Journal
IJCSM

Iraqi Journal for Computer Science and Mathematics

Journal Homepage: http://journal.esj.edu.iq/index.php/IJCM
e-ISSN: 2788-7421 p-ISSN: 2958-0544

Elementary Treatise on Melted-like Curves Derived from Center Ellipse and Circle

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DOI: https://doi.org/10.52866/ijcsm.2023.01.01.009
Received October 2022; Accepted December 2022; Available online January 2023

ABSTRACT: This study deals with a special case in the geometry of 2D special plane curves. The research treaties the development of a method for locating points and drawing curves derived from moving points on two curved paths of conic sections, namely, circle and ellipse. The paper presents geometrical methods to determine points and nodes and to draw these melted-like curves in two cases. Parametric equations are also used to determine the ordinates of the points of each curve of the two cases of the curves. This research is the initial phase for the new type of curves and awaits future studies that will deal on 3D surfaces, such as a sphere, a right cone, or a right cylinder.

Keywords: Elementary Treatise; Circle**;** Ellipse; Geometric Methods.

1. INTRODUCTION

Through the prism of classic plane geometry, 2D curves have been studied by representing the coordinates of their points on the plane. 2D planes consist of many curve types, of which conic sections, such as circles, are a popular example. This study focuses on these curves of the circle and the ellipse as the case study. Geometrically, ellipses are curves, where any point is at a constant distance from two fixed points (focus) on a straight line (the diameter) [1]. The focus and directrix of an ellipse were considered by Pappus [2]. A well-known set of curves is derived from an ellipse [3,4,5]. The pedal curve of an ellipse, with its focus as pedal point, is a circle [6]. The evolute of the ellipse with equation is the Lamé curve [7]. Huygens and Tschirnhaus (1678) [8,9], Johann Bernoulli [9,10], Jacob Bernoulli [11,12], de L'Hôpital [13], and Lagrange [14] studied about elliptic curves. However, the case in which the curvatures of an ellipse and a circle shared a center point is not yet engaged geometrically in such view.
A set of stretched curves is derived from moving points on two curvature paths of conic sections, a circle and an ellipse. On this basis, a set of new forms of closed curves is developed and two geometrical construction methods are presented. We confirmed geometrical procedures that allow obtaining more properties of harmonic curves. The figures are designed by AutoCAD, and the sphere and related curves are drawn.

2. GEOMETRICAL PROPERTIES

In this study, we deal with two conic curvatures, which share the same center point of origin. We consider the construction procedures of curves and their geometrical properties to familiarize with the above notions. The projection of curvatures on the unit circle is presented rather than the spherical curvatures given that the study focuses on the 2D plane curves produced only from this nominated case of circle and ellipse (Fig. 1).

Mathematically, all melted-like curves (MLC) resulted from the orthogonal projection of a fixed point at a line segment drawn from the center of two conic curvatures; a circle and an ellipse. A new form of closed curves of MLC is produced, and a construction method is presented. In this method, a unit circle with center point (A) is considered, and then an ellipse with the same center point whose major axis equals to the radius of the unit circle is constructed.

The proposed method is studied to produce these curves by two cases according to the position of the major axis of the ellipse related to the x-axis, (Fig. 1). The ellipse minor axis (a) is varied along the *y*-axis, whereas the major axis (b) is fixed at *x*-axis.

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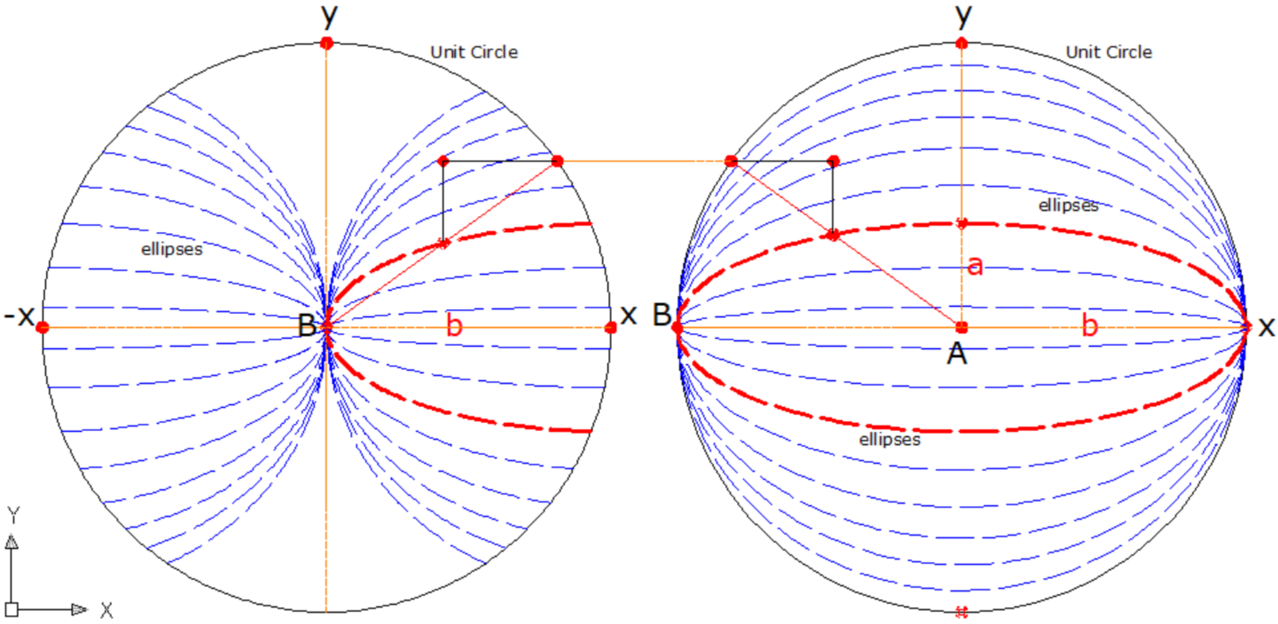

FIGURE 1- Two projections of the unit circle and ellipses whose major axis equals to the radius value

2.1 GEOMETRIC CONSTRUCTION

Building on the main features, which are illustrated in Figure 1. A construction method to draw a new form of curves is presented. The essential step is that the line segment from center point (*A*) is drawn to intersect the circumference of the unit circle at point (*P*), passing through the ellipse circumference at point (*C*). Thus, by drawing two perpendiculars (a horizontal from point (*P*) (line *PB*) and a vertical from (*C*), (line *CB*)), a set of intersection points (*B*) can be obtained, representing the points of these special MLC (Fig. 2).

2.2 First Case of MLC

In this case, each MLC is drawn by a fixed ellipse (the drawing curvature), where the major axis (*b*) lies on the x-axis, (horizontal circle), and point (*A*) is the center point of the ellipse located on the origin point (0,0).

The construction procedures of this case are as follows (Fig. 3):

- A circle with radius (*b*) is drawn, and its center point serves as the origin point (*A*).
- An ellipse (the drawing curvature), whose major axis lies on *x*-axis and equals to the radius (*b*), is drawn; thus, the minor axis (*a*) of the ellipse lies along the sides of the *y*-axis.
- From point (*A*), a segment line is drawn to intersect the ellipse at point (*C*), passing through the circle at point (*P*).
- From point (*P*), a horizontal line (parallel to x-axis) is drawn to intersect the perpendicular line already drawn from point (*C*).
- The intersection point of these perpendicular lines is point (*B*), which is a point on the curve.
- By repeating the previous steps on another ellipse with different lengths of minor axis, a set of points, which belong to a special curve produced by this method, is determined (Fig. 3).

The results obtained in the previous paragraphs are combined. The main property of this circle is its equality with the ellipse (the drawing curvature) whose major axis tangents the circle at points (*C*) and (*D*); it carries the minor axis (*a*), which is the main parameter of MLC (Fig. 4).

