Conjugate GN'-function for n variable

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ABSTRACT:

In 1969 S.K.Skaff introduced the generalized conjugate function. In this work the concept of a generalized conjugate function for n variable is defined and some of its important properties are examined. Many of the standard results which hold for N-functions and conjugate functions of a real variable will be generalized here.

الخلاصة:

في العام 1969 قدم S.K.Skaff تعميم الدالة المرافقة في هذا العمل عرفنا الدالة المرافقة إلى n من المتغيرات ودرسنا الخواص والنتائج التي تحققت للدالة-N

1. Introduction and Basic Concepts:

From the functional analysis as a function space, Orlicz spaces appeared in the first of the 30^{th} by W.R. Orlicz in Orlicz paper [1]. Many theorems and properties about conjugate function for GN*-function is introduced in [2,3, 4, and 5].

We have considered the investigation of a new definition of regarding generalized conjugate functions and discussed their properties.

Definition 1.1: [7]

Let $M(t,x,y)$ be a real valued non-negative function defined on $T\times E^n\times E^n$ such that:

(i) $M(t,x,y)=0$ if and only if *x,y* are the zero vectors $x,y\in E^n$, $\forall t\in T$

(ii) $M(t,x,y)$ is a continuous convex function of x,y for each t and a measurable function of t for each x,y,

(iii) For each $t\in T$, $\lim\limits_{\substack{\|x\|=\infty \\ \|y\|=\infty}} \dfrac{M(t,x,y)}{\|x\|\|y\|}=\infty$, and

(iv)There are constants $d\geq 0$ and $d_1\geq 0$ such that

$$\inf_t \inf_{\substack{c\geq d \\ c'\geq d_1}} k(t,c,c')>0 \qquad (1.1.1)$$

Where

$$k(t,c,c')=\frac{\underline{\underline{M}}(t,c,c')}{\overline{\overline{M}}(t,c,c')},$$

$$\overline{M}(t,c,c') = \sup_{\substack{|x|=c \\ |y|=c'}} M(t,x,y), \underline{M}(t,c,c') = \inf_{\substack{|x|=c \\ |y|=c'}} M(t,x,y)$$

and if $d \rangle 0$ and $d_1 > 0$, then $\overline{M}(t,d,d_1)$ is an integrable function of t. We call the function satisfying the properties (i)-(iv) a generalized N*-function or a GN*-function.

Definition 1.2:[7]

Let $M(t,x_1,x_2,...,x_n)$ be a real valued non-negative function defined on $\underbrace{T \times E^n \times E^n \times \;\; ... \;\; \times E^n}_{n\,-\,times}$ such that

(i) $M(t,x_1,x_2,...,x_n) = 0$ if and only if $x_1,x_2,...,x_n$ are the zero vectors $x_1,x_2,...,x_n \in E^n$, $\forall t \in T$

(ii) $M(t,x_1,x_2,...,x_n)$ is a continuous convex function of $x_1,x_2,...,x_n$ for each t and a measurable function of t for each $x_1,x_2,...,x_n$,

(iii) For each $t \in T$, $\lim\limits_{\substack{\|x_1\|=\infty \\ \|x_2\|=\infty \\ \|x_n\|=\infty}} \dfrac{M(t,x_1,x_2,...,x_n)}{\|x_1\|\|x_2\|...\|x_n\|} = \infty$, and

(iv)There are constants $d_1 \geq 0, d_2 \geq 0,..., d_n \geq 0$ such that

$$\inf_t \inf_{\substack{c_1 \geq d_1 \\ c_2 \geq d_2 \\ c_n \geq d_n}} k(t,c_1,c_2,...,c_n) > 0 \qquad (1.2.1)$$

