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Φ-Holomorphic Sectional Curvature and Generalized Sasakian Space Forms for a Class of Kenmotsu Type

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Abstract.

In this paper, we constructed an example for a warped product of the Hermitian manifold and real line for a class of Kenmotsu type. On the associated G-structure space, the conditions required for the mentioned class to be of constant pointwise Φ-holomorphic sectional curvature tensor are obtained. New classes of almost contact metric manifolds introduced and their relationships with our class are found. We studied the generalized Sasakian-space-forms and the conditions that satisfied our class, new classes and Einstein manifold are investigated.

Keywords: Almost contact metric manifolds, Einstein manifold, Kenmotsu manifold, Generalized Sasakian space form, warped product space, Φ -holomorphic sectional curvature tensor.

AMS Classification: 53D10, 53D15.

1 Introduction

In 1969, Tanno [21] investigated the Φ-holomorphic sectional (ΦHS-) curvature in the Sasakian manifolds. The necessary and sufficient conditions required for the Kenmotsu manifolds to be of constant Φ HS-curvature determined by Kenmotsu [14]. In 1991, Sinha and Srivastava [18] discussed curvatures on the Kenmotsu manifolds of constant ΦHS-curvature. In 2003, the Russian author Kirichenko [16] derived the corresponding condition for the almost contact metric (ACR-) manifolds of pointwise constant ΦHS-curvature in the associated G-structure (AGS-) space which is similar to the condition of almost Hermitian manifolds of ΦHS-curvature that used by Abood and Abdulameer [2]. In 2008, Falcitelli [10] studied the pointwise constant Φ HS-curvature of Locally conformal cosymplectic manifolds. In 2012, Kirichenko and Kharitonova [17] gave the constancy ΦHS-curvature conditions of locally conformal almost cosymplectic manifolds in AGS-space. On the other hand, the generalized Sasakian (GS-) space forms studied by Alegre et al. [3]. In 2006, Kim [15] established the necessary condition for the GS-space forms that have the Killing characteristic vector field to be conformally flat. Alegre and Carriazo [4] studied the contact metric and trans-Sasakian GS-space forms. In 2010, Falcitelli [11] proved that locally conformal $C_6 -$ manifold is a GS-space form if and only if, it satisfies the $k -$ nullity condition and has pointwise constant Φ HS-curvature. In 2012, Sular and Özgür [19] verified the Contact CR-warped product submanifolds in GS-space forms under specific conditions. Finally, in 2015, De and Loo [7] classified GS-space forms.

2 Preliminaries

We denote by M^{2n+1}, $X(M)$, $End\big(X(M)\big)$ and ∇, the smooth manifold M of dimension $2n+1$, Lie algebra of smooth vector fields of M, the set of all endomorphisms on $X(M)$, and Riemannian connection respectively.

Definition 2.1 [16] An ACR-manifold is a smooth manifold M^{2n+1} furnished with the structure (Φ, ξ, η, g), where $\Phi \in End\big(X(M)\big)$, $\xi \in X(M)$, η is the dual of ξ, and g is the Riemannian metric such that

$$\begin{cases} \Phi(\xi) = 0; \quad \eta(\xi) = 1; \quad \eta \circ \Phi = 0; \\ \Phi^2 = -id + \eta \otimes \xi; \\ g(\Phi X, \Phi Y) = g(X, Y) - \eta(X)\eta(Y); \\ \forall \ X, Y \in X(M). \end{cases}$$

Moreover, if $d\eta \neq 0$ then $(M^{2n+1}, \Phi, \xi, \eta, g)$ is called a contact metric manifold.

