \beta 2) \in R^4.$

3 Analysis of bifurcation

We have that the point $a \in E$ is a solution of the equation (8) if and only if

$$a = \sum_{i=1}^{2} \bar{x}_i e_i + \phi(\bar{x}, \bar{\lambda}),$$

where $\bar{x}$ is a solution of equation

$$\Theta_1(x,\hat{\lambda}) = 0 \tag{14}$$

also, the Discriminant set of equation (8) is convenient to the Discriminant set of the equation (14). This section holds of the determination of the Discriminant set of equation (14) and then the determination of solutions of equation (14) as $\hat{\lambda}$ varied. There are two ways to determine the Discriminant set,

1. By finding a relation between the parameters and variables which are in the equation (14).
2. By finding the parameters equation, which is, the equation of the form,

$$h(\hat{\lambda}) = 0, \qquad \hat{\lambda} = (\tilde{\lambda}_1, \tilde{\lambda}_2, \beta_1, \beta_2) \in R^4$$

such that $\hat{\lambda} = (\tilde{\lambda}_1, \tilde{\lambda}_2, \beta_1, \beta_2)$ in which equation (14) has degenerate solutions that satisfy the equation $h(\hat{\lambda}) = 0$, where $h : R^2 \to R$ is a map. To study the Discriminant set of the equation (9) it is appropriate to fixed the values of $\tilde{\lambda}_1$, $\tilde{\lambda}_2$ and then find all sections of Discriminant set in the $\beta_1\beta_2 - plane$. To do this, we use the following parameterization

$$\beta_1 = x_1^2 + x_2^2$$

$$\beta_2 = 2x_1x_2 + \tilde{\lambda}_1 x_1 + \tilde{\lambda}_2 x_2$$

and then we describe the Discriminant set for equation (14) in the $\beta_1\beta_2 - plane$ for some values of $\tilde{\lambda}_1$, $\tilde{\lambda}_2$ with the number of regular solutions in all region in the following figures, (all figures were drawn by Maple 2016).

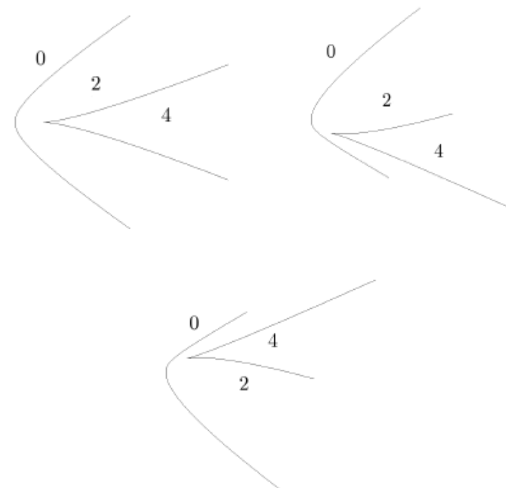


Figure 1: Material functions, Newtonian vs Inelastic, n-variation, $k = 1$

Figure 2: Describe the Discriminant set of equation 14

The above graphics are realized when $\tilde{\lambda}_1, \tilde{\lambda}_2$ are in intervales

$$(\tilde{\lambda}_1, \tilde{\lambda}_1 + 5) \quad where \quad \tilde{\lambda}_1 = -30...10$$

and $(\tilde{\lambda}_1, \tilde{\lambda}_1 + 2)$ $where$ $\tilde{\lambda}_1 = -20...18$ and $(\tilde{\lambda}_1, \tilde{\lambda}_1 + 8)$ $where$ $\tilde{\lambda}_1 = -30...8$ In the drawing above the complement of Discriminant set $\Omega = R^4/\Sigma$ which a union of the sets Ω_0 and Ω_2 and

$\Omega_4, when, \Omega$=$\Omega_0 \cup \Omega_2 \cup \Omega_4$ such that when $\hat{\lambda} \in \Omega_0$ then the equation (14) does not have regular solutions, and if $\hat{\lambda} \in \Omega_2$ then the equation (14) have two regular solutions, and when $\hat{\lambda} \in \Omega_4$ then the equation (14) have four regular solutions .

