Existence of Solution for Some Nonlinear Eigenvalue Problems

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Abstract

The existence of solutions for a non - linear eigenvalue problems is well studied and proved for an even dimension *n*. In this article we will study the case of odd dimension n > 1 for the family of quasi-homogenous and we will gi ve some examples for the case when the imension of the space either $n = 3$ or $n= 5$. We study the conditions for which we can prove the existence of non trivial solution for each case.

Keywords: nonlinear eigenvalue problems, spectra, trace, quasi-homogenous operators

وجود الحلول لبعض مسائل القيم الذاتية اللاخطية

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الخلاصة

وجود الحلول لمسائل القيم الذاتية اللاخطية قد درست جيدا بالنسبة للمسائل المعرفة على فضاءات ببعد زوجي. في هذا البحث سندرس وجود الحلول للفضاءا ذات البعد الفردي أكبر من 1 لعوائل مؤثرات شبه متجانسة و سنعطي أمثلة في حالة بعد الفضاء يساوي 3، 5. كذلك سنعطي شروط وجود الحلول الغير تافهة لكل حالة.

الكلمات المفتاحية: القيم الذاتية اللاخطية، الطيف، الاثر، المؤثرات شبه المتجانسة.

Introduction

In this article we interest by studying the existence of the non-trivial solutions for a nonlinear eigenvalue problem with a quasi-homogenous operator defined on an odd dimensional space. In 2003, Helffer, Robert and Wang have proved the existence of eigenvalues on every even dimension. The problem for the case of odd n, the case n = 1 was studied by [19], [18] and [3]. For the case odd n > 1 was studied in [1] (Doctorate thesis 2009), in which the author proved the existence of non-trivial eigenvalues for some quadratics families of operators for the cases of dimensions n = 1, 3, 5 and 7 and a conjecture was given for the upper odd dimensions. In section 3, we recall the results given in the thesis [1], in which Aboud has studied the quasi-homogeneous family $L_P(\lambda)$ of type (k1, k2, ℓ 1, ℓ 2) for any dimension (even or odd). The aim of this article is to study the following more general type of quasi-homogenous operators (section 4):

$L(\lambda) = H_0 + \lambda H_1 + \lambda^2 I,$ (1)

where H_0 and H_1 are pseudodifferential operators with the symbols H_0 et H_1 and of order M et $\frac{M}{2}$, respectively and of type (k,ℓ)such that:

$k = (k_1, k_2)$, $\ell = (\ell_1, \ell_2)$

and we prove the existence of non trivial eigenvalues for any odd dimension also we give some examples for the case of dimensions n = 3, 5.

For the case n = 3 we give the following example:

$L_P(\lambda) = D_x^4 - D_y^2 + (P(x,y) - \lambda)^2,$

with P a positive quasi-homogenous polynomial of degree M and type (k_1, k_2) i.e.

$P(p^{k1}x, p^{k2}y) = p^M P(x, y)$, $x \in \mathbb{R}^2$, $y \in \mathbb{R}$.

and

$L(\lambda) = |x|^6 - D_2^X + \lambda\, |x|^3 + \lambda^2$, $x \in \mathbb{R}^3$.

For the case n = 5 we will study the following example:

$L(\lambda) = D_x^4 + D_y^4 + |x|^2 + |y|^4 + \lambda(\sqrt{|x|^2 + |y|^4}) + \lambda^2$,

$x \in \mathbb{R}^2$, $y \in \mathbb{R}^3$. For each case we will prove the existance of non-trivial solutions by proving general results and by using the theorem of Lidskii.

Preliminaries

In this section we give some well-known results. Let T be a compact operator on H, where H is a separable Hilbert space. If $r(T) \neq 0$ (spectral radius) one order the non-zero eigenvalues of Tin a decreasing sequence in module $|\lambda_1(T)| \geq |\lambda_2(T)| \geq \ldots \geq |\lambda_n(T)| \geq \cdots$, every eigenvalue be repeated following its algebraic multiplicity. For proving the existence of non-trivial eigenvalues, we use the following theorem:

Theorem 1 (Theorem of Lidskii 1958)

For every $T \in C^1(H)$ we have $Tr(T) = \sum_{j\geq 1} \lambda_j(T)$.

