A Nonstandard Generalization of Envelopes

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Received on:27/6/2007 **Accepted on:4/11/2007**

الملخص

الهدف من هذا البحث هو إعطاء صيغة معممة جديدة لتعريف الغلاف وذلك بإعطاء تعريف غير قياسي للغلاف عائلة من المستقيمات معرفة في إحداثيات الاسقاطية المتجانسة بواسطة : u(t)x+ v(t)y w(t)z = 0 وكذلك تم تطبيق التعريف الجديد على منحنيات المخروطية وذلك بالبحث عن معادلة عائلة المنحنيات التي يكون غلافها منحنياَ مخروطياَ معروفاَ.

ABSTRACT

The generalized envelopes are studied by a given nonstandard definition of envelope of a family of lines defined in a projective homogenous coordinates **PHC** by: $u(t)x + v(t)y + w(t)z = 0$. The new nonstandard concepts of envelope are applied to conic sections. Our goal in this paper is hat for a given conic section curve $f(x,y)=0$, we search for the family of lines in which f is its envelope.

Keywords: infinitesimals, monad, envelope

1- Introduction:

The following definitions and notations are needed throughout this paper.

Every concept concerning sets or elements defined in the classical mathematics is called **standard** [7].

Any set or formula which does not involve new predicates "standard, infinitesimals, limited, unlimited…etc" is called **internal,** otherwise it is called **external.** [7]

A real number $\boldsymbol{x}$ is called **unlimited** if and only if $|\boldsymbol{x}| > \boldsymbol{r}$ for all positive standard real numbers; otherwise it is called **limited** [4].

A real number $\boldsymbol{x}$ is called **infinitesimal** if and only if $|\boldsymbol{x}| < \boldsymbol{r}$ for all positive standard real numbers $\boldsymbol{r}$ [6], [4].

Two real numbers x and y are said to be **infinitely close** if and only if $x - y$ is infinitesimal and denoted by $x \cong y$ [8].

If x is a limited number in $\mathbb{R}$, then it is infinitely close to a unique standard real number, this unique number is called the **standard part** of x or **shadow** of x denoted by $st(x)$ or ${}^0x$ [6], [8].

***Theorem 1.1* :(*Extension Principle*)** [3]

Let $\mathbf{X}$ and $\mathbf{Y}$ be two standard sets, ${}^s\mathbf{X}$ and ${}^s\mathbf{Y}$ be the subsets constitute of the standard elements of $\mathbf{X}$ and $\mathbf{Y}$, respectively. If we can associate with every $x \in {}^s\mathbf{X}$ a unique $y = f(x) \in {}^s\mathbf{Y}$ then there exists a unique standard $y^* \in \mathbf{Y}$ such that $\forall^{st} x \in \chi$, $y^* = f(x)$

Let α and β be any two infinitesimal numbers and $r \neq 0$ is a limited real number, then:

1. $\alpha \cdot r$ is an infinitesimal.
2. $\alpha \cdot \beta$ is an infinitesimal.
3. $\alpha + r$ is limited.
4. $\alpha + \beta$ is an infinitesimal (in general the sum of any arbitrary finite number of infinitesimal numbers is infinitesimal) [6].

The **projective plane** over $\mathbb{R}$, denoted by $P_{\mathbb{R}}^2$ is the set

$P_{\mathbb{R}}^2 = \mathbb{R}^2 \cup$ {one point at ∞ for each equivalence classes of parallel lines },we denoted it by (***PHP***) [2].

The **projective homogeneous coordinates** of a point $p(x, y) \in \mathbb{R}^2$ are $[xa, ya, a]$, where a is any nonzero number, we denoted it by (***PHC***), in this sense the projective homogeneous coordinates of any point is not unique [2].

A curve n is called **envelope** of a family of curves g_a depending on a parameter a , if at each of its points, it is tangent to at least one curve of the family, and if each of its segments is tangent to an infinite set of these curves[2].

By a **parameterized differentiable curve**, we mean a differentiable map $\gamma : I \to \mathbb{R}^3$ of an open interval $\mathbf{I} = (a, b)$ of the real line $\mathbb{R}$ in to $\mathbb{R}^3$ such that: $\gamma(t) = (x(t), y(t), z(t)) = x(t)e_1 + y(t)e_2 + z(t)e_3$, and $x, y,$ and z are differentiable at t ; it is also called spherical curve[2].