Where

$$k(t,c_1,c_2,...,c_n) = \frac{\underline{M}(t,c_1,c_2,...,c_n)}{\overline{M}(t,c_1,c_2,...,c_n)},$$

$$\overline{M}(t,x_1,c_2,...,c_n) = \sup_{\substack{|x_1|=c_1 \\ |x_2|=c_2 \\ |x_n|=c_n}} M(t,x_1,x_2,...,x_n),$$

$$\underline{M}(t,c_1,c_2,...,c_n) = \inf_{\substack{|x_1|=c_1 \\ |x_2|=c_2 \\ |x_n|=c_n}} M(t,x_1,x_2,...,x_n)$$

and if $d_1 > 0, d_2 > 0, \ldots, d_n > 0$, then $\overline{M}(t,d_1,d_2,...,d_n)$ is an integrable function of t. We call the function satisfying the properties (i)-(iv) a generalized N'-function or a GN'-function.

Definition 1.3:

Let $M(t,x,y)$ be a GN*-function. Then we call $M^*(t,x,y)$ the conjugate function of $M(t,x,y)$ if for each *t* in *T*

$$M^*(t,x,y) = \sup_{\substack{z \ in \ E^n \\ w \ in \ E^n}} \{\sup\{zx, wy\} - M(t,z,w)\}. \qquad (1.3.1)$$

The notation *zx* represents the scalar product of the vectors *z* and *x,*and also *wy* represents the scalar product of vectors *w* and *y*.

Let us observe that if $zx \le 0$ and $wy \le 0$ in (1.3.1) then

$$\sup_{\substack{z \ in \ E^n \\ w \ in \ E^n}} \{\sup\{zx, wy\} - M(t,z,w)\} \le 0.$$

This means we could, equivalently, restrict the definition to those *z,w* for which $zx \ge 0$ and $wy \ge 0$.Moreover, the equation (1.3.1) yields immediately for each *t* in *T* that

$$\sup_{\substack{z \in E^n \\ w \in E^n}} \{zx, wy\} \le M(t,z,w) + M^*(t,x,y) \qquad (1.3.2)$$

for all *x,y,z* and *w* in E^n. Inequality (1.3.2) could have been used as a definition of the conjugate function.

Fenchel [4] states that to every $z,w \ in \ E^n$ such that $M'(t,z,w;r) < \infty$ for all *z,w* for which it is defined, there is at least one point *(x,y)* in $E^n \times E^n$ such that the equality holds in (1.3.2). However, by [6, Th.5.2] when applied to GN*-functions, we know for *z,w* in E^n that $M'(t,z,w;r) < \infty$ for all *r*. Therefore, the supremum in (1.3.2) is attained for at least one point.

Definition 1.4:

Let $M(t,x_1,x_2,...,x_n)$ be a GN'-function. Then we call $M^*(t,x_1,x_2,...,x_n)$ the conjugate function of $M(t,x_1,x_2,...,x_n)$ if for each *t* in *T*

$$M^*(t,x_1,x_2,...,x_n) = \sup_{\substack{z_1 \ in \ E^n \\ z_2 \ in \ E^n \\ z_n \ in \ E^n}} \{\sup\{z_1x_1, z_2x_2, ...z_nx_n\}$$

$$-M(t,z_1,z_2,...,z_n)\}. \qquad (1.4.1)$$

The notation z_ix_i represents the scalar product of the vectors z_i and x_i for all *i*=1 to *n*.

Let us observe that if $z_i x_i \le 0$ for all i=1 to n in (1.4.1) then

$$\sup_{\substack{z_1 \text{ in } E^n \\ z_2 \text{ in } E^n \\ z_n \text{ in } E^n}} \{\sup\{z_1 x_1, z_2 x_2, \ldots z_n x_n\} - M(t, z_1, z_2, \ldots, z_n)\} \le 0.$$

This means we could, equivalently, restrict the definition to those z_i for all i=1 to n for which $z_i x_i \ge 0$ for all i=1 to n. Moreover , the equation (1.4.1) yields immediately for each t in T that

$$\sup\{z_1 x_1, z_2 x_2, \ldots, z_n x_n\} \le M(t, z_1, z_2, \ldots, z_n) + M^*(t, x_1, x_2, \ldots, x_n) \qquad (1.4.2)$$

for all $x_1, x_2, \ldots, x_n, z_1, z_2, \ldots, z_n$ in E^n. Inequality (1.4.2) could have been used as a definition of the conjugate function.

