



Nano $\beta\theta$ – Open Sets in Nano Topological Spaces

Rzgar U. Ahmad, Sami A. Hussein

Department of mathematics, College of Basic Education, Salahaddin University-Erbil, Iraq

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Corresponding Author:

Name: Rzgar U. Ahmad

E-mail:

Tel:

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ABSTRACT

This paper's primary objective is to present and study a different class of nano sets known as nano $\beta\theta$ -open sets in nano topological spaces. Additionally, introduced some nano topological concepts by using nano $\beta\theta$ -open set.

المجموعات المفتوحة النانوية من النمط بيتا ثيتا في الفضاءات التوبولوجية النانوية

رزگار عمر احمد، سامي علي حسين

قسم الرياضيات، كلية التربية الاساسية، جامعة صلاح الدين-اربيل، العراق

الملخص

الهدف الأساسي من هذه الدراسة هو تقديم ودراسة فئة مختلفة من مجموعات نانو المعروفة باسم مجموعات النانو المفتوحة من النمط $\beta\theta$ في الفضاءات النانو التوبولوجية. بالإضافة إلى ذلك، تم تقديم بعض المفاهيم التوبولوجية النانوية باستخدام مجموعة النانو المفتوحة من النمط $\beta\theta$.

Introduction

The concepts of β -open sets or semipreopen sets, which were investigated by Abd El Monsef et al. and Andrijevi [1], respectively, are among the most well-known notions and sources of inspiration. Besides. In 2003, Noiri [2] introduced the concept of β -closure of a set to introduce the notions of $\beta\theta$ -open and $\beta\theta$ -closed sets which provide a formulation of the " $\beta\theta$ -closure" of a set in a topological space, using the notions " $\beta\theta$ -open and $\beta\theta$ -closed sets". Using $\beta\theta$ -closed sets, Caldas, Jafari, and Ekici expanded on Noiri's work and established additional concepts. On the other hand, the study of nano topological spaces introduced by Lellis Thivagar and Richard [3], which defined

approximations and the boundary region of a subset of the finite universe using an equivalence classes as well as defined nano closed sets, nano interior, and nano closure operator. ($N\alpha$ -open, nano semi open, nano preopen, and nano regular open)sets are the weak variants of nano open sets that he has introduced. Nano open sets refer to the constituent parts of nano topological space. Its root is the Greek word "Nanos," which in modern science refers to a dwarf that is an order of magnitude measuring one billionth. Due to its size and the fact that it has no more than five elements, the topology is known as a nano topology. Additionally, in 2015, Revathy and Ilango [4]

expanded on the earlier research by proposing the idea of $N\beta$ -open sets. In this work, the researchers introduce and explore a new notion known as $N\beta\theta$ -open sets, which are both stronger than $N\beta$ -open sets as well as weaker than nano regular β -open sets. further types of $N\beta\theta$ -open set are investigated in various cases of approximations. In this work, we symbolized nano topological space by $(U, \tau_R(X))$ or U which, unless explicitly stated, does not consider any separation axioms.

We focus on gathering definitions and theorems that are utilized in the sequel in order to make this work as independently as possible. Nearly all of these sections are very well.

Definition 1.1. [4] The *nano β – open* set is a subset S of U where $S \subset NCl[NInt(NClS)]$.

Definition 1.2. [3] The *nano regular – open* (resp., *nano pre – open, nano semi – open, nano α – open*) set is a subset S of U where $S = NInt(NClS)$ (resp., $S \subset NInt(NClS)$, $S \subset NCl(NIntS)$, $S \subset NInt[NCl(NIntS)]$).

Definition 1.3. [5] The *nano regular semi – open* set is a subset S of U where $S = NsInt(NsClS)$.

Definition 1.4. [6] The *nano θ – open* (resp., *nano semi θ – open, nano pre θ – open*) set is a subset S of U , $\exists$ *nano open* (resp., *nano semi open, nano preopen*) set G where $x \in G \subset NClG \subset S$ (resp., $x \in G \subset NSClG \subset S$, $x \in G \subset NPClG \subset S$).

Definition 1.5. [7] The *nano δ – open* (resp., *nano pre δ – open*) set is a subset S of U , $\exists$ *nano open* (resp., *nano preopen*) set G where $x \in G \subset NInt(NClG) \subset S$ (resp., $x \in G \subset NPInt(NPClG) \subset S$).

The family of all *nano β – open* (resp., *nano regular – open, nano pre – open, nano semi – open, nano regular semi – open, nano α – open, nano θ – open, nano δ – open, nano semi θ – open, nano pre θ – open, nano pre δ – open*) sets of a nano space $(U, \tau_R(X))$ symbolized by $N\beta O(U, X)$ (resp., $NRO(U, X)$, $NPO(U, X)$, $NSO(U, X)$, $NRSO(U, X)$, $N\alpha O(U, X)$, $N\theta O(U, X)$, $N\delta O(U, X)$, $NS\theta O(U, X)$, $NP\theta O(U, X)$, $NP\delta O(U, X)$).