Every point on the formed MLC shares the same value of *y*-axis of the ellipse, whereas the *x*-axis of this MLC shares the same value as that of the unit circle. The frontal and right-side projections of the unit circle plot six nominated curve samples obtained by various (*a*), where angle (*u*) and ray (AP) completed a full round (2π). In addition, all the resulting curve cases in Figure 6 show a clear symmetry on the *x*- and *y*-axes, intersecting the circumference of the circle at four points. Moreover, all the resulting curves are tangent on two vertices of the ellipse at a distance of twice the large axis of the ellipse (*b*). When the ratio between (*b*) and (*a*) decreases, the MLC inflation points head closely to the circle circumference (Fig. 4 and 5).

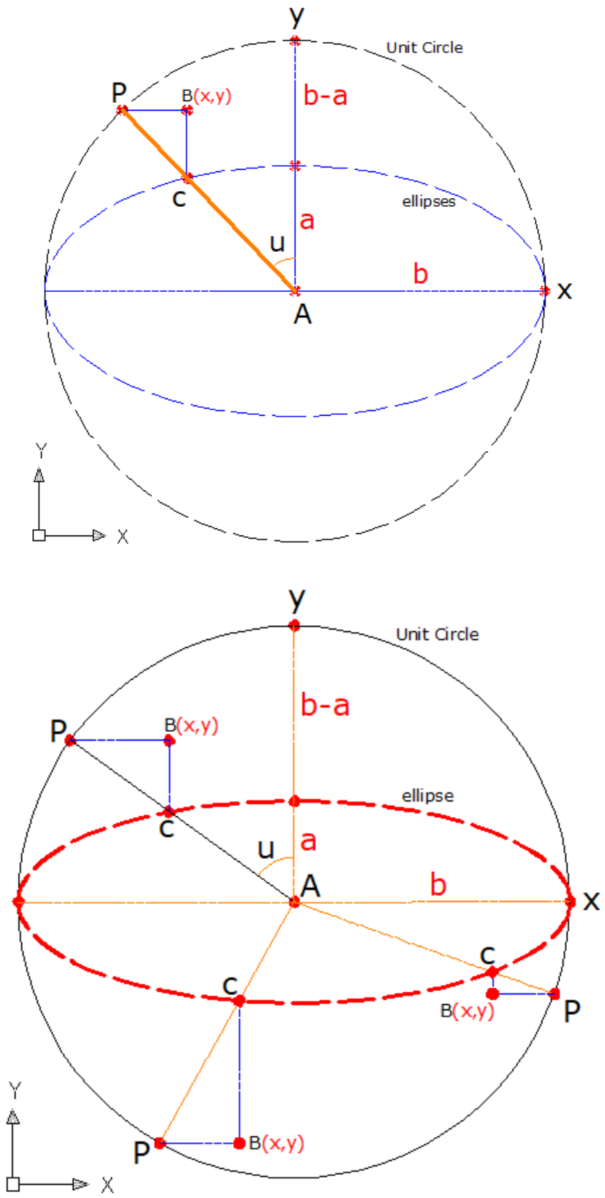

FIGURE 2. Plot of a circle and an ellipse around the origin as center, (A), two perpendiculars (PB), and (BC), from the intersection points (P) and (C).

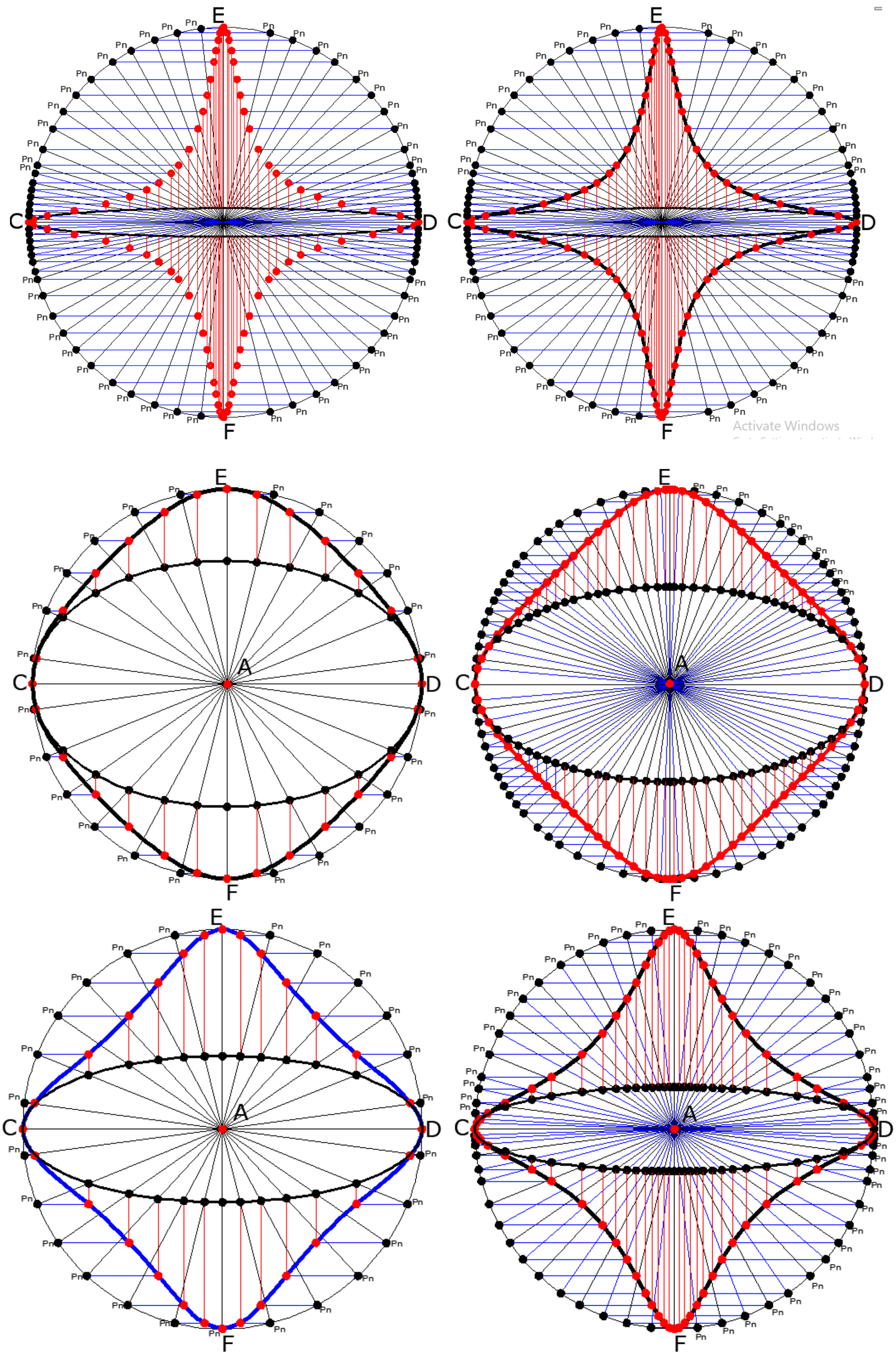

FIGURE 3. Set of points determined by plotting MLC, which tangents the unit circle at four points (E, F, C, and D).