2. An Envelope of a Family of Lines in a Plane

We consider $\mathbb{R}^2$ as a subset of ***PHP***, let $\{L_t\}$ be a family of lines in ***PHC*** space defined by:

$$u(t)\mathbf{X} + v(t)\mathbf{Y} + w(t)\mathbf{Z} = 0 ,$$

and suppose that the ordered pairs $(u(t),v(t)),(u(t),w(t)),(v(t),w(t))$,where u, v, w are standard functions defined on an interval sub set of $\mathbb{R}$.

The purpose is to associate a standard curve which is coincident with the envelope to the family $\{L_t\}$.

Suppose that t ranges over the interval $E \subset \mathbb{R}$ so that for every $t \in E$, there exists $\alpha > 0$ such that $\forall\ s \in [t-\alpha, t+\alpha]$, $L_t \neq L_s$

This is equivalent to $L_t \neq L_{t+\varepsilon}\ \forall\ t \in E$, where e is an infinitesimal real number.

Also, at each standard t, we can associate two lines L_t and $L_{t+\varepsilon}$ such that $L_t \neq L_{t+\varepsilon}$, where L_t and $L_{t+\varepsilon}$ are taken in ***PHP***.

Let $\gamma(t)$ be the envelope curve of the family $\{L_t\}$. By using the **principle of extension** we have:

There exists a unique standard application $a : E \to P_R^2$ such that $\gamma(t) \equiv \alpha(t)\ \forall^{st} t \in E$.

Now, let the families $\{L_t\}$ and $\{L_{t+\varepsilon}\}$ be given as follows:

$$\left.\begin{aligned} L_t &: u(t)X + v(t)Y + w(t)Z = 0 \\ L_{t+\varepsilon} &: u(t+\varepsilon)X + v(t+\varepsilon)Y + w(t+\varepsilon)Z = 0. \end{aligned}\right\} \quad \dots(2.1)$$

Then the intersection point of $\{L_t\}$ and $\{L_{t+\varepsilon}\}$ in ***PHC*** is given by:

$$\begin{aligned} X_\varepsilon(t) &= v(t+\varepsilon)w(t) - v(t)w(t+\varepsilon) \\ Y_\varepsilon(t) &= w(t+\varepsilon)u(t) - w(t)u(t+\varepsilon) \\ Z_\varepsilon(t) &= u(t+\varepsilon)v(t) - u(t)v(t+\varepsilon) \end{aligned}$$

Suppose that the functions u, v, and w are differentiable functions each of order at least n, then by expanding each of $u(t+\varepsilon)$, $v(t+\varepsilon)$, and $w(t+\varepsilon)$ using Taylor development, we get

$$\left.\begin{aligned} X_\varepsilon(t) &= v(t+\varepsilon)w(t) - v(t)w(t+\varepsilon) \\ &= (v'(t)w(t) - w'(t)v(t))\,\varepsilon + \cdots + (v^{(n)}(t)w(t) - w^{(n)}(t)v(t))\frac{\varepsilon^n}{n!} + \delta_1\varepsilon^n \\ Y_\varepsilon(t) &= w(t+\varepsilon)u(t) - w(t)u(t+\varepsilon) \\ &= (w'(t)u(t) - u'(t)w(t))\,\varepsilon + \cdots + (w^{(n)}(t)u(t) - u^{(n)}(t)w(t))\frac{\varepsilon^n}{n!} + \delta_2\varepsilon^n \\ Z_\varepsilon(t) &= u(t+\varepsilon)v(t) - u(t)v(t+\varepsilon) \\ &= (u'(t)v(t) - v'(t)u(t))\,\varepsilon + \cdots + (u^{(n)}(t)v(t) - v^{(n)}(t)u(t))\frac{\varepsilon^n}{n!} + \delta_3\varepsilon^n \end{aligned}\right\} \quad \dots(2.2)$$

where δ_1, δ_2, δ_3 are infinitesimals. In general, put

$$\left.\begin{aligned} p_n(t) &= v^{(n)}(t)w(t) - w^{(n)}(t)v(t) \\ r_n(t) &= w^{(n)}(t)u(t) - u^{(n)}(t)w(t) \\ q_n(t) &= u^{(n)}(t)v(t) - v^{(n)}(t)u(t) \end{aligned}\right\} \quad \ldots(2.3)$$

The following cases are related to the last assumption

Case1. If $q_1(t) \neq 0$ and $p_1(t)$and $r_1(t)$ are not both zero, then the ***PHC*** points of envelope curve $\gamma(t)$, $(p_1(t), r_1(t), q_1(t))$, are independent on ε, and the triple $(p_1(t), r_1(t), q_1(t))$ represents the classical definition of an envelope curve.