Theorem 3.1. *The bifurcation equation which is corresponding to the equation (7) is a nonlinear system of two nonlinear equations given in the form*

$$A_1y_1^2 + A_2y_2^2 + A_3y_1 - A_4y_2 = 0$$

$$B_1y_1y_2 + B_2y_1 + B_3y_2 = 0$$

The solutions of equation (7) are in one-to-one correspondence for the solutions of the above system . It is noted that the solutions of the system

$$\left.\begin{aligned} x_1^2 + x_2^2 - \beta_1 = 0 \\ 2x_1x_2 + \tilde{\lambda}_1x_1 + \tilde{\lambda}_2x_2 + \beta_2 = 0 \end{aligned}\right\} \tag{15}$$

can be classified as equilibrium points of a dynamical system

$$\left.\begin{aligned} \dot{x_1} = x_1^2 + x_2^2 - \beta_1 \\ \dot{x_2} = 2x_1x_2 + \tilde{\lambda}_1x_1 + \tilde{\lambda}_2x_2 + \beta_2 \end{aligned}\right\} \tag{16}$$

so that the behavior of the system (15) near the regular solutions is similar to the behavior of the dynamical system (16) near the equilibrium points (x_1, x_2).

By fixed the values of β_1 and β_2 in each set $A_i, i = 1, ..., 3$ and $B_i, i = 1, ..., 3$ and $C_i, i = 1, ..., 3$ we can classify the equilibrium points of system (16).

The jacobian matrix of system (16) is given by Where$\lambda_1 = 3$ and $\lambda_2 = 5$.

In the set A_1 we choose $\beta_1 = 100.5$ and $\beta_2 = 50.1$ For these values we have the following four real solutions (equilibrium points) of system (16)

$P_1 = (9.473579334, -3.278916681)$,

$P_2 = (-0.001457108568, -10.02496872)$,

$P_3 = (-4.540682511, 8.937684396)$,

$P_4 = (-9.931439715, 1.366201008)$.

After testing the eigenvalues of the jacobian matrix of system (16) at each point we found that each of the points P_1 is node unstable and p_4 node stable and p_2,p_3 saddle unstable.

In the set A_2 we choose $\beta_1 = 55.5$ and $\beta_2 = 50.1$ For these values we have the following two real solutions (equilibrium points) of system (16)

$P_1 = (1.132829170, -7.363198902)$,

$P_2 = (6.341913702, -3.908980761)$.

After testing the eigenvalues of the jacobian matrix of system (16) at each point we found that the point P_1 is saddle unstable and the point P_2 is node unstable . Where $\tilde{\lambda}_1 = -3$ and $\tilde{\lambda}_2 = 5$. In the set B_1 we choose $\beta_1 = 100.5$ and $\beta_2 = 25.1$ For these values we have the following four real solutions (equilibrium points) of system (16)

$P_1 = (10.02300550, 0.1983955249)$,

$P_2 = (-1.078503213, -9.966786383)$,

$P_3 = (-4.719295349, 8.844673619)$,

$P_4 = (-9.225206936, 3.923717242)$.

After testing the eigenvalues of the jacobian matrix of system (16) at each point we found that each of the points P_1 is node unstable and P_4 node stable and P_2, P_3 saddle unstable. In the set B_2 we choose $\beta_1 = 25.5$ and $\beta_2 = 25.1$ For these values we have the following tow real solutions (equilibrium points) of system (16)

$P_1 = (5.004848767, -0.6719291764)$,

$P_2 = (-0.1135148564, -5.049739710)$.

After testing the eigenvalues of the jacobian matrix of system (16) at each point we found that point P_1

is node unstable and the point P_2 is saddle unstable .

Where $\tilde{\lambda}_1 = 0$ and $\tilde{\lambda}_2 = 5$. In the set C_1 we choose $\beta_1 = 100.5$ and $\beta_2 = 30.1$ For these values we have the following four real solutions (equilibrium points) of system (16)

$P_1 = (9.951841503, -1.208656567)$,

$P_2 = (-0.9913538544, -9.975831671)$,

$P_3 = (-4.149125623, 9.126048245)$,

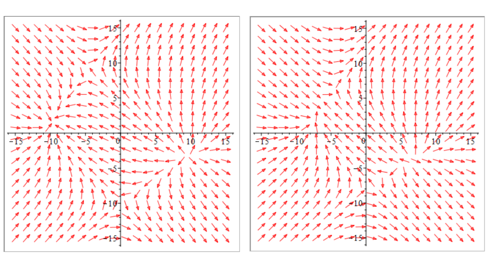
$P_4 = (-9.811362025, 2.058439994)$.