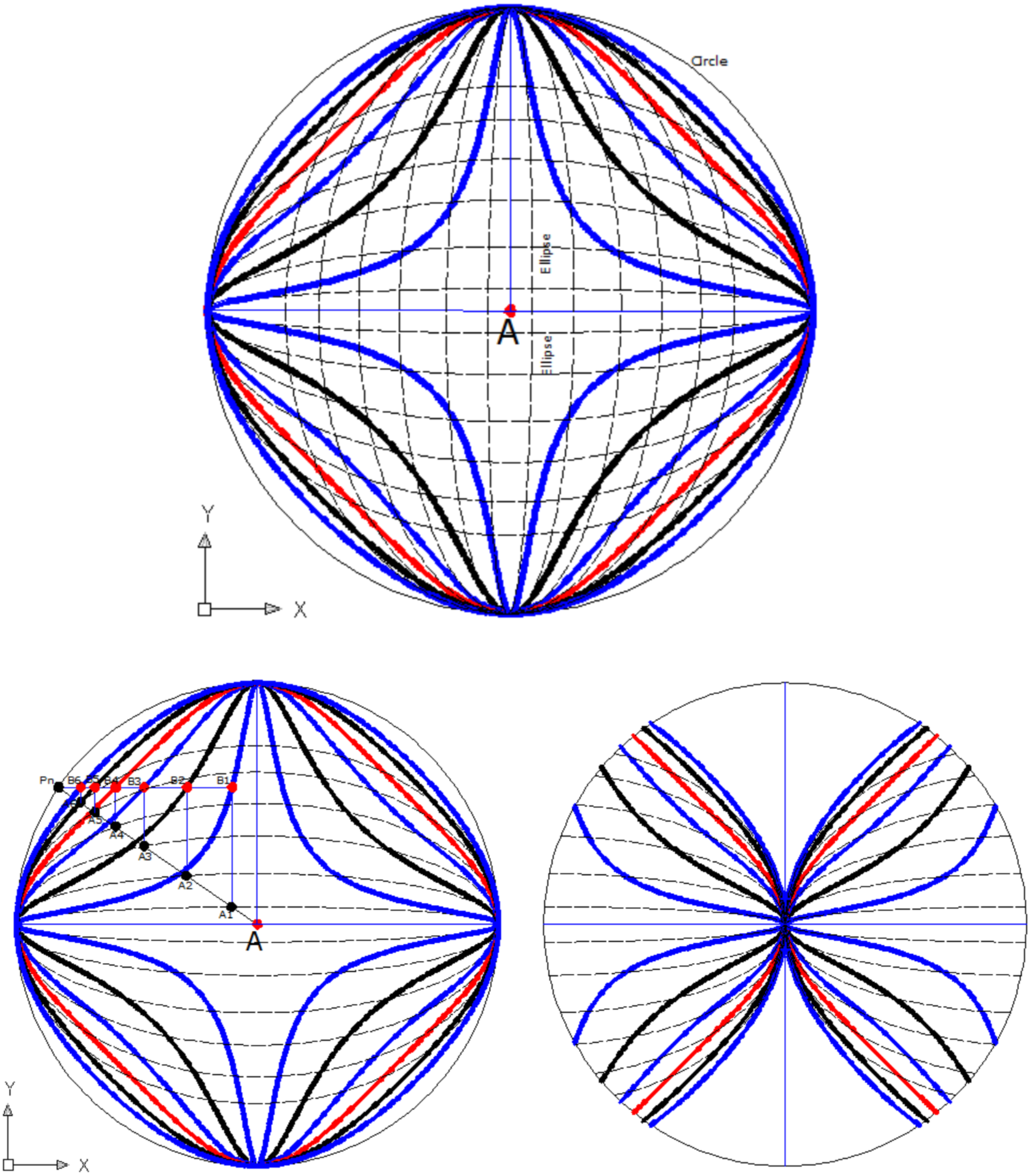

FIGURE 4. **Projected frontal and right side of a set of MLC locus for (*b* = 5.3, 5, 4, 3, 2, and 1.6*a*).**

3. PARAMETRIC EQUATION OF THE MLC

The shape of MLC is limited by *x* and *y* coordinates of a circle and ellipse. The ellipse has a major along the *x*-axis, which is (2*b*), and a minor on *y*-axis, which is (2*a*). The unit circle has a diameter denoted by (*b*), which is equal to the major of the ellipse given that the center of the circle and the ellipse is at the origin. The ray drawn from the center point to any point on the ellipse indicates that points (B) are points on the formed MLC. The cusp appears when the ratio between (*a*) and (*b*) increases in the point of inflection; here, at *u* = 45 (Figs. 4 and 5).

$$AP1 = \boldsymbol{b}\,, \tag{1}$$
$$\boldsymbol{x} = \boldsymbol{b}\ \cos \mathrm{u}\ , \tag{2}$$

The length of the radius vector from the center point (A) of the ellipse to any point on the ellipse, (AA_2), which is drawn in a given direction; $AA_2 = r$, then;

$$\boldsymbol{y} = r\ \sin u\,, \tag{3}$$

$$\frac{x^2}{\boldsymbol{b}^2} + \frac{y^2}{\boldsymbol{a}^2} = 1\,,\ a > 0, \tag{4}$$

$$\frac{r^2\ \cos^2 u}{\boldsymbol{b}^2} + \frac{r^2\ \sin^2 u}{\boldsymbol{a}^2} = 1\,, \tag{5}$$

Given equation (3), then:

$$\boldsymbol{y} = \boldsymbol{ab}\ \ \sin \mathrm{u} \,/\, \sqrt{\boldsymbol{a}^2\ cos^2 u\ +\ \boldsymbol{b}^2\ sin^2 u}\ \ , \tag{6}$$

Geometrically, all MLCs are in contact with the circumference of the ellipse and do not intersect at any point along the ellipse. The bulge in the MLC's path is accrued in two regions, all of which are at the contact area with the ellipse, whereas the concavity is defined in the two corresponding regions from both sides of the curve parallel to the ends of the positive and negative *x*-coordinates (Figs. 5 and 6). Mathematically, the MLC is symmetrical, consisting of four parts located in each quadrant of the plotted circle, whereas the MLC's length is determined by a direct relationship of variables represented by the ratio between the length of the two coordinates of the ellipse, (*b*) and (*a*), and angle (*u*).