Fenchel [9] states that to every $z_1, z_2, \ldots, z_n$ *in* E^n such that $M'(t, z_1, z_2, \ldots, z_n; r) < \infty$ for all $z_1, z_2, \ldots, z_n$ for which it is defined, there is at least one point $(x_1, x_2, \ldots, x_n)$ in $E^n \times E^n \times \ldots \times E^n$ such that the equality holds in (1.4.2). However, by [6, Th.5.2] when applied to GN'-functions, we know for $z_1, z_2, \ldots, z_n$ in E^n that $M'(t, z_1, z_2, \ldots, z_n; r) < \infty$ for all r. Therefore, the supremum in(1.4.1) is attained for at least one point.

Theorem 1.5:

Let $M(t, x, y)$ be a GN*-function for which

$$\lim_{\substack{|x|=0 \\ |y|=0}} \frac{M(t, x, y)}{|x||y|} = 0$$

for each t in T. Then $M^*(t, x, y)$ satisfies the properties (i)-(iii) of definition 1.1. Moreover, if $M(t, x, y) = M(t, -x, -y)$, then

$$M^*(t, x, y) = M^*(t, -x, -y)$$

Proof:

Condition (i) for $M^*(t, x, y)$ follows directly from the same condition for $M(t, x, y)$ and the equation in the hypothesis. Convexity follows from the inequality

$$\begin{aligned} M^*(t, ax+bz, ay+bw) &= \sup_{z', w'}\{\sup\{axz', ayw'\} - aM(t, z', w') + \\ &\quad + \sup_{z', w'}\{abzz', bww'\} - bM(t, z', w')\} \\ &\le aM^*(t, z', w') + bM^*(t, z', w') \end{aligned}$$

where $a+b=1, a\geq 0, b\geq 0$. Measurability in t also follows from the same property for $M(t,x,y)$. Finally , if we substitute

$z=\frac{kx}{|x|}, w=\frac{k_1 y}{|y|}, k>1, k_1>1$ in (1.3.2) we arrive at

$$\frac{M^*(t,x,y)}{|x||y|}\geq \sup\{\sup\{\ \frac{k}{|y|},\frac{k_1}{|x|}\} - \frac{M(t,\frac{kx}{|x|},\frac{k_1 y}{|y|})}{|x||y|}.\} \qquad (1.5.1)$$

However, $M(t,\frac{kx}{|x|},\frac{k_1 y}{|y|})$ is bounded on every compact set in E^n (see[6,Th.2.5]). Letting $|x|$ and $|y|$ tend to infinity in (1.5.1) results in property (iii).

Suppose $M(t,x,y)$ is an even function of x and y. Then

$$M^*(t,x,y)=\sup_{z,w\in E^n}\{\sup\{-zx,-wy\}-M(t,-z,-w)\}=$$

$$\sup_{z,w\in E^n}\{\sup\{z(-x),w(-y)\}-M(t,z,w)=M^*(t,(-x),(-y))$$

Finally, we give conditions when $M(t,x,y)$ is the conjugate function of $M^*(t,x,y)$.

Theorem 1.6:

Suppose $M(t,x,y)$ is a GN*-function for which $M'(t,x,y;z)$ is a linear in z. Then $M(t,x,y)$ is the conjegate function of $M^*(t,x,y)$.