After testing the eigenvalues of the jacobian matrix of system (16) at each point we found that each of the points P_1 is node unstable and p_4 node stable and p_2,p_3 saddle unstable. In the set C_2 we choose $\beta_1 = 50.5$ and $\beta_2 = 50.1$ For these values we have the following tow real solutions (equilibrium points) of system (16)

$P_1 = (1.065313185, -7.026030728)$,

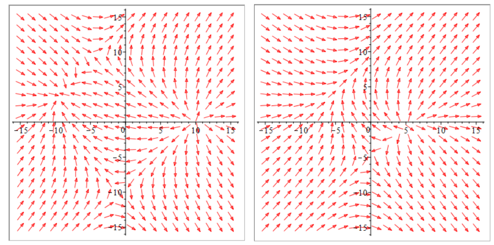
$P_2 = (6.544378826, -2.769676114)$.

After testing the eigenvalues of the jacobian matrix of system (16) at each point we found that the point P_1 is node unstable and the point P_2 is saddle unstable .



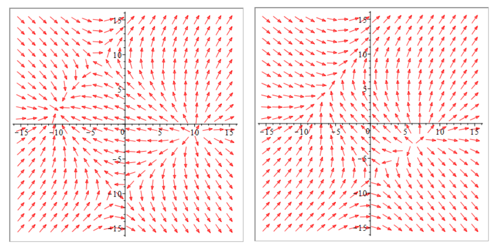
(a) :$\beta_1 = 100.5, \beta_2 = 50.1$ (b) :$\beta_1 = 55.5, \beta_2 = 50.1$

Figure 3: The phase where $\tilde{\lambda}_1 = 3, \tilde{\lambda}_2 = 5$



(a) :$\beta_1 = 100.5, \beta_2 = 25.1$ (b) :$\beta_1 = 25.5, \beta_2 = 25.1$

Figure 4: The phase where $\tilde{\lambda}_1 = -3, \tilde{\lambda}_2 = 5$



(a) :$\beta_1 = 100.5, \beta_2 = 30.1$ (b) :$\beta_1 = 50.5, \beta_2 = 50.1$

Figure 5: The phase where $\tilde{\lambda}_1 = 0, \tilde{\lambda}_2 = 5$

Theorem 3.2. *The polar form of the dynamical system (18) is*

$$\dot{r} = r^2(\cos(\theta) + 2\cos(\theta)\sin^2(\theta)) + r(\tilde{\lambda}_1\cos(\theta)\sin(\theta)$$

$$+\tilde{\lambda}_2\sin^2(\theta)) - \beta_1\cos(\theta) + \beta_2\sin(\theta),$$

$$\dot{\theta} = \frac{-1}{r}[r^2(\sin(\theta) + 2\sin(\theta)\cos^2(\theta)) - r(\tilde{\lambda}_1\cos^2(\theta)$$

$$+\tilde{\lambda}_2\cos(\theta)\sin(\theta)) - \beta_1\sin(\theta) - \beta_2\cos(\theta)]$$

proof :

Let $x_1 = r\cos(\theta)$ *and* $x_2 = r\sin(\theta)$

then from above assume we have

$$\dot{x_1} = \dot{r}\cos(\theta) - r\sin(\theta)\dot{\theta} \tag{17}$$

$$\dot{x_2} = \dot{r}\sin(\theta) + r\cos(\theta)\dot{\theta} \tag{18}$$

then from (17) and (18) we have

$$\dot{r} = \dot{x_1}\cos(\theta) + \dot{x_2}\sin(\theta) \tag{19}$$

and

$$\dot{\theta} = -1/r(\dot{x_1}\sin(\theta) - \dot{x_2}\cos(\theta) \tag{20}$$

then by substitute (17) and (18) in (19) and (20) respectively, then we have

$$\dot{r} = r^2(\cos(\theta) + 2\cos(\theta)\sin^2(\theta)) + r(\tilde{\lambda}_1\cos(\theta)\sin(\theta)$$
$$+\tilde{\lambda}_2\sin^2(\theta)) - \beta_1\cos(\theta) + \beta_2\sin(\theta),$$
$$\dot{\theta} = \frac{-1}{r}[r^2(\sin(\theta) + 2\sin(\theta)\cos^2(\theta)) - r(\tilde{\lambda}_1\cos^2(\theta)$$
$$+\tilde{\lambda}_2\cos(\theta)\sin(\theta)) - \beta_1\sin(\theta) - \beta_2\cos(\theta)]$$

this represents the polarity of the above system .

conclusion

In this paper, we are studied imperfect bifurcation of the Camassa-Holm equation using local method of Lyapunov-Schmidt. We showed that the bifurcation equation corresponding to the Camassa-Holm equation is given by the nonlinear system of two algebraic equations. According to the finite-dimensional reduction theorem, there is a one-to-one correspondence between the solutions of the Camassa-Holm equation and the solutions of the nonlinear system, so that the existence of the solutions of the Camassa-Holm equation depends on the existence of the solutions of the nonlinear system.

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