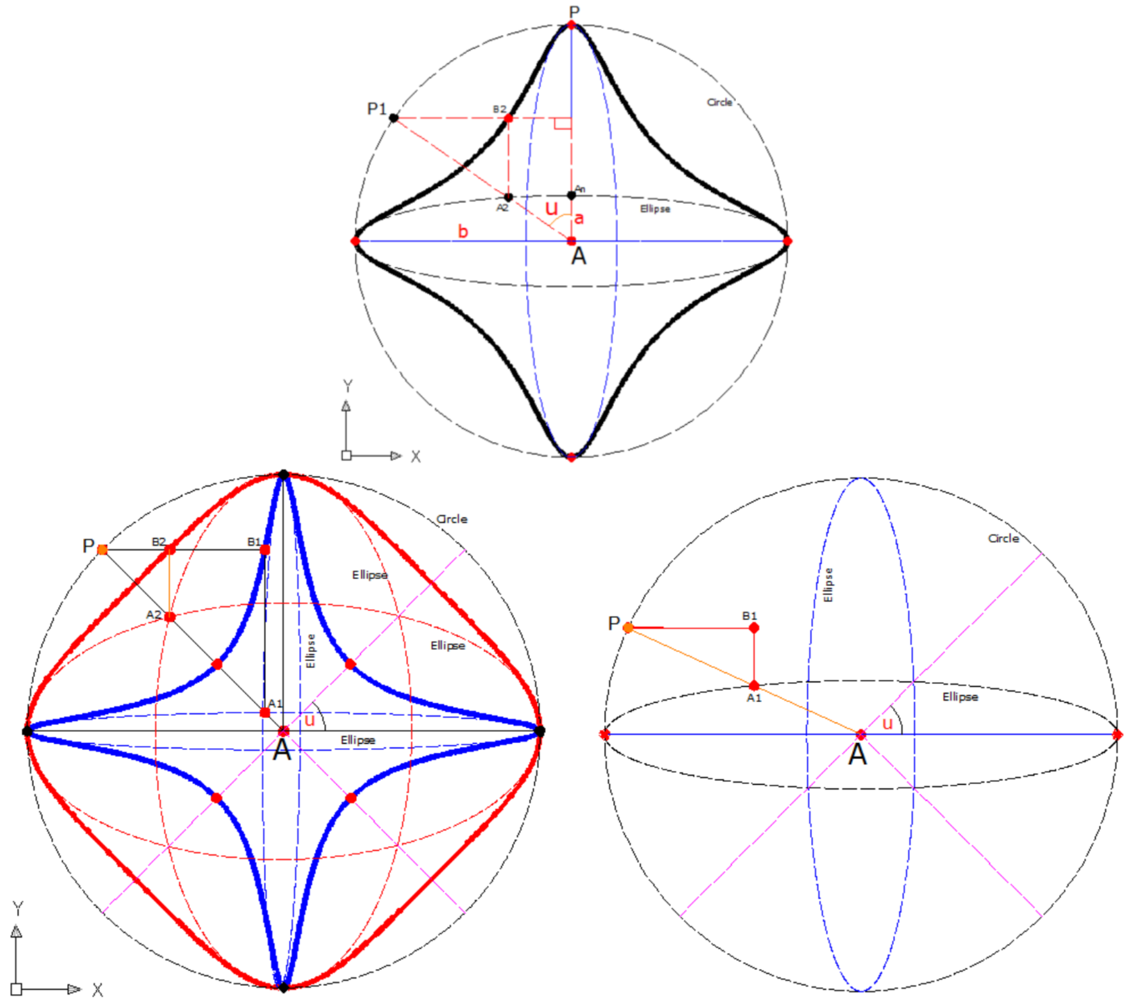

Figure 5. Construction of the MLC's locus for $b = 5.3a$

3.1 MLC's Second Case

The MLC's second study case in this research has been conducted by drawing a circle with a radius equal to the length of the great axis of the ellipse, (b), which is drawn at the origin, that is also the center of the circle. In second case of MLC, the curve is vertically drawn by a fixed ellipse (drawing curvature), where the major axis lies on the x-axis, (b), and point (A) is the center point located on the origin. All curves in this case are extended bars parallel to the major axis of the ellipse. The convexity region is at the vertices of the ellipse and tangentially at those points, depending on the length of the great axis of the ellipse.

The diversity of the MLC - nets may not be consistently oriented to that of the ellipses, which are in unit circle and exhibit this feature intimately, due to their combinatorial structure. Similar to the first case, the bulge is lateral and tangent to the ellipse and is determined by the value of the minor axis of the ellipse. All curves in this case have four curvatures that begin at a distance equal to the length of the small axis of the ellipse down to the point of contact with the ellipse. An example of a checkerboard MLC-net is displayed in Fig. 6.

In general, MLC-net may be constructed in a single manner. This terminology of MLC-net is due to the fact that the intersection points of the MLC-net lie on two centered curvatures, which are confocal to the underlying conic. Thus, the Euclidean notions and homothetic MLC-nets form an actual 2D family of nets.