Proof:

Using (2.2), we get:

$$X_\varepsilon(t) = (v'(t)w(t) - w'(t)v(t))\varepsilon + \ldots + (v(n)(t)w(t) - w(n)(t)v(t))\frac{\varepsilon^n}{n!} + \delta_1\varepsilon^n$$

$$= \varepsilon(v'(t)w(t) - w'(t)v(t)) + \ldots + (v(n)(t)w(t) - w(n)(t)v(t))\frac{\varepsilon^{n-1}}{n!} + \delta_1\varepsilon^{n-1}$$

Taking the shadow of the of the last equation we obtain

$${}^oX_\varepsilon(t) = \varepsilon(v'(t)w(t) - w'(t)v(t))$$

In the same way, we have

$${}^oY_\varepsilon(t) = \varepsilon(w'(t)u(t) - u'(t)w(t)),$$

and $${}^oZ_\varepsilon(t) = \varepsilon(u'(t)v(t) - v'(t)u(t))$$

Therefore,

$$(X_\varepsilon(t), Y_\varepsilon(t), Z_\varepsilon(t)) > ({}^oX_\varepsilon(t), {}^oY_\varepsilon(t), {}^oZ_\varepsilon(t))$$
$$= (\varepsilon(v'(t)w(t) - w'(t)v(t)), \varepsilon(w'(t)u(t) - u'(t)w(t)), \varepsilon(u'(t)v(t) - v'(t)u(t)))$$

Now, using the properties of the ***PHC***, we deduce that any point of the form $(\lambda a, \lambda b, \lambda c)$ is equivalent with the point (a, b, c) for any parameter λ.

Therefore, the ***PHC*** of $\gamma(t)$ is $(X_\varepsilon(t), Y_\varepsilon(t), Z_\varepsilon(t))$, and it is equal to

$$(v'(t)w(t) - w'(t)v(t), w'(t)u(t) - u'(t)w(t), u'(t)v(t) - v'(t)u(t))$$
$$= (p_1(t), r_1(t), q_1(t)) \quad \ldots (2.4)$$

That is, $(p_1(t), r_1(t), q_1(t))$ represents the classical definition of an envelope curve which does not depend on ε.

Moreover, the Cartesian coordinates of points of γ are given by

$$(x(t),y(t))=\left(\frac{X_\varepsilon(t)}{Z_\varepsilon(t)},\ \frac{Y_\varepsilon(t)}{Z_\varepsilon(t)}\right)$$
$$=\left(\frac{v'(t)w(t)-w'(t)v(t)}{u'(t)v(t)-v'(t)u(t)},\ \frac{w'(t)u(t)-u'(t)w(t)}{u'(t)v(t)-v'(t)u(t)}\right) \quad \dots (2.5)$$

This is the classical form of the envelope curve of the family of straight lines.

Case2. If $q_1(t)=0$, $p_1(t)$, and $r_1(t)$ are not both zeroes, then the ***PHC*** points of the envelope curve $\gamma(t)$ are infinitely large, and the corresponding tangents $\{L_t\}$ are asymptotes of $\gamma(t)$.

Case3. If $p_1(t)=r_1(t)=q_1(t)=0$ and $p_2(t) \neq 0$,$r_2(t)\neq 0$ =$q_2(t) \neq 0$ then, the ***PHC*** points of the envelope curve $\gamma(t)$ are $(p_2(t),r_2(t), q_2(t))$, and $(p_1(t),r_1(t),q_1(t))$ is an inflection point of the envelope curve $\gamma(t)$.