We get directly the following corollary:

Corollary 1 *If there exists $k \geq 1$ such that $Tr(T^k) \neq 0$, then $\sigma(T) \neq \{0\}$.*

Definition 1 *Let X be a Banach space, an application $f : \mathbb{R}^n \setminus \{0\} \to X$, and a multi-index*

$$\underline{h} \in \mathbb{N}^n,\ \underline{h} = (h1, \cdots, hn)$$

and $d \in \mathbb{C}$. One say that f is quasi-homogenous of type $\underline{h}$ and of degree d if

$$f(p^{h1}x_1 \cdots p^{hn}x_n) = p^d(x_1, \cdots, x_n),$$

for each $p > 0$ and $(x_1, \cdots, x_n) \in \mathbb{R}^n \setminus \{0\}$.

We give now the definition of quasi-homogenous symbol.

Definition 2 *For the two multi-indices*

$$\underline{h} \in \mathbb{N}^n,\ \underline{h} = (h1, \cdots, hn)$$

and

$$\underline{k} \in \mathbb{N}^n,\ \underline{k} = (k1, \cdots, kn)$$

we set

$$M = p.p.cm\{h1, \cdots, hn, k1, \cdots, kn\}.$$

For, a, C^∞ function such that

$$a : R^{2n} \setminus \{0\} \to \mathcal{L}(H)$$

with

$$a(p^{h1}x_1, \cdots, p^{hn}x_n, p^{h1}\mathcal{E}_1, \cdots, p^{hn}\mathcal{E}_n)$$

where $\mathscr{L}$(H) is the space of continues linear operators from H to H , $x \in \mathbb{R}^n$, $(x_1, \cdots, x_n)$ and $\xi \in \mathbb{R}^n$, $(\xi_1, \ldots, \xi_n)$ and

$$\emptyset_{\underline{h},\underline{k}}(x, \xi) = \emptyset_{\underline{h},\underline{k}}(x, \xi) = (1 + \sum_{i=1}^{n} |x_i|^{2pi} + \sum_{i=1}^{n} |\xi_i|^{2pi})$$

Where $p_i = \frac{M}{h_i}$, $q_i = \frac{M}{k_i}$, $1 \leq i \leq n$, $(x, \xi) \in \mathbb{R}^{2n}$.

1 Conditions of trace class and Hilbert-Schmidt class for symbols

In the following we give a result which help us to know if the operator is of trace class (or of Hilbert-Schmidt class) by using the formula of the associated symbol. For the proof of the following lemmas see [23]:

Lemma 1.1 *Let $\sigma^w(L)$ be the Weyl symbol of L. One has L is an operator Hilbert Schmidt iff $\sigma^w(L) \in \mathcal{L}^2(\mathbb{R}^{2n})$.*

Lemma 1.2 *Let $\sigma^w(L)$ be the Weyl symbol of L. One has $L \in Cp$ if:*

$$\sum_{0\le|\alpha|+|\beta|\le k(p)} \left(\iint \left|D_x^\alpha D_\xi^\beta \sigma(L)\right|^p dx d\xi\right)^{\frac{1}{p}},$$

with k(p) even integer > $2n((\frac{2}{p}) - 1)$ if $p \in [1, 2]$, k(p) even integer > $3n(1 + (\frac{2}{p}))$ if $p\in(2,\infty)$.

We recall the following lemma (for the proof see [1]):

Lemma 1.3 *For an operator L the symbol $\sigma(L)$ quasi homogenous of order M and of type:*

$$(\underline{k}, \underline{\ell}) = (k1, k2, \ell_1, \ell_2)$$

such that

$$\sigma(L)(p^{k1}x, p^{k2}y, p^{l1}\xi, p^{l2}\eta) = p^M(x,y,\xi,\eta),$$

i.e.

$$\sigma(L) \in S^M_{\underline{k,l}} ,$$

where $(x, y) \in \mathbb{R}^{n1} \times \mathbb{R}^{n2}$, $(\xi,\eta) \in \mathbb{R}^{n1} \times \mathbb{R}^{n2}$ and $n1 + n2 = n.$ then :

i) L is of trace class if $\sigma(L) \in S^M_{\underline{k,l}}$ with the condition:

$$M + (n_1k_1 + n_2k_2 + \ell_1n_1 + \ell_2n_2) < 0. \qquad (2)$$

ii) L is of Hilbert-Schmidt class if $\sigma(L) \in S^M_{\underline{k,l}}$ with the condition:

$$2M + (n_1k_1 + n_2k_2 + \ell_1n_1 + \ell_2n_2) < 0. \qquad (3)$$

Theorem 1.2 (Theorem of Composition)