Abood and Abass [1] defined a new class of ACR-manifold characterized by the following identity:

$$\nabla_X(\Phi)Y - \nabla_{\Phi X}(\Phi)\Phi Y = -\eta(Y)\Phi X; \qquad \forall \ X, Y \in X(M); \qquad (1)$$

and they found the following components:

$$\begin{cases} R^a_{0c0} = -\delta^a_c \ ; \\ R^a_{bcd} = 2A^a_{bcd} \ ; \\ R^a_{\hat{b}c\hat{d}} = B^{abd}{}_c - B^{ab}{}_h \ B^{hd}{}_c \ ; \\ R^a_{\hat{b}cd} = 2\big(B^{ab}{}_{[cd]} - \delta^a_{[c} \ \delta^b_{d]}\big); \\ R^a_{bc\hat{d}} = A^{ad}_{bc} - B^{ah}{}_c \ B_{bh}{}^d \\ \qquad -\delta^a_c \ \delta^d_b; \\ \qquad a, b, c, d = 1, \dots, n \ ; \\ \qquad \hat{} = \cdot + n \ ; \end{cases} \qquad (2)$$

where R^i_{jkl} are the components of the Riemannian curvature tensor R of ACR-manifold satisfied (1) in the AGS-space. Moreover, the equalities below are hold

$$\begin{cases} A^{ad}_{[bc]} - B^{ad}{}_{[cb]} - B^{ah}{}_{[b}B_{|h|c]}{}^d = 0; \\ A^{[bc]}_{ad} + B_{ad}{}^{[cb]} - B_{ah}{}^{[b}B^{|h|c]}{}_d = 0; \\ A^{acd}_b - B^{a[cd]}{}_b + B^{a[c}{}_h B^{|h|d]}{}_b = 0; \\ A^b_{acd} + B_{a[cd]}{}^b - B_{a[c}{}^h B_{|h|d]}{}^b = 0; \\ A^a_{[bcd]} = 0; \qquad A^{[bcd]}_a = 0. \end{cases} \qquad (3)$$

The symbol $[\cdot \,|\, \cdot \,|\cdot]$ denotes the anti-symmetric operator of the including indexes except $|\cdot|$.

In [16], Kirichenko calculated the components of g and Φ in the AGS-space as the following:

$$\begin{cases} g_{00} = 1; \quad g_{0t} = 0; \quad g_{ab} = 0; \\ \Phi^0_0 = 0; \quad \Phi^0_t = 0; \quad \Phi^{\hat{a}}_b = 0; \\ g_{\hat{a}b} = \delta^a_b; \qquad g_{\hat{a}\hat{b}} = \overline{g_{ab}} \ ; \\ \Phi^a_b = \sqrt{-1} \ \delta^a_b; \qquad \Phi^{\hat{a}}_{\hat{b}} = \overline{\Phi^a_b} \ ; \end{cases} \qquad (4)$$

where $g_{ij} = g_{ji}$, $\Phi^i_j = -\Phi^{\hat{i}}_{\hat{j}}$, $i, j = 0,1,\dots,2n$, $t = 1,\dots,2n$, $\hat{0} = 0$, $\overline{A}$ is the complex conjugate of A, and $\hat{\hat{j}} = j$.

Further, in [1], the authors established the components of the Ricci tensor r of ACR-manifold verified (1) in the AGS-space as follows:

$$\begin{cases} r_{00} = -2n \ ; \quad r_{a0} = 0 \ ; \\ r_{ab} = -2A^{c}_{abc} + B_{cab}{}^{c} \\ \quad -B_{ca}{}^{h} \ B_{hb}{}^{c} \ ; \\ r_{\hat{a}b} = -2\left(n\delta^{a}_{b} + B^{ca}{}_{[bc]}\right) + A^{ac}_{cb} \\ \quad -B^{ah}{}_{b} \ B_{ch}{}^{c} \ ; \end{cases} \tag{5}$$

where $r_{ij} = r_{ji}$.

Definition 2.2 [1] An ACR-manifold $(M^{2n+1}, \Phi, \xi, \eta, g)$ with Ricci tensor r, is called

1. Einstein manifold, if $r_{ij} = \lambda g_{ij}$, where λ is an Einstein constant.
2. η-Einstein manifold, if $r_{ij} = \lambda g_{ij} + \mu \eta_i \eta_j$, where λ, μ are scalars.
3. of Φ-invariant Ricci tensor, if $r_{a0} = r_{ab} = 0$.

3 Example for ACR-manifold as warped product space

Bishop and O'Neill [5] introduced the idea of warped product spaces to constructed an example on the manifold of negative curvature.