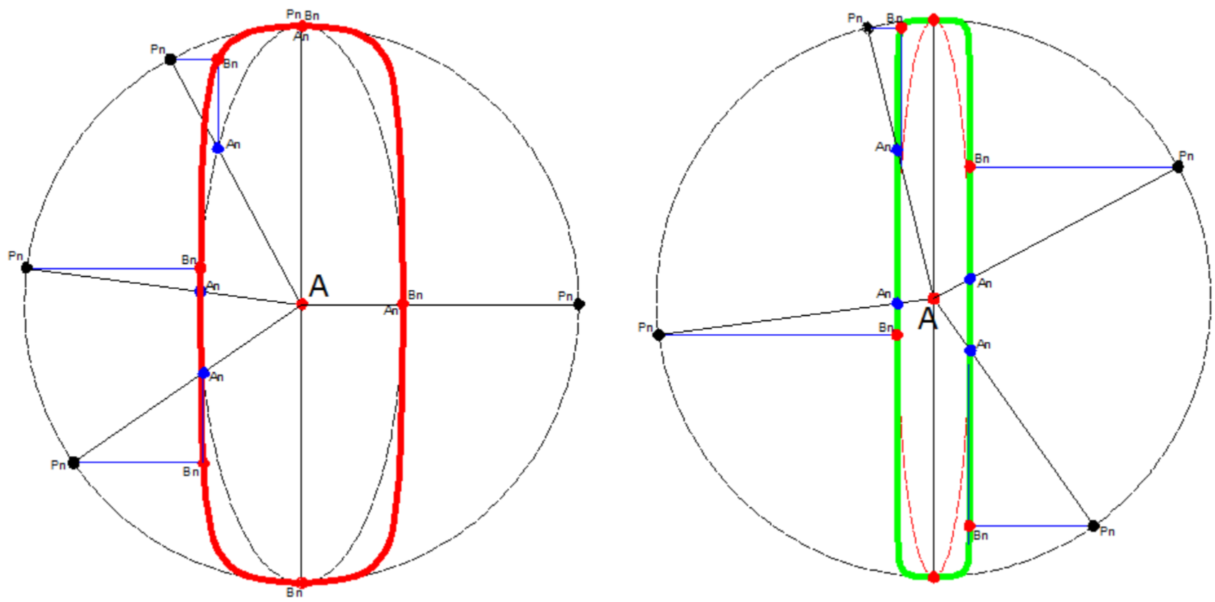

FIGURE 6. Construction of a set of MLC locus for *a* = 1.3, 2, 3 and 5*b*

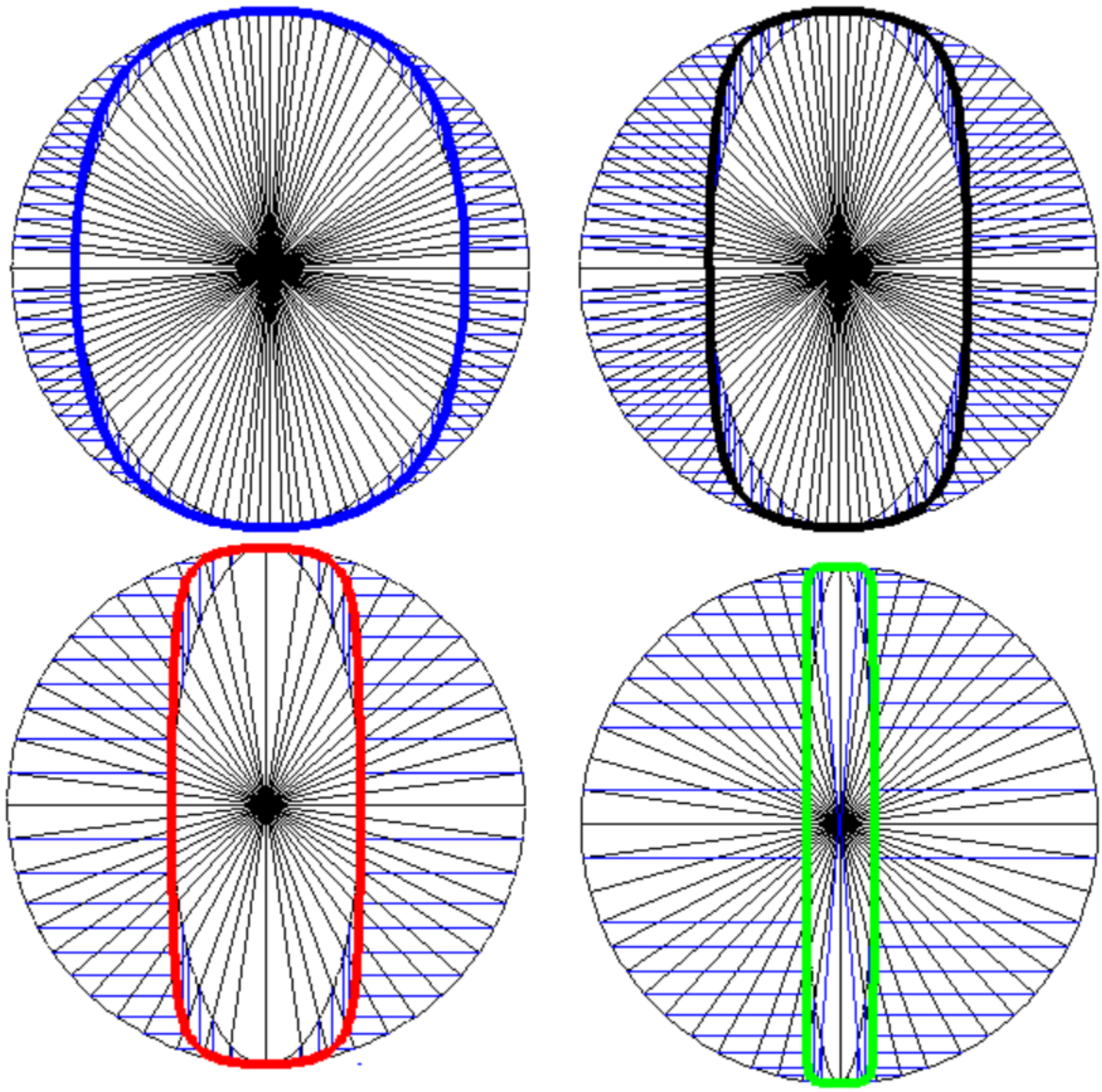

FIGURE 7. Construction of a set of MLC locus for *a* = 1.3, 2, 3 and 5*b*

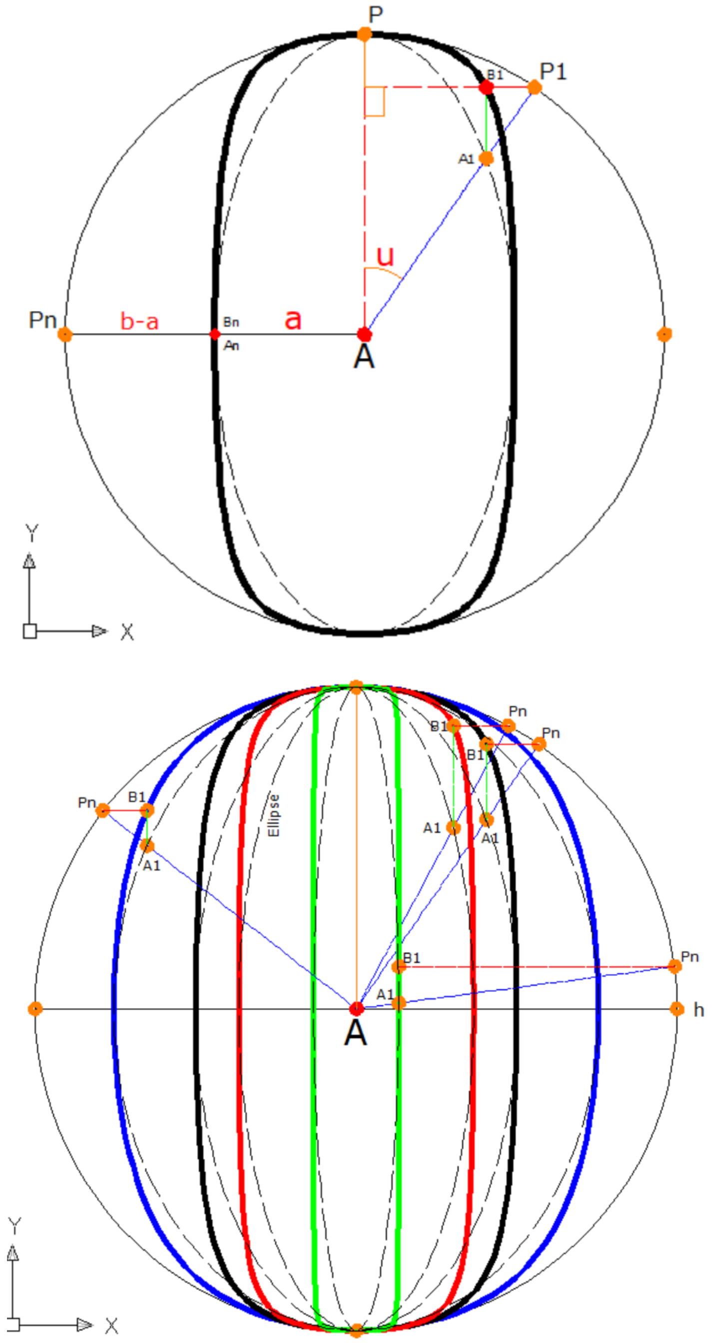

FIGURE 8. Construction of a final set of MLC locus for *a* = 1.3, 2, 3 and 5*b*

3.2 MLC'S GEOMETRIC PROPERTIES

The MLC also shares an interesting property to ellipse. All focus points of MLC in the second case are located on the major axis of their ellipse with the same distance from the unit circle center, (A). The distance is measured by the following equation, where (*a*) and (*b*) are the major and minor axes of the ellipse, respectively (Fig.8):

$$MLC's\ \text{focus} = \sqrt{2\boldsymbol{ab} - \boldsymbol{a}^2}\ , \tag{7}$$

Geometrically, MLCs share the same position of focus point as its drawing ellipse. Figure 8 shows the MLC foci point determined according to the elliptical variable of (*a*), (*b*), and (*u*), and then the distance of (*a*) from the center point along the x-axis, the tangent point, and the horizontal segment from the foci point. Thus, the intersection point (W) of the two perpendicular lines and the ray from the focus assist in constructing the right triangle ($\Delta F1AW$). A circle with radius (AF1) is set as the measuring circle, and its circumference value determines the MLC perimeter (Fig. 8). Then, the horizontal distance from the intersection point (W) and the intersection point on MLC, point (M), provides the required value, which is added to the circumference value of the measuring circle. Therefore, the MLC's perimeter can be approximately computed by the following:

$$MF1 \ = \tan u \sqrt{2\boldsymbol{ab} - \boldsymbol{a}^2}, \tag{8}$$

From the right triangle ($\Delta F1AW$), segment line (WM) $= (\boldsymbol{a} - MF1)$, then:

$$MLC's \text{ perimeter} = 2\pi\sqrt{2\boldsymbol{ab} - \boldsymbol{a}^2} + 2(\boldsymbol{a} - \tan u \sqrt{2\boldsymbol{ab} - \boldsymbol{a}^2}), \tag{9}$$

$$MLC's \text{ perimeter} = 2\left[\, \boldsymbol{a} + (\pi - \tan u)\sqrt{2\boldsymbol{ab} - \boldsymbol{a}^2}\,\right], \tag{10}$$