Definition 1.6. [8] The *nano regular β – open* (resp., *nano pre – regular p – open, nano regular pre – open*) set is a subset S of U , where $S = N\beta Int(N\beta ClS)$ (resp., $S = NPInt(NpClS)$, S is nano pre-open as well as nano pre-closed). And the family of all *nano regular β – open* (resp., *nano pre – regular p – open, nano regular pre – open*)

sets of $(U, \tau_R(X))$ symbolized by $NR\beta O(U, X)$ (resp. $NPRPO(U, X)$, $NRPO(U, X)$).

Remark 1.7

$NInt[NCl(NIntA)] \subset N\beta Int(N\beta clA) \subset Ncl[NInt(NclA)]$

Theorem 1.8. Assume that $S \subset U$, then $S \in NR\beta O(U, X)$ if and only if $S \in N\beta O(U, X) \cap N\beta C(U, X)$.

Proof: Let $S \in NR\beta O(U, X)$, then $S = N\beta Int(N\beta ClS)$. By Remark 1.7, $NInt[NCl(NIntS)] \subset N\beta Int(N\beta ClS) \subset NCl[NInt(NClS)]$. Hence, $S \in N\beta O(U, X) \cap N\beta C(U, X)$. Conversely, if $S \in N\beta O(U, X) \cap N\beta C(U, X)$ then $S = N\beta Int(N\beta ClS)$. Therefore, S is $NR\beta O(U, X)$.

Theorem 1.9. [9] Assume that $S \subset X$, then $S \in \beta O(X)$ if and only if $\beta ClS \in R\beta O(X)$.

Theorem 1.10. [10] For each $S \subset X$, $\beta ClS \subset PClS$ and $\beta ClS \subset SCIS$.

Theorem 1.11. [11] A subset $S \subset X$ is δ – open (resp. $P\delta$ -open) if and only if S is union of regular open (resp. pre-regular p-open) sets in X .

Theorem 1.12. [12] For any space X , $PRPO(X) \subset R\beta O(X)$.

Theorem 1.13. [4] If $U_r(X) = U$ in $(U, \tau_R(X))$ then $N\beta O(U, X) = P(U)$.

Theorem 1.14. [4] If $U_r(X) \neq U$ then any set that intersects with $U_r(X)$ are $N\beta O(U, X)$ in U .

Theorem 1.15. [13] $S \cap B \in R\beta O(X)$, when $S \in RO(X)$ and $B \in R\beta O(X)$.

Theorem 1.16. [9] A space X is extremely disconnected if and only if $PO(X) = \beta O(X)$.

Theorem 1.17. [9] For any space X , $\theta O(X, PO(X)) = P\theta O(X)$.

Theorem 1.18. [9] If $S_i \in R\beta O(X_i)$ for $i = 1, 2$ then $S_1 \times S_2 \in R\beta O(X_1 \times X_2)$.

Theorem 1.19: For any nano topological space, $S \in N\delta O(U, X)$ if and only if $\forall x \in S$, $\exists M \in NRO(U, X)$ where $x \in M \subset S$.

Proof: Let $S \in N\delta O(U, X)$, by Definition of $N\delta$ – open set there exists a Nano open set V where $x \in V \subset NInt(NClV) \subset S$, taking $M = NInt(NClV)$, by Definition of *nano regular*, M is *nano regular* and $x \in M \subset S$. Conversely, for each $x \in S$, there exists $M \in NRO(U, X)$ where $x \in M \subset S$. Therefore, $\forall x \in S, \exists M \in NRO(U, X)$ where

$x \in M \subset NInt(NCIM) \subset S$. Hence, $S \in N\delta O(U, X)$.

Theorem 1.20. A subset S of $(U, \tau_R(X))$ is $N\delta$ – open (resp. $NP\delta$ – open) if and only if S is union of nano regular open (resp. pre-regular p – open) sets in (U, X) .

Proof. Assume that $S \in N\delta O(U, X)$ (resp. $S \in NP\delta O(U, X)$). For each $x \in S$, by Theorem 1.19, $\exists M_x \in \tau_R(X)$ (resp. $M_x \in NPRPO(U, X)$) where $x \in M_x \subset S$. Therefore, $\bigcup_{x \in S} \{x\} \subset \bigcup_{x \in S} M_x \subset S$. Hence, $S = \bigcup M$.

Conversely, suppose that S is union of nano open (resp. nano pre – regular P – open) sets $\{M_\lambda, \lambda \in \Delta\}$. Let $x \in S$ then, $x \in M_{\lambda_0}$ for some $\lambda_0 \in \Delta$ and M_{λ_0} is nano open (resp. nano pre – regular P – open, Therefore $x \in M_{\lambda_0} \subset S$. By Theorem 1.19, $S \in N\delta O(U, X)$ (resp. $S \in NP\delta O(U, X)$).

Theorem 1.21. [9] In any space X , $RO(X, \beta O(X)) = R\beta O(X)$.

Theorem 1.22. [9] Let $S \in \alpha O(X)$ and $M \in R\beta O(X)$, then $S \cap M \in R\beta O(S)$.