Case4. If $p_k(t)= r_k(t)= q_k(t)=0$ for $1\le k<n$ (n standard) and $p_n(t)$, $r_n(t)$, $q_n(t)$ are not all zeros, then the ***PHC*** points of $\gamma(t)$ are of the form $(p_n(t), r_n(t), q_n(t))$ which does not depend on ε . Thus, we get the generalized nonclassical form of the envelope curve $\gamma(t)$ as follows:

$$(x(t),y(t))=\left(\frac{X_\varepsilon(t)}{Z_\varepsilon(t)},\ \frac{Y_\varepsilon(t)}{Z_\varepsilon(t)}\right)$$
$$=\left(\frac{v^{(n)}(t)w(t)-w^{(n)}(t)v(t)}{u^{(n)}(t)v(t)-v^{(n)}(t)u(t)},\ \frac{w^{(n)}(t)u(t)-u^{(n)}(t)w(t)}{u^{(n)}(t)v(t)-v^{(n)}(t)u(t)}\right)$$

Case5. If $p_k(t)= r_k(t)= q_k(t) =0$ for any value of k, then we can not say any thing about the generalization of the envelope curve. In the following sections, by $p(t)$, $r(t)$ and $q(t)$ we mean $p_1(t)$, $r_1(t)$ and $q_1(t)$ respectively.

3. Applications to Conic Sections

We restrict our study on a family of straight lines only, other studies on envelops and singularity of envelops, for example can be found in [1]. Our goal is that for a given conic section curve $f(x,y)=0$, we search for a family of lines in which f is its envelope.

Lemma 3.1

Consider a standard family of lines $\{L_t\}$ defined by

$$u(t)X + v(t)Y + w(t)Z =0 \quad \dots (3.1.1)$$

where u, v, and w are standard real polynomials of at most second degree, then the equation of $\{L_t\}$ can be written as follows:

$A(X, Y, Z)t^2 + B(X, Y, Z)t + C(X, Y, Z)=0,$ … (3.1.2)
in which A, B, and C are linear equations of the variables X and Y and Z belonging to ***PHP***. And **conversely** every equation of the form **(*3.1.2*)** represents a family of lines of the form **(*3.1.1*)**

***Proof*:**
Obvious

Theorem 3.2

If $\{L_t\}$ is a family of lines defined by:
$u(t)x + v(t)y + w(t) =0,$
where u, v, and w are standard real polynomials of at most second degree, then the envelope of $\{L_t\}$ is a cone of the form:
$B^2(x,y) - 4A(x,y)C(x,y)=0,$ … (3.2.1)
in which A, B, and C are linear equations of the variables x and y belonging to $\mathbf{R}[x,y]$.

Moreover Equation **(*3.2.1*)** represents a general form of a second degree equation of two variables x and y and conversely.

***Proof*:**

Consider the families of lines $\{L_t\}$ and $\{L_{t+\varepsilon}\}$
By using ***Lemma 3.1*** , we get that:

L_t : $A(x,y)t^2 + B(x,y)t + C(x,y)=0$

$L_{t+\varepsilon}$: $A(x,y)(t+\varepsilon)^2 + B(x,y)(t+\varepsilon) + C(x,y)=0$

Then solving L_t and $L_{t+\varepsilon}$ as an instantaneous system to omit t^2 we get:
$2\varepsilon A(x,y)t + A(x,y)\varepsilon^2 + B(x,y)\varepsilon =0.$
Therefore,
$2A(x,y)t + A(x,y)\varepsilon + B(x,y)=0$ … (3.2.2)

Taking the shadow of **(*3.2.2*)**, we get $t = \dfrac{-B(x,y)}{2A(x,y)}$ and then putting it in L_t we obtain the required result.

For the second part, since A, B, and C are linear equations of the variables x and y belonging to $\mathbf{R}[x,y]$, so Putting

$A(x,y) = a_1x + a_2y + a_3$
$B(x,y) = b_1x + b_2y + b_3$
$C(x,y) = c_1x + c_2y + c_3$,

in Equation **(*3.2.1*)**, we get the following equation

$B^2(x,y) - 4A(x,y)C(x,y) = ax^2 + bxy + cy^2 + dx + ey + f = 0,$

where

$a = (b_1^2 - 4a_1c_1)$

$b = 2(b_1b_2 - 2(a_1c_2 + a_2c_1))$

$c = (b_2^2 - 4a_2c_2)$

$d = 2(b_1b_3 - 2(a_1c_3 + a_3c_1))$

$e = 2(b_2b_3 - 2(a_2c_3 + a_3c_2))$

$f = (b_3^2 - 4a_3c_3)$

This is a general form of second degree equation in two variables x and y.