Proof:

Since $M(t,x,y)$ is convex in x and y and $M'(t,x,y;z)$ is linear in z, we achieve for any x,x_0,y,y_0 in E^n .

$$M(t,x,y)-M(t,x_0,y_0)\geq M'(t,x_0,y_0;x-x_0)$$
$$\geq M'(t,x_0,y_0;x)-M'(t,x_0,y_0;x_0)$$

from which it follows that

$$M'(t,x_0,y_0;x_0)-M(t,x_0,y_0)\geq$$
$$\sup_{w,z\in E^n}\{\sup\{xz,wy\}-M(t,x,y)\} \qquad (1.6.1)$$

where $r^i=M'(t,x_0,y_0;e_i)$ for each $i=1,\dots,n$ and $\{e_i\}$ basis vectors for E^n . On the other hand, it is clear that

$$M'(t,x_0,y_0;x_0)-M(t,x_0,y_0)\leq$$
$$\sup_{z,w\in E^n}\{\sup\{xz,yw\}-M(t,x,y)\} \qquad (1.6.2)$$

since $M'(t,x_0,y_0;x_0)=\sup_{z,w\in E^n}\{zx_0,wy_0\}$. From combining (1.6.1) and (1.6.2) we obtain the equation

$$\sup_{z,w\in E^n}\{zx_0, wy_0\} - M(t,x_0,y_0) = M^*(t,z,w) \qquad (1.6.3)$$

Howover, by (1.3.2) we know that

$$\sup\{x_0\ z, y_0 w\} = M(t,x_0,y_0) + M^*(t,z,w) \qquad (1.6.4)$$

for all $x_0, y_0, z, w \quad in \quad E^n$. Rewriting (1.6.4) yields

$$M(t,x_0,y_0) \geq \sup_{z,w\in E^n}\{\sup\{x_0 z, y_0 w\} - M^*(t,z,w)\}. \qquad (1.6.5)$$

Since (1.6.3) holds for some *z,w* it follows that

$$\begin{aligned} M(t,x_0,y_0) &= \sup\{x_0 z, y_0 w\} - M^*(t,z,w) \leq \\ &\sup_{z,w\in E^n}\{\sup\{x_0 z, y_0 w\} - M^*(t,z,w)\} \end{aligned} \qquad (1.6.6)$$

Therefore, combining (1.6.5) and (1.6.6) produces the desired result that is

$$M(t,x_0,y_0) = \sup_{z,w\in E^n}\{\sup\{x_0 z, y_0 w\} - M^*(t,z,w)\}. \blacksquare$$

2.Conjugate GN*-function:

Theorem 2.1:

Let $M\ (t, x_1, x_2, ..., x_n)$ be a GN'-function for which

$$\lim_{\substack{|x_1|=0 \\ |x_2|=0 \\ |x_n|=0}} \frac{M(t,x_1,x_2,...,x_n)}{|x_1||x_2|..|x_n|} = 0$$

for each *t* in *T*. Then $M^*(t,x_1,x_2,...,x_n)$ satisfies the properties (i)-(iii) of definition 1.1. Moreover, if $M(t,x_1,x_2,...,x_n) = M(t,-x_1,-x_2,...,-x_n)$, then

$$M^*(t,x_1,x_2,...,x_n) = M^*(t,-x_1,-x_2,...,-x_n)$$

Proof:

Condition (i) for $M^*(t,x_1,x_2,...,x_n)$ follows directly from the same condition for $M(t,x_1,x_2,...,x_n)$ and the equation in the hypothesis. Convexity follows from the inequality

$$\begin{aligned} M^*(t, ax_1+by_1, ax_2+by_2, ..., ax_n+by_n) &= \sup_{z',w'}\{\sup\{ax_1z_1, ax_2z_2, ..., ax_nz_n\} \\ -aM(t,z_1,z_2,...,z_n) &+ \sup_{z',w'}\{by_1z_1, by_2z_2, ..., by_nz_n\} - bM(t,z_1,z_2,...,z_n)\} \\ &\leq aM^*(t,z_1,z_2,...,z_n) + bM^*(t,z_1,z_2,...,z_n) \end{aligned}$$

where $a+b=1, a\geq 0, b\geq 0$. Measurability in *t* also follows from the same property for $M(t,x_1,x_2,...,x_n)$. Finally , if we substitute