Theorem 1.23: If $(U, \tau_R(X))$ is extremely disconnected then $NP\delta O(U, X) = NP\theta O(U, X)$.

Proof: Let $S \in NP\delta O(U, X)$, then by Theorem 1.20, S is a union of $NPRP$ – open. Since, $\forall x \in S, \exists V \in NPO(U, X)$ where $x \in V \subset NPInt(NPCIV) \subset S$. Since $(U, \tau_R(X))$ is extremely disconnected then $NPCIV \in NPO(U, X), \forall V \in NPO(U, X)$. Therefore, $NPInt(NPCIV) = NPCIV$. Thus, $x \in V \subset NPCIV \subset S$. Hence, $S \in NP\theta O(U, X)$.

Theorem 1.24. [4] If $U_r(X) \neq U$ and $L_r(X) \neq \emptyset$, then any set which intersects $U_r(X)$ are nano β – open sets in U .

1. Basic Backgrounds

Definition 2.1. [5] Assume that $U \neq \emptyset$ of objects known as the universe and that $\mathcal{R}$ represents an equivalence relation of U called as the indiscernible relation. All the elements in the same equivalence class are assumed to be indiscernibility with each another. The approximation space is stated to consist of the pair $(U, \mathcal{R})$. Let $X \subset U$.

1. The lower approximation of X is symbolized by $L_r(X)$. That is, $L_r(X) = \bigcup_{x \in U} \{\mathcal{R}(x) \cap X\}$.
2. The upper approximation of X is symbolized by $U_r(X)$. That is $U_r(X) = \bigcup_{x \in U} \{\mathcal{R}(x) \cap X \neq \emptyset\}$

3. The boundary region of X is symbolized by $B_r(X)$. That is $B_r(X) = U_r(X) \setminus L_r(X)$.

Definition 2.2. [5] Assume that U represents the universe, $\mathcal{R}$ represents an equivalence relation on U and $\tau_{\mathcal{R}}(X) = \{U_r(X), L_r(X), B_r(X), U, \emptyset\}$ where $X \subset U$. Then $\tau_{\mathcal{R}}(X)$ satisfies these conditions.

1. $\emptyset$ and U belong to $\tau_{\mathcal{R}}(X)$,
2. $\bigcup A_\lambda$ is in $\tau_{\mathcal{R}}(X)$ where A_λ is any sub-collection of $\tau_{\mathcal{R}}(X)$,
3. $\bigcap A_\lambda$ is in $\tau_{\mathcal{R}}(X)$ where A_λ is any finite sub-collection of $\tau_{\mathcal{R}}(X)$.

It implies that $\tau_{\mathcal{R}}(X)$ satisfies the topology's axioms on U and is referred to as nano topology on U with respect to X . Additionally, $(U, \tau_{\mathcal{R}}(X))$ is known as the nano topological space.

in this work, $NInt(S)$ represents the nano interior of S . And $NCI(S)$ denotes the nano closure of S .

3. Nano $\beta\theta$ -Open Sets

In this section focuses on gathering definition and prove some results about Nano $\beta\theta$ -open sets.

Definition 3.1. The Nano $\beta\theta$ – open set is a subset S of U , where $\forall x \in S, \exists G \in N\beta$ – open set where $x \in G \subset N\beta CIG \subset S$ and symbolized by $N\beta\theta O(U, X)$.

Theorem 3.2. A subset S of $(U, \tau_{\mathcal{R}}(X))$ is $N\beta\theta$ – open set if and only if $\forall x \in S, \exists B \in NR\beta O(U, X)$ where $x \in B \subset S$.

Proof. Assume that $S \in N\beta\theta O(U, X)$ and $x \in S$, by Definition of $N\beta\theta$ – open set, there exists a $N\beta$ – open set G where $x \in G \subset N\beta CIG \subset S$. Assume $B = N\beta CIG$, by Theorem 1.9, $x \in G \subset N\beta CIG \subset NR\beta O(U, X) \subset S$. Hence, B is $NR\beta$ – open set and $x \in B \subset S$.

Conversely, assume $x \in S$ there exists a nano regular β – open set G where $x \in G \subset S$, by Theorem 1.8, $G \in N\beta O(U, X)$ and $G \in N\beta C(U, X)$. Since G is nano β -closed set. Therefore $N\beta CIG = G$. Hence, $x \in G \subset N\beta CIG \subset S$. Therefore, S is $N\beta\theta$ – open set.

Theorem 3.3. A subset S of $(U, \tau_{\mathcal{R}}(X))$ is $N\beta\theta$ – open set if and only if S is union of nano regular β – open sets in U .

Proof. Let $S \in N\beta\theta O(U, X)$, for each $x \in S$, by Theorem 3.2, there exists a nano regular β – open sets G_x , where $x \in G_x \subset S$. Therefore, $S = \bigcup_{x \in S} \{x\} \subset \bigcup_{x \in S} G_x \subset S$. Hence, $S = \bigcup_{x \in S} G_x$. Conversely, let $S = \bigcup_{\lambda \in \Delta} G_\lambda$, G_λ is a nano regular β – open sets in U , for each $\lambda \in \Delta$. let $x \in S$ there exists $\lambda_0 \in \Delta$, where $x \in G_{\lambda_0}$ and

G_{λ_0} is $NR\beta O(U, X)$. Consequently, $x \in G_{\lambda_0} \subset S$. by Theorem 3.2, $S \in N\beta\theta O(U, X)$.