Conversely, assuming that we have a second degree equation of two variables x and y such as:

$$ax^2 + bxy + cy^2 + dx + ey + f = 0 \qquad \dots (3.2.3)$$

By a suitable changing of coordinate's axis; if $b \neq 0$ then a rotation of axis through the angle α determined by the equation $\cot 2\alpha = \frac{A - C}{B}$ will transform Equation **(*3.2.3*)** to the following equation

$$a^* x^2 + b^* y^2 + c^* x + d^* y + e^* = 0 \qquad \dots (3.2.4)$$

Completing the square for each uncompleted square related to the variables x and y in Equation **(*3.2.4*)** and simplifying the result, we get:

$$\left[\frac{x + \frac{c^*}{2a^*}}{\sqrt{\frac{c^{*2}b^* + d^{*2}a^* - 4a^*b^*e^*}{4a^{*2}b^*}}}\right]^2 - 4\left[\frac{1}{4} - \left(\frac{y + \frac{d^*}{2b^*}}{\sqrt{\frac{c^{*2}b^* + d^{*2}a^* - 4a^*b^*e^*}{a^*b^{*2}}}}\right)^2\right],$$

Now put $\quad A(x,y) = \left[\frac{1}{2} - \frac{y + \frac{d^*}{2b^*}}{\sqrt{\frac{c^{*2}b^* + d^{*2}a^* - 4a^*b^*e^*}{a^*b^{*2}}}}\right]$

$$B(x,y) = \left[\frac{x + \frac{c^*}{2a^*}}{\sqrt{\frac{c^{*2}b^* + d^{*2}a^* - 4a^*b^*e^*}{4a^{*2}b^*}}} \right]$$

$$C(x,y) = \left[\frac{1}{2} + \frac{y + \frac{d^*}{2b^*}}{\sqrt{\frac{c^{*2}b^* + d^{*2}a^* - 4a^*b^*e^*}{a^*b^{*2}}}} \right],$$

If $a^*, b^* = 0$ or $\frac{c^{*2}b^* + d^{*2}a^* - 4a^*b^*e^*}{4a^*b^{*2}} < 0$ then we obtain undefined or imaginary values which are unacceptable cases in real homogenous projective plane.

Thus we assume that $a^*, b^* \neq 0$ and $\frac{c^{*2}b^* + d^{*2}a^* - 4a^*b^*e^*}{4a^*b^{*2}} \geq 0$.

Hence we get the required result.

Remark 3.3

If the last conditions of the previous theorem are not valid or $a^* = b^*$ we deduce that Equation **(*3.2.4*)** represents a standard conic section which can be transferred to the form **(*3.2.1*)** as it is shown in the following table:

Conic Section	General Form	Standard Form	$A(x,y)$	$B(x,y)$	$C(x,y)$
Circle	$x^2+y^2+ax+by+c=0$	$x^2+y^2=r^2$	$r/2-x/2$	y	$r/2+x/2$
Parabola	$y^2+ax+by+c=0$	$y^2=4ax$	a	y	x
Ellipse	$ax^2+by^2+cx+dy+e=0$	$x^2/a^2+y^2/b^2=1$	$1/2-x/2a$	y/b	$1/2+x/2a$
Hyperbolic	$ax^2+by^2+cx+dy+e=0$	$x^2/a^2-y^2/b^2=1$	$x/2a-y/2b$	1	$x/2a+y/2b$

Example 3.4

The circle $x^2+y^2=1$ is an envelope curve of the family of lines $(1-t^2)x+(2t)y+(t^2+1)=0$, such as shown in the ***Figure 3.1***
By applying ***Theorem 3.2*** to the equation of the given circle we get

$$y^2-4(1/2-x/2)(1/2+x/2)=0$$

Therefore $(1/2-x/2)t^2+yt+1/2+x/2=0$,
which is an equation of a family of lines.