$z_1 = \frac{k_1 x_1}{|x_1|}, z_2 = \frac{k_2 x_2}{|x_2|}, \ldots, z_n = \frac{k_n x_n}{|x_n|}, k_1 > 1, k_2 > 1, \ldots, k_n > 1$ into (1.4.2) we arrive at

$$\frac{M_*(t, x_1, x_2, \ldots, x_n)}{|x_1||x_2|\ldots|x_n|} \geq \sup\{\sup\{ \frac{k_1}{|x_1|}, \frac{k_2}{|x_2|}, \ldots, \frac{k_n}{|x_n|}\} - M(t, \frac{k_1 x_1}{|x_1|}, \frac{k_2 x_2}{|x_2|}, \ldots, \frac{k_n x_n}{|x_n|})\}$$

(2.1.1)

However, $M(t, \frac{k_1 x_1}{|x_1|}, \frac{k_2 x_2}{|x_2|}, \ldots, \frac{k_n x_n}{|x_n|})$ is bounded on every compact set in E^n (see[6,Th.2.5]). Letting $|x_1|, |x_2|, \ldots, |x_n|$ tend to infinity in (2.1.1) results in property (iii).

Suppose $M(t, x_1, x_2, \ldots, x_n)$ is an even function of $x_1, x_2, \ldots, x_n$. Then

$M^*(t, x_1, x_2, \ldots, x_n) = \sup_{z,w \in E^n} \{\sup\{-z_1 x_1, -z_2 x_2, \ldots, -z_n x_n\} - M(t, -z_1, -z_2, \ldots, -z_n)\} =$

$\sup_{z,w \in E^n} \{\sup\{z_1(-x_1), z_2(-x_2), \ldots, z_n(-x_n)\} - M(t, z_1, z_2, \ldots, z_n)$

$= M^*(t, (-x_1), (-x_2), \ldots, (-x_n))$

Finally, we give conditions when $M(t, x_1, x_2, \ldots, x_n)$ is the conjugate function of . $M(t, x_1, x_2, \ldots, x_n)$

Theorem 2.2:

Suppose $M(t, x_1, x_2, \ldots, x_n)$ is a GN*-function for which $M'(t, x_1, x_2, \ldots, x_n; z)$ is a linear in *z*. Then $M(t, x_1, x_2, \ldots, x_n)$ is the conjegate function of $M^*(t, x_1, x_2, \ldots, x_n)$.

<u>Proof:</u>

Since $M(t, x_1, x_2, \ldots, x_n)$ is convex in $x_1, x_2, \ldots$ and x_n and $M'(t, x_1, x_2, \ldots, x_n; z)$ is linear in *z*, we achieve for any u8 $x_1, x_2, \ldots, x_n, x_{01}, x_{02}, \ldots, x_{0n}$ in E^n .

$$M(t, x_1, x_2, \ldots, x_n) - M(t, x_{01}, x_{02}, \ldots, x_{0n}) \geq M'(t, x_{01}, x_{02}, \ldots, x_{0n}; x_1 - x_{01})$$
$$\geq M'(t, x_{01}, x_{02}, \ldots, x_{0n}; x_1) - M'(t, x_{01}, x_{02}, \ldots, x_{0n}; x_{01})$$

from which it follows that

$$M'(t, x_{01}, x_{02}, \ldots, x_{0n}; x_{01}) - M(t, x_{01}, x_{02}, \ldots, x_{0n}) \geq \sup_{y_1, y_2, \ldots, y_n \in E^n} \{\sup\{x_1 y_1, x_2 y_2, \ldots, x_n y_n\} - M(t, x_1, x_2, \ldots, x_n)\} \qquad (2.2.1)$$

where $r^i = M'(t, x_{01}, x_{02}, ..., x_{0n}; e_i)$ for each i=1,..., n and $\{e_i\}$basis vectors for E^n . On the other hand, it is clear that