Definition 3.1 [5] Suppose that (B, g_B) and (F, g_F) are Riemannian manifolds and $f : B \to B$ is a positive smooth function. The Riemannian manifold $(B \times F, g)$ is called a warped product manifold and denoted by $B \times_f F$, if

$$g(X,Y) = g_B(\pi_*(X), \pi_*(Y)) \\ + f^2(\pi(p)) g_F(\psi_*(X), \psi_*(Y))$$

for all X, Y belong to the tangent space $T_p(M)$, where $M = B \times F$, and $\pi : M \to B$, $\psi : M \to F$ are projection. Moreover, π_* and ψ_* are the differential map of π and ψ respectively.

Regarding the idea of Bishop and O'Neill, Kenmotsu [14] confirmed that his class is identical to $\mathrm{R} \times_f F$, where F is a Kähler manifold and $f(t) = ce^t$ with positive constant c. Further, Dileo and Pastore [8] proved that almost Kenmotsu is a warped product of the real line and almost Kähler manifold and the converse is true. Recently, Erken et al. [9] produced nearly Kenmotsu manifold from the warped product $\mathrm{R} \times_f H$, where H is a nearly Kähler manifold and f as defined by Kenmotsu [14]. So, according to the previous results we try to give an example on ACR-manifold satisfies (1).

To introduce such example, we suppose that (N^{2n}, J, h) is an almost Hermitian manifold of class $W_3 \oplus W_4$ (see Gray and Hervella [13]), then N satisfies the following identity:

$$D_X(J)Y - D_{JX}(J)JY = 0; \quad \forall \ X, Y \in X(N), \tag{6}$$

where D is the Riemannian connection of N with respect to the metric h. We take $M = \mathrm{R} \times_f N$, where $f(t) = e^t$ defined on R. If N has local coordinates $(x_1, x_2, \ldots, x_{2n})$, then M has local coordinates $(t, x_1, x_2, \ldots, x_{2n})$. Now, suppose that $\xi = \frac{\partial}{\partial t}$ is the characteristic vector field of M. Let $X \in X(M)$, then $X = X_0 + \eta(X)\xi$, where $X_0 \in X(N)$ and $\eta(X_0) = 0$. Then we define the endomorphism Φ on M by $\Phi(X) = J(X_0)$. Suppose that g is the Riemannian metric of M and ∇ is the Riemannian connection of g. Then from Goldberg [12] we obtain

$$\nabla_{X_0} Y_0 = D_{X_0} Y_0 - H(X_0, Y_0)\xi; \quad \forall \ X_0, Y_0 \in X(N), \tag{7}$$

where $H(X_0,Y_0)=g(\nabla_{X_0}\xi,Y_0)$. If we put $Y=\xi$ in the equation (1) we get $\nabla_X\xi=X-\eta(X)\xi$. Since $\eta(X_0)=0$, then $H(X_0,Y_0)=g(X_0,Y_0)$ and the equation (7) becomes

$$\nabla_{X_0}Y_0=D_{X_0}Y_0-g(X_0,Y_0)\xi.$$

Moreover, for any $X,Y\in X(M)$ we have

$$\begin{aligned}\nabla_X(\Phi)Y&=\nabla_{X_0+\eta(X)\xi}\Phi(Y)\\&\quad-\Phi(\nabla_{X_0+\eta(X)\xi}(Y_0+\eta(Y)\xi))\\&=\nabla_{X_0}\Phi(Y)+\eta(X)\nabla_\xi\Phi(Y)\\&\quad-\Phi\{\nabla_{X_0}Y_0+\eta(X)\nabla_\xi Y_0\\&\quad+(\nabla_{X_0}(\eta)Y)\xi+\eta(Y)\nabla_{X_0}\xi\\&\quad+\eta(X)(\nabla_\xi(\eta)Y)\xi+\eta(X)\eta(Y)\nabla_\xi\xi\}\\&=\nabla_{X_0}\Phi(Y)+\eta(X)\nabla_\xi\Phi(Y)-\Phi(\nabla_{X_0}Y_0)\\&\quad-\eta(X)\Phi(\nabla_\xi Y_0)-\eta(Y)\Phi(\nabla_{X_0}\xi)\end{aligned}$$