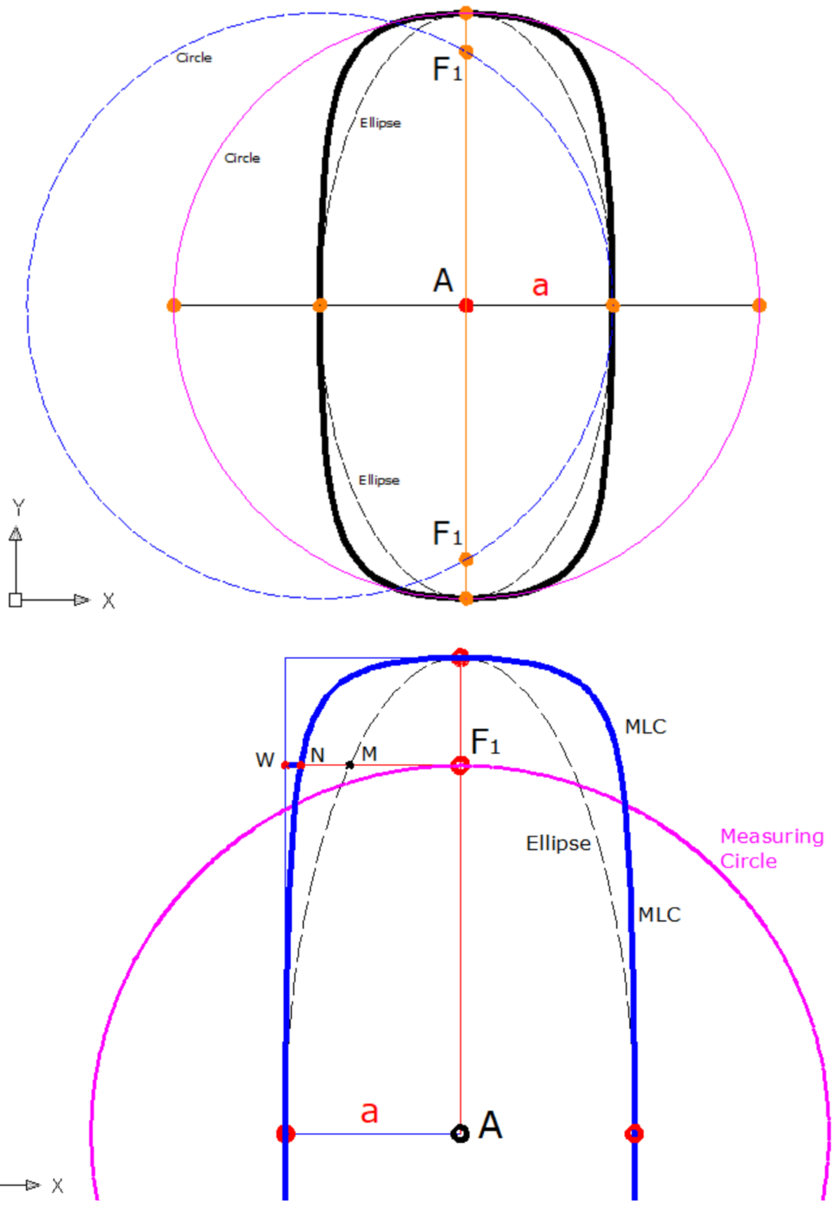

FIGURE 8. Plot of the method in which the perimeter of MLC in the first case is determined according to the elliptical variables of (*a*), (*b*), and (*u*) added to the distance from points W and N.

According to the definition of ellipse, the construction method illustrated in Figures 8 can be converted to determine the MLC's focus and the perimeter. The same procedures are applied in this case, in which the length of the segment (WM) is added to the circumference of the measuring circle whose radius is drawn from the origin and the focus (Fig. 9).

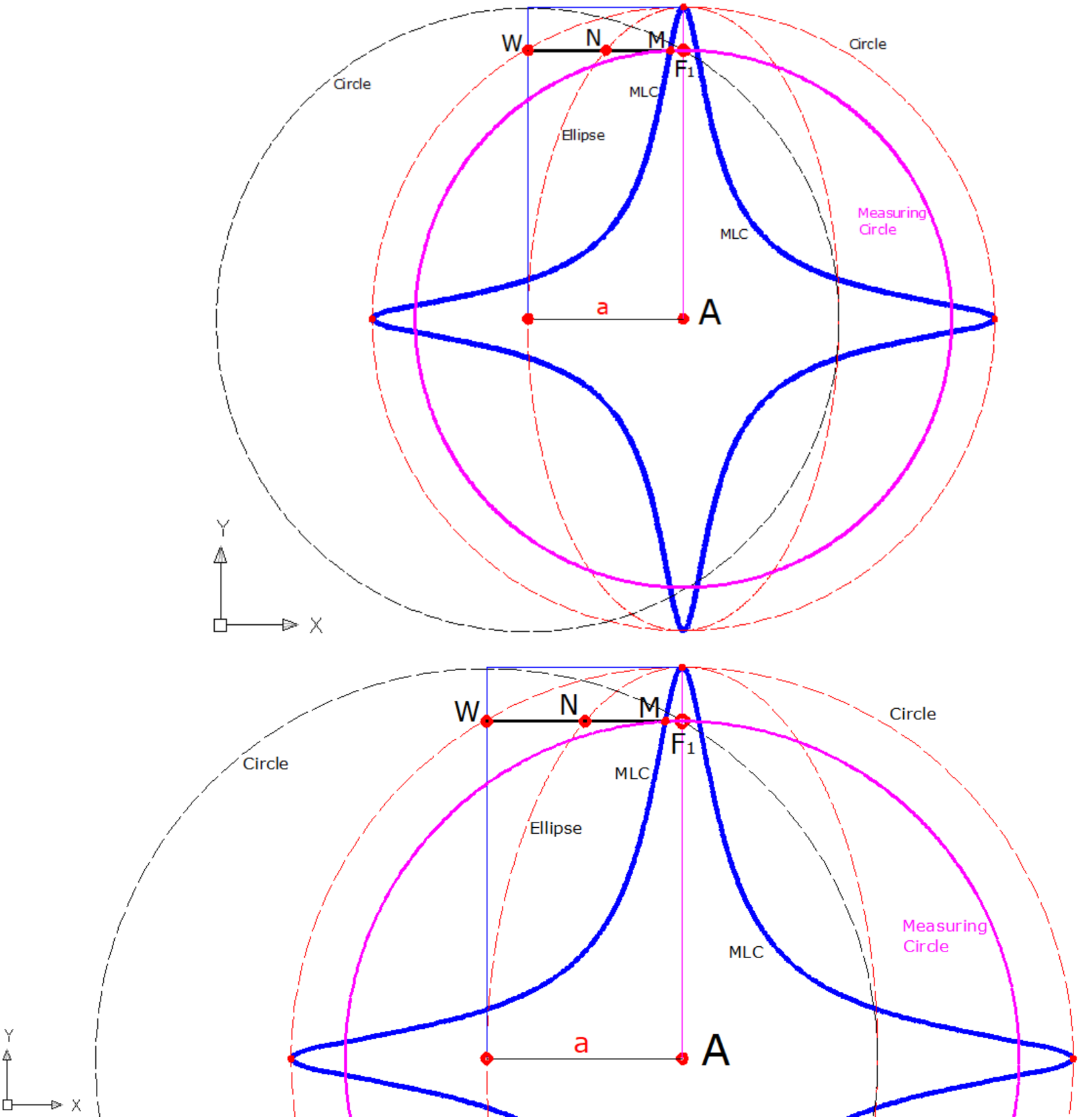

FIGURE 9. Plot of the method in which the perimeter of MLC, in the first case is determined according to the elliptical variable of (*a*), (*b*), and (*u*).