Theorem 3.4. Assume that S represents a subset of $(U, \tau_{\mathcal{R}}(X))$, $S \in N\beta O(U, X)$ if and only if $N\beta C I S \in N\beta\theta O(U, X)$.

Proof. Consider $S \in N\beta O(U, X)$, then there is $S \subset N C I [N I n t (N C I S)]$. Hence, $N\beta C I S \subset N\beta C I [N C I [N I n t (N C I S)]] = N C I [N I n t (N C I S)] \subset N C I [N I n t [N C I (N\beta C I S)]]$. Therefore, $N\beta C I S$ is $N\beta$ – open set and also $N\beta$ – closed set. Hence, $N\beta C I S \in N\beta R(U, X)$. Since, $N\beta R(U, X) \subset N\beta\theta O(U, X)$. Thus, $N\beta C I S \in N\beta\theta O(U, X)$. Conversely, if $N\beta C I S \in N\beta\theta O(U, X)$ then $S \in N\beta\theta O(U, X)$ since $S \subset N\beta C I S$. In addition, $N\beta\theta O(U, X) \subset N\beta O(U, X)$. Therefore, $S \in N\beta O(U, X)$.

Theorem 3.5. Let S be a subset of $(U, \tau_{\mathcal{R}}(X))$, then $S \in NR\beta(U, X)$ if and only if $S \in N\beta\theta O(U, X) \cap N\beta\theta C(U, X)$.

Proof: Let $S \in NR\beta(U, X)$, then $S \in N\beta(U, X)$. Therefore, $S = N\beta C I(S) = N\beta C I_{\theta}(S)$. Hence, S is $N\beta\theta$ – closed. Since, $U \setminus S \in NR\beta(U, X)$, by the argument above, $U \setminus S$ is $N\beta\theta$ – closed and hence, S is $N\beta\theta$ – open.

Therefore, $A \in N\beta\theta O(U, X) \cap N\beta\theta C(U, X)$. Conversely, if $S \in N\beta\theta O(U, X) \cap N\beta\theta C(U, X)$ then S is both $N\beta\theta$ – open and $N\beta\theta$ – closed. Therefore, by Theorem 1.8, $S \in NR\beta(U, X)$.

4. Nano $\beta\theta$ – open Sets of $U_r(X)$, $L_r(X)$, and $B_r(X)$ Approximations

In this part, we defined some properties of $N\beta\theta O(U, X)$ by deferent cases of lower, upper approximations and boundary region.

Proposition 4.1. In any $(U, \tau_{\mathcal{R}}(X))$ if $U_r(X) = U$ then $N\beta\theta O(U, X) = N\beta O(U, X) = P(U)$.

Proof. Assume that $U_r(X) = U$ then by Theorem 1.13, $N\beta O(U, X) = P(U)$. Let S be any subset of $(U, \tau_{\mathcal{R}}(X))$. Then, $\forall x \in S$, $\exists G \in N\beta O(U, X)$ where, $x \in G \subset N\beta C I G = G \subset S$. Thus, $N\beta\theta O(U, X) = P(U)$.

Proposition 4.2. In any $(U, \tau_{\mathcal{R}}(X))$, if $S \subset [U_r(X)]^c$ then S is not $N\beta\theta$ – open set.

Proof. Suppose that $S \subset [U_r(X)]^c$, therefore $S \not\subset U_r(X)$. Hence by Theorem 1.14, S is not nano β – open set in U . Which implies that, S is not nano $\beta\theta$ – open set in U .

Proposition 4.3. In any $(U, \tau_{\mathcal{R}}(X))$, if $\tau_{\mathcal{R}}(X) = \{\varphi, U\}$ then $N\beta O(U, X) = N\beta\theta O(U, X) = P(U)$.

Proof. Suppose that $\tau_{\mathcal{R}}(X) = \{\varphi, U\}$ then $\forall S \subset U$, $N\beta C I(S) = U$, implies that $N\beta C I [N\beta I n t (N\beta C I(S))] = U$. It is mean $N\beta O(U, X) = P(U)$. If $N\beta O(U, X) = P(U)$ then each set in $(U, \tau_{\mathcal{R}}(X))$ is clopen set. Therefore, $N\beta C I G = G$. Hence, $\forall x \in S$, $\exists G \in N\beta O(U, X)$ where $x \in G \subset N\beta C I G \subset S$. Thus, $N\beta\theta O(U, X) = N\beta O(U, X) = P(U)$.

Proposition 4.4. In any $(U, \tau_{\mathcal{R}}(X))$, if $L_r(X) = \varphi$ and $U_r(X) = U$ then $N\beta O(U, X) = N\beta\theta O(U, X) = P(U)$.