Since $\Phi(X)\in X(N)$, then $[\Phi(X),\xi]=0$ and from the fact that $[X,Y]=\nabla_XY-\nabla_YX$, we get $\nabla_\xi\Phi(Y)=\nabla_{\Phi(Y)}\xi=\Phi(Y)$ and $\nabla_\xi Y_0=\nabla_{Y_0}\xi=-\Phi^2(Y)$. Then from the previous discussion, we deduce that

$$\nabla_X(\Phi)Y=D_{X_0}(J)Y_0-g(X,\Phi(Y))\xi$$
$$-\eta(Y)\Phi(X). \qquad (8)$$

Now, from the equation (8) we conclude that

$$\nabla_{\Phi(X)}(\Phi)\Phi(Y)=D_{J(X_0)}(J)J(Y_0)$$
$$-g(X,\Phi(Y))\xi. \qquad (9)$$

So, by subtracting the equation (9) from the equation (8), and using the equation (6), we obtain the equation (1) and then we get the desired.

The above result proved that our class of dimension > 3 exists, therefore the method of Cabrera [6] does not apply here.

4 Φ -Holomorphic sectional curvature

Definition 4.1 [17] A Φ HS-curvature of ACR-manifold (M,Φ,ξ,η,g) in the direction of X $(X\neq 0; X\in\ker(\eta))$ is defined by

$$H(X)=\frac{g(R(X,\Phi X)X,\Phi X)}{g(X,X)^2},$$

where R is the Riemannian curvature tensor of M. Moreover, M is called has pointwise constant Φ HS-curvature if $H(X)=\gamma$, where $\gamma\in C^\infty(M)$ and γ does not depend on X.

Theorem 4.2 [17] An ACR-manifold $(M^{2n+1},\Phi,\xi,\eta,g)$ has pointwise constant Φ HS-curvature if and only if, the Riemannian curvature tensor R of M on AGS-space satisfies

$$R^{(a\ \ d)}_{(bc)}=\frac{\gamma}{2}\ \ \tilde{\delta}^{ad}_{bc}=\frac{\gamma}{2}\left(\delta^a_b\delta^d_c+\delta^a_c\delta^d_b\right);$$

where δ^a_b is the Kronecker delta, $a,b,c,d=1,2,...,n$, and $(\cdot\cdot)$ denotes the symmetric operator of the including indexes.

If we return to the ACR-manifold which satisfies the identity (1), then its Riemannian curvature tensor on AGS-space owned $R^{a\ \ d}_{bc}=R^a_{bc\hat{d}}=A^{ad}_{bc}-B^{ah}{}_c\ B_{bh}{}^d-\delta^a_c\ \delta^d_b$, (see equation (2) above). Then regarding the Theorem 4.2 above, we can state the following theorem:

Theorem 4.3 An ACR-manifold $(M^{2n+1},\Phi,\xi,\eta,g)$ furnished with the identity (1) has pointwise constant Φ HS-curvature if and only if, on AGS-space the following equality hold:

$$A^{ad}_{bc}=B_{bc}{}^{[ad]}-B_{hb}{}^a\ B^{dh}{}_c$$
$$+\frac{\gamma+1}{2}\tilde{\delta}^{ad}_{bc}.$$

Proof. Suppose that M is an ACR-manifold satisfied the identity (1) and has pointwise constant

Φ HS-curvature. Regarding the Theorem 4.2, on AGS-space of M, the following equation hold:

$$A^{(ad)}_{(bc)} = B^{(a|h|}{}_{(c}\ B_{b)h}{}^{d)} + \frac{\gamma+1}{2}\tilde{\delta}^{ad}_{bc}.$$

where $|h|$ means the index h does not act by the symmetric operator. Then using the fact $B_{bh}{}^{d} = -B_{hb}{}^{d}$ and the symmetric operator of the indexes b and c, we get

$$A^{(ad)}_{(bc)} = -B^{(a|h|}{}_{(b}\ B_{|h|c)}{}^{d)} + \frac{\gamma+1}{2}\tilde{\delta}^{ad}_{bc}.$$

Since $A^{ad}_{bc} = A^{[ad]}_{[bc]} + A^{[ad]}_{(bc)} + A^{(ad)}_{[bc]} + A^{(ad)}_{(bc)}$. Then regarding the equation (3), we get the required assertion.