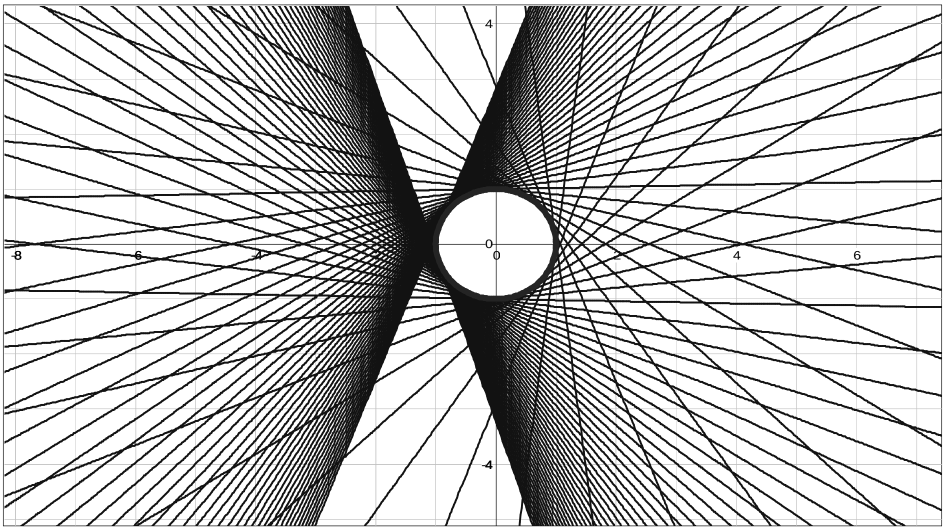

Figure 3.1

Note that we can show that the given circle equation $x^2 + y^2 = 1$ is an envelope equation of the founded family classically or by nonstandard tools. In the following, we give a nonstandard method for such purpose.

$$\left.\begin{aligned} L_t &: (1-t^2)x + (2t)y + (t^2+1) = 0 \\ L_{t+\varepsilon} &: (1-(t+\varepsilon)^2)x + 2(t+\varepsilon)y + (t+\varepsilon)^2 + 1 = 0 \end{aligned}\right\} \quad \ldots(3.4.1)$$

Solving equations L_t and $L_{t+\varepsilon}$ instantaneously, we get:

$2\varepsilon tx + \varepsilon^2 x - 2\varepsilon y - 2\varepsilon t - \varepsilon^2 = 0$.

Therefore

$2tx + \varepsilon x - 2y - 2t - \varepsilon = 0$.

Taking the shadow, we get

$2tx - 2y - 2t = 0$ **…** (3.4.2)

Now, remove the variable t form Equations **(*3.4.1*)** and **(*3.4.2*),** we get the required result.

Example 3.5

Consider the curve $x + y^2 - 1 = 0$

By applying ***Theorem 3.2*** to the given equation, we get

$y^2 - 4(1/4)(1 - x) = 0$,

now use ***Lemma 3.1*** we get

$1/4t^2 + yt + 1 - x = 0$

which is a family of lines whose envelope is the given equation, such as shown in the ***Figure 3.2***

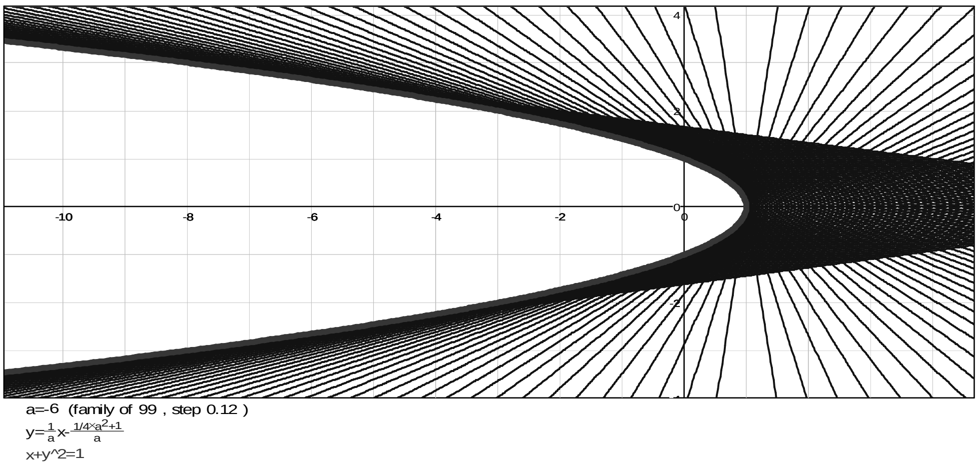

Figure 3.2

Remark: The graphs in Figs 3.1 and 3.2 are plotted with specific softwares: Omnigraph V3.1b-2005. ,Function Grapher V2.8-2006.

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