Proof. Suppose that $L_r(X) = \varphi$ and $U_r(X) = U$, then $B_r(X) = U_r(X) = U$. Therefore, nano topological space is indiscrete nano topology. And, $N\beta O(U, X) = P(U)$. By Proposition 4.3, $N\beta\theta O(U, X) = N\beta O(U, X) = P(U)$.

Proposition 4.5. In any $(U, \tau_{\mathcal{R}}(X))$, if $U_r(X) = U$ and $L_r(X) \neq B_r(X) \neq \varphi$ then $N\beta O(U, X) = N\beta\theta O(U, X)$.

Proof. Suppose that $U_r(X) = U$, then by Theorem 1.13, $N\beta O(U, X) = P(U)$. And By proposition 4.3, $N\beta\theta O(U, X) = N\beta O(U, X) = P(U)$.

Proposition 4.6. In any $(U, \tau_{\mathcal{R}}(X))$, $N\beta O(U, X) = N\beta\theta O(U, X)$ if $L_r(X) = \varphi$ and $U_r(X) = B_r(X) \neq \varphi$.

Proof. Let $L_r(X) = \varphi$ and $U_r(X) = B_r(X) \neq \varphi$. By Theorem 1.14, S is $N\beta O(U, X)$ if and only if $S \cap U_r(X) \neq \varphi$. And by Proposition of $N\beta$ – open sets, if $U_r(X) \neq U$, then any subset of $(U_r(X))^c$ is not $N\beta O(U, X)$. Therefore, every subset of U is $N\beta$ – open sets except subset of $(U_r(X))^c$. Thus, if S is $N\beta O(U, X)$, then it must be shown that S is $N\beta\theta O(U, X)$. To show that S is $N\beta\theta O(U, X)$, there must be as $\forall x \in S$, $\exists G \in N\beta O(U, X)$ where, $x \in G \subset N\beta C I G \subset S$. Now, if S is also $N\beta C(U, X)$, then $G = S$. Therefore, $\forall x \in S$, $\exists G = S \in N\beta O(U, X)$ where, $x \in G \subset N\beta C I G = G \subset S$. But, if S is not $N\beta C(U, X)$, then $\forall x \in S$, $\exists G = U \setminus \{x\}$ or $G = U \setminus \{x\} \cup \{x, a\}$, where $\forall x \in U_r(X)$ and $\forall a \in (U_r(X))^c$. First, if $G = \{x\}$, since $\exists y \in G^c \subset U_r(X)$ then G^c is also $N\beta O(U, X)$. Therefore, $(G^c)^c = G$ is $N\beta C(U, X)$. Thus, G is also $N\beta C(U, X)$. Hence, $\exists G = \{x\} \in N\beta O(U, X)$, $N\beta C I G = \{x\}$. $x \in G \subset N\beta C I G \subset S$. Consequently, S is $N\beta\theta O(U, X)$. Second, If $G = \{x, a\}$. $\{a\} \subset (U_r(X))^c$. Therefore, $U_r(X) \subset U \setminus \{a\} = \{a\}^c$. Thus, there exists at least one element of G^c in $U_r(X)$. Accordingly, G^c is $N\beta O(U, X)$. $(G^c)^c = G$ is $N\beta C(U, X)$. $\forall x \in S$, $\exists G \in N\beta O(U, X)$ where, $x \in G \subset N\beta C I G = G \subset S$. Consequently, S is $N\beta\theta O(U, X)$.

Proposition 4.7. In any $(U, \tau_{\mathcal{R}}(X))$, if $B_r(X) = \varphi$ and $U_r(X) = L_r(X) \neq \varphi$ then $N\beta O(U, X) \neq N\beta\theta O(U, X)$.

Proof. It is clear.

Corollary 4.8. If $L_r(X) = \varphi$ or $B_r(X) = \varphi$ and $U_r(X) \neq U$, then S is $N\beta\theta O(U, X)$ if and only if $S \cap U_r(X) \neq \varphi$.

Proof. Propositions 4.6 and 4.7 provide evidence for the claim.

Proposition 4.9. For any $(U, \tau_{\mathcal{R}}(X))$, if $X = \{x\} \in U/R$ then $N\beta\theta O(U, X)$ are φ and U .

Proof. Suppose that $X = \{x\} \in U/R$ then $L_r(X) = U_r(X) = \{x\}$ and $B_r(X) = \varphi$. Hence, $\tau_{\mathcal{R}}(X) = \{\varphi, U, U_r(X)\}$. Since $(U_r(X))^c$ is not intersect with $U_r(X)$ thus, by Theorem 1.14, $(U_r(X))^c$ is not $N\beta$ – open set. So, any $S \in N\beta O(U, X)$, $N\beta Cl = U \not\subset \{x\}$. Therefore, nano $\beta\theta$ – open set are φ and U .

Proposition 4.10. If $U_r(X) \neq U$ and $L_r(X) \neq \varphi$ then $N\beta\theta O = N\beta O$ and every proper subset of U which intersects $U_r(X)$ is nano $\beta\theta$ – open set in U .