Theorem 4.4 If M is an Einstein manifold satisfies the identity (1) and has pointwise constant Φ HS-curvature γ, then $\gamma = -1$.

Proof. Suppose that M is an Einstein manifold satisfied the identity (1). Then from [1], we have

$$B^{ca}{}_{[bc]} = 0; \quad A^{ac}_{cb} = B^{ah}{}_{b}\ B_{ch}{}^{c}. \qquad (10)$$

Since M has pointwise constant Φ HS-curvature γ, then from the Theorem 4.3, we get

$$A^{ac}_{cb} = B_{cb}{}^{[ac]} - B_{hc}{}^{a}\ B^{ch}{}_{b} + \frac{(\gamma+1)(n+1)}{2}\delta^{a}_{b}. \qquad (11)$$

Now, combine the equations (10) and (11), we obtain

$$B^{ah}{}_{b}\ B_{ch}{}^{c} - B_{ch}{}^{a}\ B^{ch}{}_{b} = \frac{(\gamma+1)(n+1)}{2}\delta^{a}_{b}.$$

Then the contracting of the above equation with respect to the indexes a and c, we conclude the result.

Corollary 1 The Einstein manifold of pointwise constant Φ HS-curvature and satisfied the identity (1) is locally isometric to the warped product $\mathbb{R}\times_f \mathbb{C}^n$.

Proof. The result follows from the above theorem and Tanno [20].

According to Vanhecke [22], we can define new classes of ACR-manifolds as the following:

Definition 4.5 An ACR-manifold $(M^{2n+1},\Phi,\xi,\eta,g)$ is called of class

G_1 if $\forall\ X,Y,Z,W \in \ker(\eta)$, then $R(\Phi X,\Phi Y,\Phi Z,\Phi W) = R(X,Y,Z,W)$;

G_2 if $\forall\ X,Y,Z,W \in \ker(\eta)$, then $R(X,Y,\Phi Z,\Phi W) = R(X,Y,Z,W)$;

G_3 if $\forall\ X,Y,Z,W \in \ker(\eta)$, then $R(\Phi X,Y,Z,\Phi W) = R(X,Y,Z,W)$;

G_4 if $\forall\ X,Y,Z,W \in X(M)$, then $R(\Phi^2 X,\Phi^2 Y,\Phi^2 Z,\Phi^2 W) = R(X,Y,Z,W)$.

Moreover, ACR-manifold of class G_1 and G_4 can be called classes of Φ-invariant and Φ^2-invariant Riemann curvature tensor respectively.

Theorem 4.6 In the AGS-space, the ACR-manifold $(M^{2n+1},\Phi,\xi,\eta,g)$ is of class

1. G_1 if and only if, $R_{\hat{a}bcd} = 0$;
2. G_2 if and only if, $R_{\hat{a}\hat{b}cd} = 0$;
3. G_3 if and only if, $R_{\hat{a}bc\hat{d}} = 0$;
4. G_4 if and only if, $R_{a0b0} = R_{\hat{a}0b0} = R_{a0bc} = R_{\hat{a}0bc} = R_{a0\hat{b}c} = 0$.