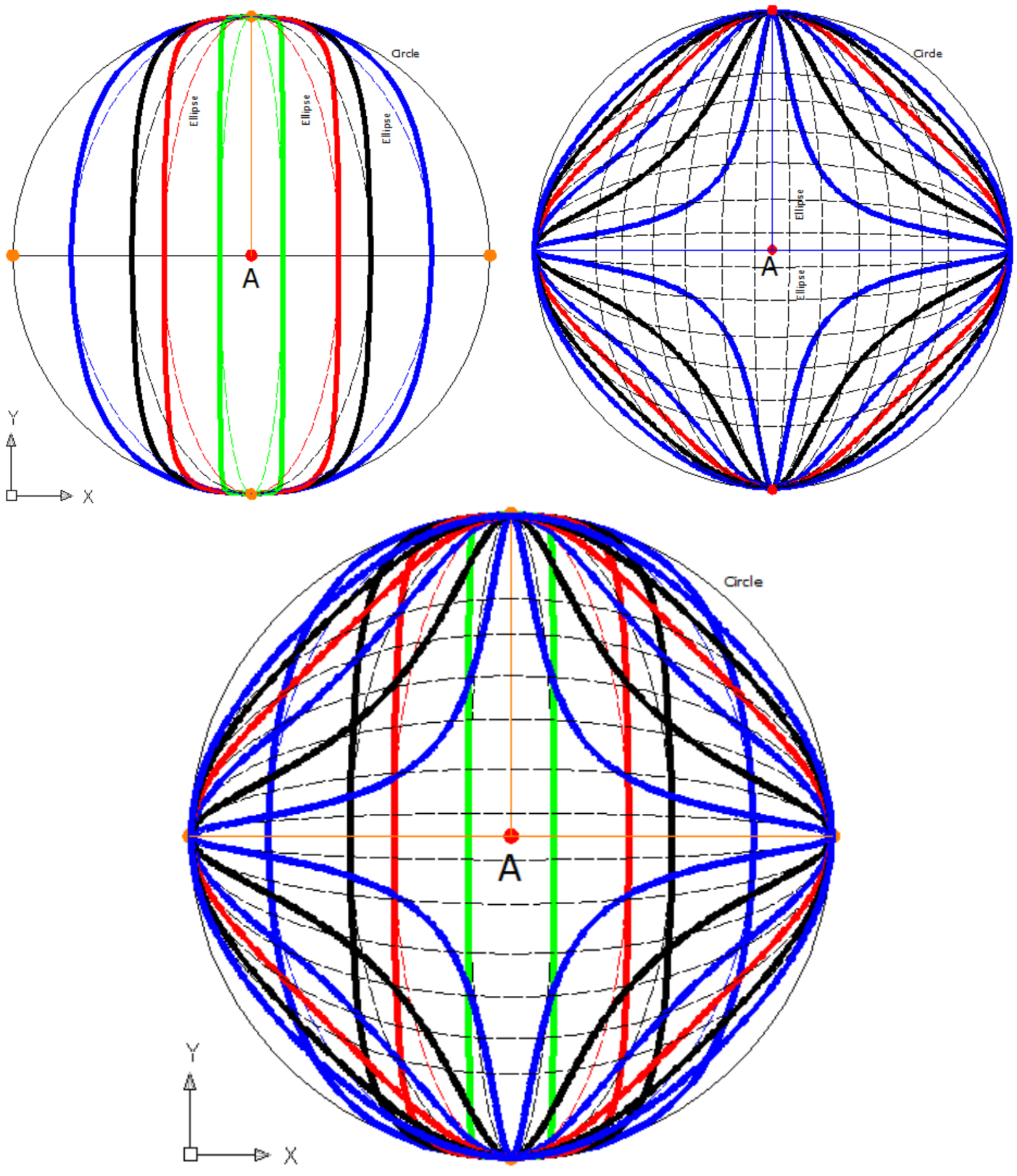

FIGURE 10. **Compilation of the MLCs in two cases**

Figure 9 shows the compilation of the MLCs in two cases on the drawing circle with different models of (*a*) and (*b*) coordinates of the ellipses and a full rotation angle (*u*) of the ray (AP) from the origin. The geometry expressions provide the implicit form of this interesting MLCs. With the slope defined and evaluated for the parameter values (*a*), (*b*), and (*u*), the MLCs can be easily found on the spherical surface. Future research may shed light on these 3D aspects (Index A).

CONCLUSIONS

The main objective of this study is to investigate a new form of curvatures produced in terms of two conic curvatures. Explicit formulas for the segment series at an ellipse band, whose focus point is the center of the unit circle, are introduced. Two cases of these curvatures in the plane in the geometry of oriented circles and lines are also investigated. Then, algebraic applications to commutative reciprocity laws are provided.

Two geometric methods were presented to determine points and coordinates for drawing two cases of curves based on the circle and ellipse curves. Also, resulting curves is classified into two types; the first being those curves that resulted from the intersection of a ray from a fixed point that extends to intersect with the circumference of an ellipse and passing through the circumference of the unit circle. The length of the major axis of the ellipse and the angle of inclination of the ray drawn from the fixed point, are variables that determine the geometrical properties of each case of the resulting curves.

In this paper, a unit circle with a radius equals to the length value of the great axis of an ellipse was studied. Future researches can help to focus on studying these forms of curves at a three-dimensional level by shifting it to the projection state of points located on the 3-dimensional surface of a sphere. The metabolite results of this spherical case would be an extension of the scope of this paper.

Acknowledgements:

I am grateful to my family for their support and patience. This work would not have been written without them. I am also thankful to professor Nesreen Kurdi from the College of Engineering, University of Thi-Qar, for his valuable comments, remarks, and suggestions that have improved the quality of my paper.

Author's Declaration: - Conflicts of Interest: **None**.

I hereby confirm that all the Figures, Tables, and measurements in the manuscript are mine. All obtained figures and geometrical aspects in this paper have been drawn and measured by AutoCAD program, (Version 2007) to increase the accuracy of results.

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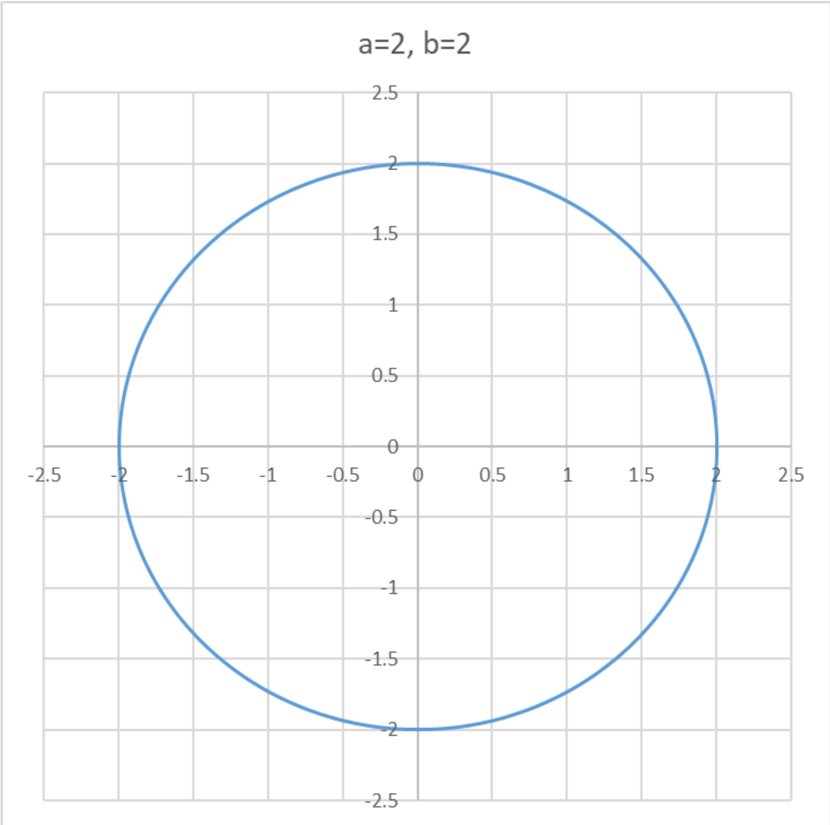

Figure 1. Plot of a sample with a value of (*a*=*b*=2).

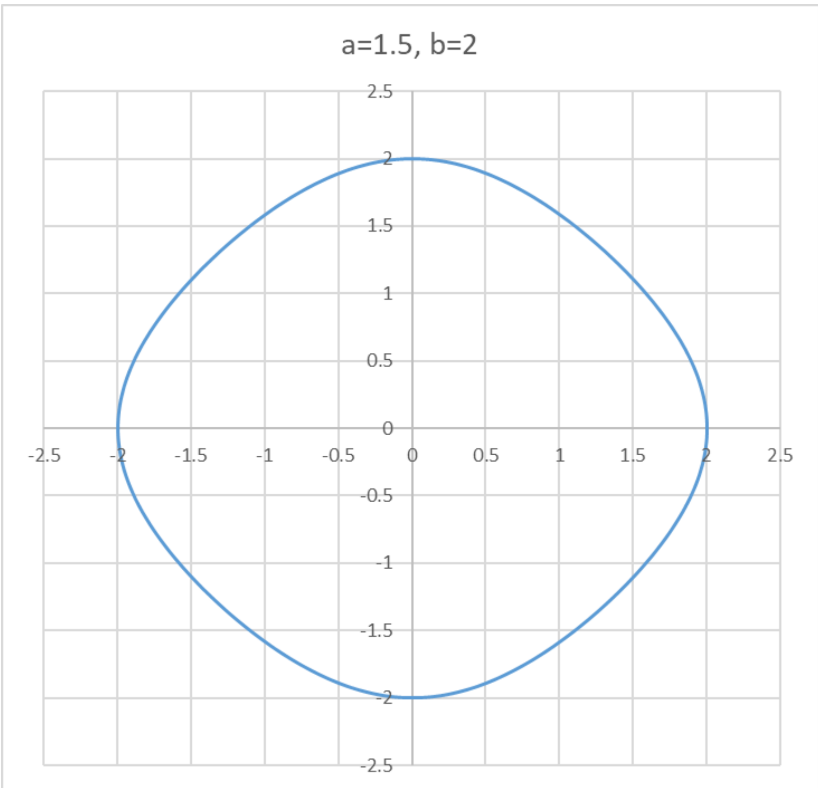

Figure 2. Plot of a sample with a value of (*a*=1.5, *b*=2).