Proof. Directly from Theorem 1.24, proposition 4.5 and Corollary 4.8.

Proposition 4.11. In any $(U, \tau_{\mathcal{R}}(X))$, $N\beta\theta O = P(U)$ if $U_r(X) = L_r(X) = U$.

Proof. Let $U_r(X) = L_r(X) = U$. Then $\tau_{\mathcal{R}}(X) = \{\varphi, U\}$. Implies that by Proposition 4.3, $N\beta O(U, X) = N\beta\theta O(U, X) = P(U)$.

5. Relationship Between $N\beta\theta$ – open sets and other sets

In this section, we explain the relation between $N\beta\theta$ – open sets and some nearly nano open sets.

Theorem 5.1. Every $N\theta$ – Open set is $N\beta\theta$ – open.

Proof. Assume that $G \in N\theta O(U, X)$. $\forall x \in G$, $\exists H \in \tau_{\mathcal{R}}(X)$, where $x \in H \subset NClH \subset G$. Since nano open set is also nano β – Open set. $H \in N\beta O(U, X)$ and $N\beta ClH \subset NClH$. $x \in H \subset N\beta ClH \subset G$. Hence, G is $N\beta\theta O(U, X)$.

Theorem 5.2. Every $N\beta\theta$ – open set is $N\beta$ – open set.

Proof. Assume that $H \in N\beta\theta O(U, X)$, $\forall x \in H$. $\exists G \in N\beta O(U, X)$ where $x \in G \subset N\beta ClG \subset H$. $U\{x, x \in H\} \subset UG_x \subset H$. So, $H = UG_x$, where $UG_x \in N\beta O(U, X)$ And union of $N\beta$ – open sets is also $N\beta$ – open set. Hence, $H \in N\beta O(U, X)$.

Theorem 5.3. For any $(U, \tau_{\mathcal{R}}(X))$, if $X = \{x\} \in U/R$ then X is nano β – open set but not nano $\beta\theta$ – openset.

Proof. Suppose that $X = \{x\}$ then $L_r(X) = U_r(X) = \{x\}$ and $B_r(X) = \varphi$. Therefore, $\tau_{\mathcal{R}}(X) = \{\varphi, U, U_r(X)\}$. By Theorem 1.14, any subset S of U is $N\beta$ – open set if $S \cap U_r(X) \neq \varphi$. Therefore, $U_r(X) = \{x\}$ is $N\beta$ – open set. On the other hand, if $S \cap U_r(X) \neq \varphi$ then $S^c \cap U_r(X) = \varphi$. $U_r(X) \not\subset S^c$ implies that $U_r(X) \subset S$. Henceforth, $N\beta ClU_r(X) = N\beta Cl\{x\} = U \not\subset \{x\}$. Therefore, $x \in U_r(X) \subset N\beta ClU_r(X) \not\subset \{x\}$. Consequently, $\{x\}$ is not $N\beta\theta$ – open set.

Theorem 5.4. Any union of $N\beta\theta$ – open sets in a nono topological space is $N\beta\theta$ – open set.

Proof. Let $\{G_{\lambda}\} \in N\beta\theta O(U, X)$, there has to be shown that $G = \bigcup G_{\lambda}$ is also $N\beta\theta$ – open of U . Let $x \in \bigcup G_{\lambda}$ then $x \in G_{\lambda_0}$, for some $\lambda_0 \in \Delta$. And $G_{\lambda_0} \in N\beta\theta O(U, X)$, $\exists H \in N\beta(U, X)$, where $x \in H \subset N\beta ClH \subset G_{\lambda_0} \subset \bigcup G_{\lambda}$. Consequently, $\bigcup G_{\lambda} \in N\beta\theta O(U, X)$.

Remark 5.1. If S and B are two $N\beta\theta$ – open sets, then $S \cap B$ may not be $N\beta\theta$ – open set in general as shown in the example below.

Example 5.1. Assume that $U = \{p_1, p_2, p_3, p_4\}$ with $U/X = \{\{p_1\}, \{p_4\}, \{p_2, p_3\}\}$, and assume that $X = \{p_1, p_2\}$ then $\tau_{\mathcal{R}}(X) = \{\varphi, U, \{p_1\}, \{p_1, p_2, p_3\}, \{p_2, p_3\}\}$, and $N\beta O(U, X) = P(U) \setminus \{p_4\}$. Consider $S = \{p_1, p_4\}$, and $B = \{p_2, p_4\}$. S is $N\beta\theta O(U, X)$ since there exists $G = \{p_1, p_4\} \in N\beta O(U, X)$ where $x \in G \subset N\beta ClG \subset S$, and B is $N\beta\theta O(U, X)$ because there exists $H = \{p_2, p_4\} \in N\beta O(U, X)$ where $x \in H \subset N\beta ClH \subset B$. But $S \cap B = \{p_4\}$. Since, $\nexists G \in N\beta O(U, X)$, where $x \in G \subset N\beta ClG \subset \{p_4\}$. therefore $\{p_4\}$ is not $N\beta\theta O(U, X)$.