Proof. Since $R(X,Y,Z,W) = R_{ijkl}\ X^i\ Y^j\ Z^k\ W^l$, where $i,j,k,l = 0,1,\ldots,2n$ and for short we set $i,j,k,l = 0,a,\hat{a}$, where $a = 1,2,\ldots,n$ and $\hat{a} = a+n$. Then we have M of class G_1 if and only if,

$R(\Phi X,\Phi Y,\Phi Z,\Phi W) = R(X,Y,Z,W)$; $\forall\ X,Y,Z,W \in \ker(\eta)$. Then the above equation equivalent to

$$R_{rstu}\ (\Phi X)^r\ (\Phi Y)^s\ (\Phi Z)^t\ (\Phi W)^u = R_{ijkl}\ X^i\ Y^j\ Z^k\ W^l,$$

where i,j,k,l,r,s,t,u do not vanish because $X,Y,Z,W \in \ker(\eta)$. Then the last equation simplifies to

$$R_{rstu}\ \Phi^r_i\ \Phi^s_j\ \Phi^t_k\ \Phi^u_l\ X^i\ Y^j\ Z^k\ W^l =$$

$$R_{ijkl}\ X^i\ Y^j\ Z^k\ W^l.$$

Then $R_{rstu}\ \Phi_i^r\ \Phi_j^s\ \Phi_k^t\ \Phi_l^u = R_{ijkl}$ and from the values of the indexes and Φ in the equation (4) attain the result. Similarly, if M of class G_2 or G_3.

Now, if M of class G_4, then we have

$$R(\Phi^2X,\Phi^2Y,\Phi^2Z,\Phi^2W)=R(X,Y,Z,W).$$

The above equation can be written in the following form:

$$0=\eta(X)\eta(Z)R(\xi,Y,\xi,W)$$
$$+\eta(X)\eta(W)R(\xi,Y,Z,\xi)+\eta(Y)\eta(Z)R(X,\xi,\xi,W)$$
$$+\eta(Y)\eta(W)R(X,\xi,Z,\xi)-\eta(X)R(\xi,Y,Z,W)$$
$$-\eta(Y)R(X,\xi,Z,W)$$
$$-\eta(Z)R(X,Y,\xi,W)-\eta(W)R(X,Y,Z,\xi).$$

If we replace X,Y,Z,W,ξ by the indexes $i,j,k,l,0$ respectively in the last equation, we get

$$0=\eta_i\eta_k R_{0j0l}+\eta_i\eta_l R_{0jk0}+\eta_j\eta_k R_{i00l}$$
$$-\eta_l R_{ijk0}+\eta_j\eta_l R_{i0k0}-\eta_i R_{0jkl}$$
$$-\eta_j R_{i0kl}-\eta_k R_{ij0l}.$$

So, if we take the values of i,j,k,l as above and use the properties of the Riemannian curvature tensor, then we obtain the result.

Corollary 2 The ACR-manifold $(M^{2n+1},\Phi,\xi,\eta,g)$ which satisfies (1) can not be of class G_4.

Proof. Suppose that M satisfies the identity (1), then from the equation (2), we have

$$R_{\hat{a}0b0}=R^a_{0b0}=-\delta^a_b\neq 0.$$

Then from the Theorem 4.6, we arrive to the substance of this corollary.

Corollary 3 The ACR-manifold $(M^{2n+1},\Phi,\xi,\eta,g)$ which satisfies (1) can be of class

1. G_1 if and only if, $A^a_{bcd}=0$; or equivalently $B_{bcd}{}^a=B_{bc}{}^hB_{hd}{}^a$;
2. G_2 if and only if, $B^{ab}{}_{[cd]}=\delta^a_{[c}\ \delta^b_{d]}$;
3. G_3 if and only if,

$$A^{ad}_{bc}=B^{ah}{}_c\ B_{bh}{}^d+\delta^a_c\ \delta^d_b.$$

Proof. The results follow from Theorem 4.6 and the equations (2).

Corollary 4 If the ACR-manifold $(M^{2n+1},\Phi,\xi,\eta,g)$ satisfies the identity (1) and of *class* G_3, *then* M *of class* G_2.

Proof. The assertion of the present corollary follows from Corollary 3 and the first part of the equation (3).

Corollary 5 If the ACR-manifold $(M^{2n+1},\Phi,\xi,\eta,g)$ is a manifold of class G_3, then M has vanishing Φ HS-curvature tensor H.