$$M'(t, x_{01}, x_{02}, ..., x_{0n}; x_{01}) - M(t, x_{01}, x_{02}, ..., x_{0n}) \leq$$

$$\sup_{y_1, y_2, ..., y_n \in E^n} \{\sup\{x_1 y_1, x_2 y_2, ..., x_n y_n\} - M(t, x_1, x_2, ..., x_n)\} \qquad (2.2.2)$$

since $M'(t, x_{01}, x_{02}, ..., x_{0n}; x_{01}) = \sup_{y_1, y_2, ..., y_n \in E^n} \{x_{01} y_1, x_{02} y_2, ..., x_{0n} y_n\}$. From combining (2.2.1) and (2.2.2) we obtain the equation

$$\sup_{y_1, y_2, ..., y_n \in E^n} \{x_{01} y_1, x_{02} y_2, ..., x_{0n} y_n\} - M(t, x_{01}, x_{02}, ..., x_{0n}) \qquad (2.2.3)$$

$$= M^*(t, y_1, y_2, ..., y_n)$$

Howover, by (1.4.2) we know that

$$\sup\{x_{01} y_1, x_{02} y_2, ..., x_{0n} y_n\} = \qquad (2.2.4)$$

$$M(t, x_{01}, x_{02}, ..., x_{0n}) + M^*(t, y_1, y_2, ..., y_n)$$

for all $x_{01}, x_{02}, ..., x_{0n}, y_1, y_2, ..., y_n$ *in* E^n . Rewriting (2.2.4)yields

$$M(t, x_{01}, x_{02}, ..., x_{0n}) \geq \sup_{z, w \in E^n} \{\sup\{x_{01} y_1, x_{02} y_2, ..., x_{0n} y_n\} \qquad (2.2.5)$$

$$- M^*(t, y_1, y_2, ..., y_n)\}.$$

Since (3.4.9) holds for some $y_1, y_2, ..., y_n$ it follows that

$$M(t, x_{01}, x_{02}, ..., x_{0n}) = \sup\{x_{01} y_1, x_{02} y_2, ..., x_{0n} y_n\} - M^*(t, y_1, y_2, ..., y_n) \leq$$

$$\sup_{z, w \in E^n} \{\sup\{x_{01} y_1, x_{02} y_2, ..., x_{0n} y_n\} - M^*(t, y_1, y_2, ..., y_n)\} \qquad (2.2.6)$$

Therefore, combining (2.2.3) and (2.2.4) produces the desired

$$M(t, x_{01}, x_{02}, ..., x_{0n}) = \sup_{z, w \in E^n} \{\sup\{x_{01} y_1, x_{02} y_2, ..., x_{0n} y_n\} - M^*(t, y_1, y_2, ..., y_n)\}.$$

References

1. A.Battor, O.Benderski, S.Yaskolko (1990):"Measurable fields of Orlicz spaces". Thesis reports.XV All-union school on the theory of operators in function spaces.Vljanovsk.(Russian)
2. A.Brondstes (1964):"Conjugate convex functions in topological vector spaces".Mat.Fys.Medd.Dansk.Vid.Silsk.34, 3-27.
3. W.Fenchel (1949):"On conjugate convex functions".Canad.J.Math.1,73-77
4. W.Fenchel (1953):"Convex cones, sets and function. Lecture notes".Prirceton univ.
5. W.A.J.Luxemburg and A.C.Zaanen (1956):"Conjugate space of Orlicz spaces".Indag.Math.18, 217-228.

6. M.S.Skaff (1968):"Vector valued Orlicz spaces". Thesis, Uiversity of California, Los Angeles.
7. M.S.Skaff(1969):"Vector valued Orlicz spaces generalized N-functions " .Pacific Journal of Mathematics vol.28.No.1
8. Wang Sheng-Wang (1963):"Convex function of several variables *and vector valued Orlicz spaces*".Bull.Aead.POLOn.Sc.ser.math. Astr.et Ph 11, 279-284.