Theorem 5.5. For any $(U, \tau_{\mathcal{R}}(X))$, $NS\theta O(U, X) \subset N\beta\theta O(U, X)$.

Proof. Let $S \in NS\theta O(U, X)$ and $x \in S$, $\exists G \in NSO(U, X)$ where, $x \in G \subset NSClG \subset S$. By Theorem 1.10, $N\beta ClG \subset NSClG$. Therefore, $x \in G \subset N\beta ClG \subset S$. Since, $NSO(U, X) \subset N\beta O(U, X)$. Consequently $S \in N\beta\theta O(U, X)$.

Theorem 5.6. For any $(U, \tau_{\mathcal{R}}(X))$, $NP\theta O(U, X) \subset N\beta\theta O(U, X)$.

Proof. Follows directly from Theorem 1.10 and their definitions.

Theorem 5.7. For any $(U, \tau_{\mathcal{R}}(X))$, $NP\delta O(U, X) \subset N\beta\theta O(U, X)$.

Proof. Assume that $S \in NP\delta O(U, X)$, by Theorem 1.11, S is union of nano pre-regular p-open sets in (U, X) . Besides by Theorem 1.12, $NPRPO(U, X) \subset NR\beta O(U, X)$ and by Theorem 3.3, $S \in N\beta\theta O(U, X)$.

The diagram below represents the relationship in nano topological space between the class of $N\beta\theta$ – open sets and several associated classes of nano open sets.

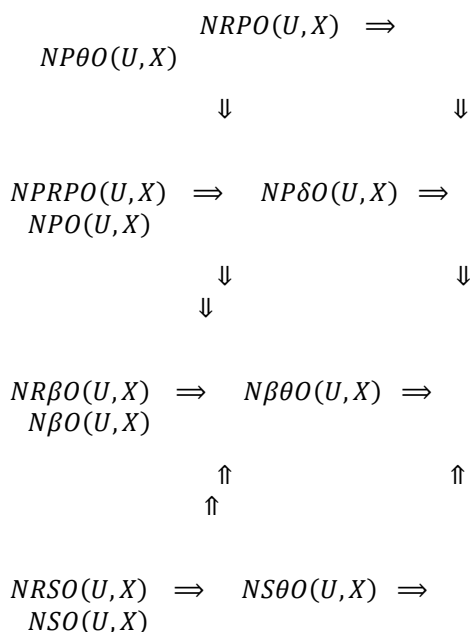


Diagram 1

It should be noted that the inverse of implications untrue in general as shown in the following examples

Example 5.2. In Example 5.1, let $S = \{p_1, p_3\}$ and $G = \{p_1, p_3\}$, then $N\beta CIG = \{p_1, p_3\}$. Accordingly, $x \in S, \exists G$ is $N\beta O(U, X)$, where $x \in G \subset N\beta CIG \subset S$. Hence, S is $N\beta\theta O(U, X)$. But if $x = p_1 \in S$, consider $H = \{p_1, p_3\}$ then $N\beta CIG = U$. Hence, $p_1 \in \{p_1, p_3\} \subset N\beta CIG \not\subset S$. Therefore, S is not $N\beta O(U, X)$ when S is $N\beta\theta O(U, X)$.

Example 5.3. Assume that $U = \{p_1, p_2, p_3\}$ with $U/R = \{\{p_1\}, \{p_2, p_3\}\}$, and consider that $X = \{p_1\}$ then $\tau_{\mathcal{R}}(X) = \{\varphi, U, \{p_1\}\}$ and $N\beta O(U, X) = \{\{p_1\}, \{p_1, p_2\}, \{p_1, p_3\}, \varphi, U\}$. $N\beta C(U, X) = \{\{p_2, p_3\}, \{p_3\}, \{p_2\}, U, \varphi\}$. Assume that $S = \{p_1\}$ is $N\beta$ – open set. Assume that $G = \{p_1\}$ then $N\beta CIG = U$. $p_1 \in G \subset N\beta CIG \not\subset S$. Consequently, S is $N\beta$ – open set but it is not $N\beta\theta$ – open set.

Example 5.4. In Example 5.1,

$NPO(U, X) = \{\varphi, U, \{p_1\}, \{p_2\}, \{p_3\}, \{p_1, p_2\}, \{p_1, p_3\}, \{p_1, p_2, p_3\}, \{p_1, p_2, p_4\}, \{p_1, p_2, p_3, p_4\}\}$.

$NPC(U, X) =$

$\{U, \varphi, \{p_2, p_3, p_4\}, \{p_1, p_3, p_4\}, \{p_1, p_2, p_4\}, \{p_3, p_4\}, \{p_2, p_4\}, \{p_4\}, \{p_3\}\}$. Assume that $S = \{p_2, p_3, p_4\}$. Since, $\forall x \in S, \exists G = \{p_2, p_3, p_4\} \in N\beta O(U, X)$ where $x \in G \subset N\beta CIG \subset S$. Therefore, $S \in N\beta\theta O(U, X)$. But $S \notin NP\delta O(U, X)$, $\nexists G \in NPO(U, X)$ where $x \in G \subset NPInt(NPCIG) \subset S$. Consequently, S is $N\beta\theta O(U, X)$ but not $NP\delta O(U, X)$.