Proof. Suppose that M of class G_3, then for all $X\in\ker(\eta)$, we get

$$H(X)=\frac{R(\Phi X,X,X,\Phi X)}{(g(X,X))^2}$$
$$=\frac{R(X,X,X,X)}{(g(X,X))^2}=0.$$

5 Generalized Sasakian space forms

In this section, we introduce the definition of generalized Sasakian (GS-) space forms as mentioned in [3] and derive the conditions of our class to be GS-space forms.

Definition 5.1 [3] An ACR-manifold (M,Φ,ξ,η,g) is called GS-space forms if there exist three functions f_1,f_2,f_3 on M such that

$$R(X,Y)Z=f_1\{g(Y,Z)X-g(X,Z)Y\}$$
$$+f_2\{g(X,\Phi Z)\Phi Y-g(Y,\Phi Z)\Phi X$$
$$+2g(X,\Phi Y)\Phi Z\}$$
$$+f_3\{\eta(X)\eta(Z)Y-\eta(Y)\eta(Z)X$$
$$+g(X,Z)\eta(Y)\xi-g(Y,Z)\eta(X)\xi\};$$

for all $X,Y,Z\in X(M)$, where R is the Riemannian curvature tensor of M. Moreover, such manifold is denoted by $M(f_1,f_2,f_3)$.

Suppose that

$R(X,Y,Z,W)=g(R(Z,W)Y,X)$ and $\Omega(X,Y)=g(X,\Phi Y)$. Then on the AGS-space, we have

$$\begin{cases}\Omega_{00}=\Omega_{a0}=\Omega_{\hat{a}0}=0;\\ \Omega_{ab}=\Omega_{\hat{a}\hat{b}}=0;\\ \Omega_{\hat{a}b}=\sqrt{-1}\ \ \delta_b^a\ \ ;\ \ \Omega_{ij}=-\Omega_{ji}.\end{cases}\qquad(12)$$

Moreover, the components of Riemannian curvature tensor of the GS-space forms on the AGS-space given by

$$\begin{aligned}R_{ijkl}&=f_1\{g_{ik}\ g_{jl}-g_{il}\ g_{jk}\}\\&+f_2\{\Omega_{il}\ \Omega_{kj}-\Omega_{lj}\ \Omega_{ik}+2\Omega_{ij}\ \Omega_{kl}\}\\&+f_3\{\eta_j\ \eta_k\ g_{il}-\eta_j\ \eta_l\ g_{ik}+\eta_i\ \eta_l\ g_{jk}\\&-\eta_i\ \eta_k\ g_{jl}\}.\end{aligned}\qquad(13)$$

Theorem 5.2 An ACR-manifold $(M^{2n+1},\Phi,\xi,\eta,g)$ satisfies the identity (1) is GS-space forms if and only if, on AGS-space M attains the following:

1. $f_3=f_1+1;\quad A^a_{bcd}=0;$
2. $A^{ad}_{bc}=B^{ah}{}_c\ B_{bh}{}^d+(f_2+f_3)\delta^a_c\ \delta^d_b+2f_2\ \delta^a_b\ \delta^d_c;$
3. $B^{ad}{}_{[cb]}=(f_3-f_2)\delta^a_{[c}\ \delta^d_{b]};$ $\quad B^{abd}{}_c=B^{ab}{}_h\ B^{hd}{}_c.$

Proof. Using the equations (2), (4), (12) and (13), we get the requirements.

Theorem 5.3 An ACR-manifold $M(f_1,f_2,f_3)$ has pointwise constant ΦHS-curvature equal to γ if and only if, $\gamma+f_1+3f_2=0$.

Proof. $M(f_1,f_2,f_3)$ has pointwise constant ΦHS-curvature equal to γ if and only if, $\forall\ X\in\ker(\eta)$

$$R(\Phi X,X,X,\Phi X)=\gamma(g(X,X))^2.$$

But the Riemann curvature tensor of $M(f_1,f_2,f_3)$ satisfies the following:

$$R(\Phi X,X,X,\Phi X)=-(f_1+3f_2)(g(X,X))^2;$$

$\forall\ X\in\ker(\eta)$. Then the subtracting of the above equations confirm the result.