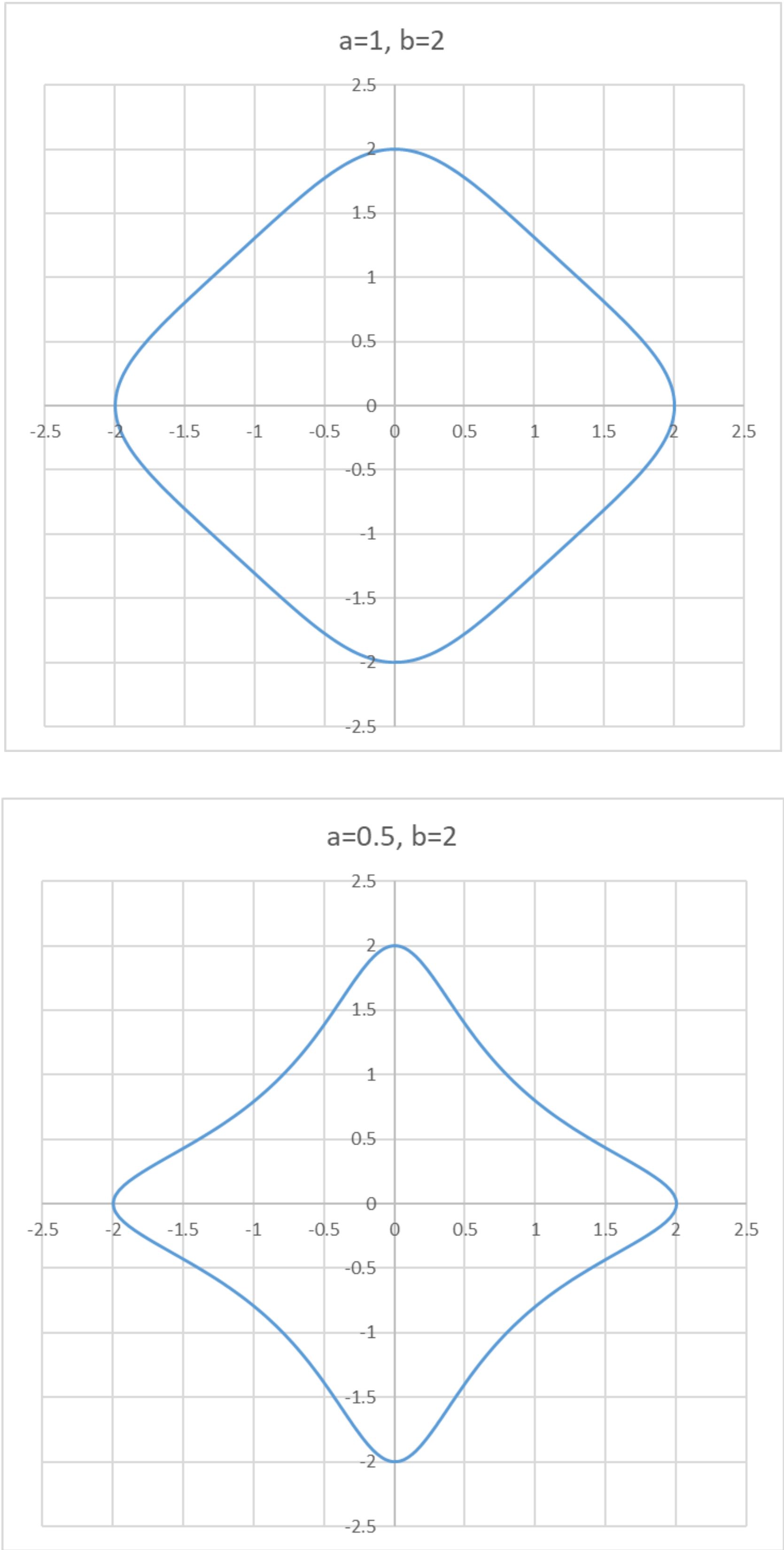

Figure 3. Plot of a sample with a value of (*a*=1 and 0.5, and *b*=2).

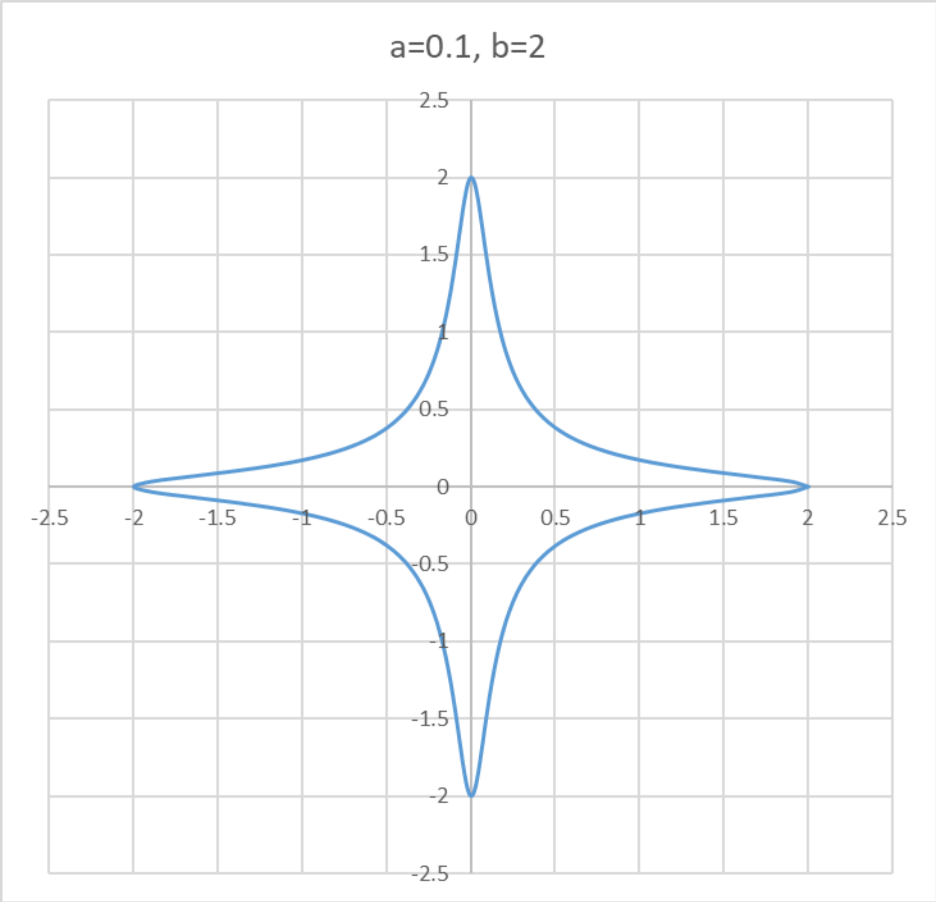

Figure 4. Plot of a sample with a value of (a= 0.1, and b=2).