Example 5.5. In Example 5.1,

$NSO(U, X) =$

$\{\varphi, U, \{p_1\}, \{p_1, p_4\}, \{p_2, p_3\}, \{p_1, p_2, p_3\}, \{p_2, p_3, p_4\}\}$. Assume that $S = \{p_2\}$. S is $N\beta\theta O(U, X)$, Since, $\forall x \in S, \exists G = \{p_2\} \in N\beta O(U, X)$ where $x \in G \subset N\beta CIG \subset S$. But $S \notin NS\theta O(U, X)$, since $\nexists G \in NSO(U, X)$ where $x \in G \subset NSCIG \subset S$. Consequently, S is $N\beta\theta O(U, X)$ but not $NS\theta O(U, X)$.

Example 5.6. Assume that $U = \{p_1, p_2, p_3, p_4\}$ with $U/R = \{\{p_1\}, \{p_2\}, \{p_3, p_4\}\}$, and assume that $X = \{p_3\}$, then $\tau_{\mathcal{R}}(X) = \{\varphi, U, \{p_3, p_4\}\}$ and

$N\beta O(U, X) = \{\varphi, U, \{p_3\}, \{p_4\}, \{p_1, p_3\}, \{p_1, p_4\}, \{p_2, p_3\}, \{p_2, p_4\}, \{p_3, p_4\}, \{p_1, p_2, p_3, p_4\}\}$

Now, consider $S = \{p_2, p_3\}$. To show that S is $N\beta\theta O(U, X)$, it needs evidence to prove that $\forall x \in S, \exists G \in N\beta O(U, X)$ where $x \in G \subset N\beta CIG \subset S$. $p_3 \in S, \exists G = \{p_3\} \in N\beta O(U, X)$. and $N\beta CIG = \{p_3\} \subset S$. Therefore, $p_3 \in G \subset N\beta CIG \subset S$.

$p_4 \in S, \exists G = \{p_4\} \in N\beta O(U, X)$ and $N\beta CIG = \{p_4\} \subset S$. Therefore, $p_4 \in G \subset N\beta CIG \subset S$. Hence, S is $N\beta\theta O(U, X)$. But, to show S is $NR\beta O(U, X)$, it must be proven that $S = N\beta Int(N\beta CIG)$. $N\beta CIG = U$ which implies that $N\beta Int(N\beta CIG) = U$. Therefore, $S \neq N\beta Int(N\beta CIG)$. Hence, S is $N\beta\theta O(U, X)$ but not $NR\beta O(U, X)$.

Theorem 5.8. If $S_i \in N\beta\theta O(U_i, X)$ for $i=1,2$ then $S_1 \times S_2 \in N\beta\theta O(U_1 \times U_2)$.

Proof: Let $S_i \in N\beta\theta O(U_i, X)$ for $i=1,2$, by Theorem 3.3, $S_i = \bigcup_{\ell_i \in \Delta_i} V_{\ell_i}$ and $V_{\ell_i} \in NR\beta O(U_i)$. By Theorem 1.18, $V_{\ell_1} \times V_{\ell_2} \in NR\beta O(U_1 \times U_2)$, $\forall \ell_1 \in \Delta_1$ and $\ell_2 \in \Delta_2$. Since, $S_1 \times S_2 = \bigcup_{\ell_1 \in \Delta_1} V_{\ell_1} \times \bigcup_{\ell_2 \in \Delta_2} V_{\ell_2} = \bigcup_{\ell_1 \in \Delta_1} \bigcup_{\ell_2 \in \Delta_2} V_{\ell_1} \times V_{\ell_2}$. By Theorem 3.3, $S_1 \times S_2 \in N\beta\theta O(U_1 \times U_2)$.

Theorem 5.9. Intersection $N\beta\theta$ – open set and NR – open set is $N\beta\theta$ – open set

Proof. Let $S \in NRO(U, X)$ and $V \in N\beta\theta O(U, X)$, so by Theorem 3.3, $V = \bigcup_{\lambda \in \Delta} V_{\lambda}$, where $V_{\lambda} \in NR\beta O(U, X)$. Hence, by Theorem 1.15, $S \cap V_{\lambda} \in NR\beta O(U, X)$. Since, $NR\beta O(U, X) \subset N\beta\theta O(U, X)$ then $S \cap V_{\lambda} \in N\beta\theta O(U, X)$.

$N\beta\theta O(U, X)$. And by Theorem 5.4, $\bigcup_{\lambda \in \Delta} S \cap V_\lambda \in N\beta\theta O(U, X)$. Hence, $S \cap V \in N\beta\theta O(U, X)$.

CONCLUSION

In this study, we defined a new type of nano open set in nano topological spaces called (Nano $\beta\theta$ –

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open sets) and compared it with other types of nano open sets. We have also studied the features of this set, and obtained new results.

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