Theorem 5.4 If $M(f_1,f_2,f_3)$ has pointwise constant ΦHS-curvature and satisfies the equation (1), then $f_2=f_3=\frac{1}{4};\quad f_1=\frac{-3}{4}$.

Proof. Combine the value of A^{ad}_{bc} from the Theorem 4.3 with its value in the Theorem 5.2, we get

$$B_{bc}{}^{[ad]}-B_{hb}{}^a\ \ B^{dh}{}_c+\frac{\gamma+1}{2}\tilde{\delta}^{ad}_{bc}$$
$$=$$
$$B^{ah}{}_cB_{bh}{}^d+(f_2+f_3)\delta^a_c\delta^d_b+2f_2\ \ \delta^a_b\delta^d_c.$$

The above equation reduces to

$$B_{bc}{}^{[ad]}+\frac{\gamma+1}{2}\tilde{\delta}^{ad}_{bc}=(f_2+f_3)\delta^a_c\ \delta^d_b+2f_2\ \delta^a_b\ \delta^d_c.$$

Regarding the Theorems 5.2 and 5.3, we have $B_{bc}{}^{[ad]}=(f_3-f_2)\delta^{[a}_b\ \delta^{d]}_c$ and $\gamma=-(f_1+3f_2)$ respectively, and substitute them in the above equation gives $f_2=f_3=\frac{1}{4}$. Then using $f_3=f_1+1$, we get the result.

Theorem 5.5 $M(f_1,f_2,f_3)$ is of class

1. G_1 constantly;
2. G_2 if and only if, $n=1$ or $f_1=f_2$;
3. G_3 if and only if, $f_1=f_2=0$;
4. G_4 if and only if, $f_1=f_3$.

Proof. Taking the equation (13) into account, we conclude that

$$R_{\hat{a}bcd}=0;$$
$$R_{\hat{a}\hat{b}cd}=(f_1-f_2)\{\delta^a_c\delta^b_d-\delta^a_d\delta^b_c\};$$
$$R_{\hat{a}bc\hat{d}}=(f_1+f_2)\delta^a_c\delta^d_b+2f_2\ \delta^a_b\delta^d_c;$$
$$R_{a0b0}=R_{a0bc}=R_{\hat{a}0bc}=R_{a0\hat{b}c}=0;$$
$$R_{\hat{a}0b0}=(f_1-f_3)\delta^a_b.$$

Compare the above equations with the Theorem 4.6, we deduce the results.

On the AGS-space can determine the components of the Ricci tensor of $M(f_1,f_2,f_3)$ from the equation (13) as follows:

$$r_{jk} = -g^{il} R_{ijkl} = (2nf_1 + 3f_2 - f_3) g_{jk} - (3f_2 + (2n-1) f_3) \eta_j \eta_k ;$$

where g^{il} are the components of g^{-1}. Then we deduce the following theorem:

Theorem 5.6 $M(f_1, f_2, f_3)$ is η -Einstein manifold with $\lambda = 2nf_1 + 3f_2 - f_3$ and $\mu = -(3f_2 + (2n-1) f_3)$.

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انحناء مقطعي Φ – هولومورفي وتعميم نماذج فضاء ساساكي لفئة من نوع كينموتسو

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المستخلص:

في هذا البحث، انشأنا مثال لفئة من نوع كينموتسو عبارة عن ضرب مشوه من المنطوي الهرميشي والمستقيم الحقيقي. حصلنا على الشروط المطلوبة للفئة المذكورة لكي تمتلك تنسر انحناء مقطعي Φ – هولومورفي ثابت نقطياً على الفضاء مترابط البنية – G. قدمنا فئات جديدة لمنطويات مترية متصلة تقريبياً و وجدنا علاقاتها مع فئتنا. درسنا تعميم نماذج الفضاء ساساكي والشروط التي تجعله يحقق فئتنا و ناقشنا منطوي اينشتاين و الفئات الجديدة.

الكلمات المفتاحية: منطويات اتصال متري تقريبي، منطوي اينشتاين، منطوي كينموتسو، تعميم نماذج فضاء ساساكي، فضاء ضرب مشوه، تنسر انحناء مقطعي Φ – هولومورفي.