One suppose that ϕ, φ are weight functions, if $A \in \mathcal{L}^{\mu1}_{\phi,\varphi}$ with symbol $a \in S^{\mu1}_{\phi,\varphi}$ and $B \in \mathcal{L}^{\mu2}_{\phi,\varphi}$ of symbol $b \in S^{\mu2}_{\phi,\varphi}$, then $AB \in \mathcal{L}^{\mu1+\mu2}_{\phi,\varphi}$. Moreover, the symbol a.b of AB admits an asymptotic expression

$$a.b \asymp \sum_{\alpha,\beta} \Gamma(\alpha,\beta)(\partial_\xi^\alpha D_x^\beta a)(\partial_\xi^\beta D_x^\alpha b),$$

in the following sense:

for an integer number N, one has

$$a.b - \sum_{|\alpha,\beta|<N} \Gamma(\alpha,\beta)(\partial_\xi^\alpha D_x^\beta a)\left(\partial_\xi^\beta D_x^\alpha b\right) \in S^{\mu 1+\mu 2-(N,N)},$$

where $\Gamma(\alpha,\beta) = \left(\frac{1}{2}\right)^\alpha (-\frac{1}{2})^\beta \frac{1}{\alpha!\beta!.}$

2. Functional analysis of the problem

Let H be a complex Hilbert space and let:

L (λ) = H0 + λH1 + ... + λ^{m-1}Hm−1 + λ^m , (4)

where L is a family of non-bounded operators on H, and $\lambda \in$ C.

In addition, H0 is a closed operator with a dense domain D (H_0).

The operators H_1, ...,H_{m-1} are defined on D (H_0).

One considers the following hypothesis:

Hypothesis (H_1): $H_0$is a self-adjoint positive operator of the domain D (H_0) in H.

Hypothesis (H_2): For each integer j, 0 ≤ j ≤ k − 1, the operators

$H_0^{-1}H_j$ *and* $H_0^{-1}H_j$ *are* bounded in H.

To give the third hypothesis, we give the following definition:

Definition 1.3 *Let* H_1, H_2*be two complex Hilbert space and T:* $H_1 \rightarrow H_2$ *a compact operator. We denote by* $(\mu_j(T))j \geq 1$ *the decreasing sequence of eigenvalues of* $(T * T)^{\frac{1}{2}}$ *where each eigenvalue repeated corresponding to its multiplicity. Let p strictly positive real number. We say that:*

$$T \in C^p(H_1, H_2)$$

If:

$$\sum_{j=1}^{\infty} \mu j(T)^p < +\infty$$

where C^p*denotes the class of Schatten.*

Hypothesis (H_3): There exists a real p > 0 such that:

$$H_0^{\frac{-1}{m}} \in C^P(H).$$

Hence, we have the following result, for the proof see [1].

Proposition 1.1 *Under the previous hypothesis one has:*

i)L (λ) define a closed operator of domain D(H0).

ii)If $L(\lambda)^{-1}$ exists, then it is compact.

iii)$\forall\lambda \in C, L(\lambda)$ is an operator with index and Ind (L (λ)) = 0.

3 Quasi-homogenous operators of type (k1, k2, ℓ_1, ℓ_2) [1]

In this section we recall some results given in [1], we reformulate these results for a general case of homogenousity and we will generalize these results in section 4 for a more general kind of operators.

We consider the operator $L_P(\lambda)$ which is studied by Aboud in [1]:

$$L_P(\lambda) = a(Dx,Dy) + (P(x,y) - \lambda)^2, \qquad (5)$$

where P is a positive quasi-homogenous polynomial of order M and of type (k1, k2) and a is a pseudo-differential operator, positive, quasi-homogenous of order M and of type ($\ell_1\ell_2$) i.e.:

$$P(p^{k1}x, p^{k2}y) = p^M P(x,y),$$

$$a(p^{\ell 1}D_x, p^{\ell 2}D_y) = p^{M'} a(Dx,Dy).$$

where x $\in \mathbb{R}^{n1}, y \in \mathbb{R}^{n2}, \mathbb{R}^{n1} \times \mathbb{R}^{n2} = \mathbb{R}^n$.We start by taking an arbitrary n and in this article

we interest by the case of ***n*** odd.

In [1], Aboud has studied this kind of quasi-homogenous family of operators LP (λ) of type ($\frac{k_1}{M}, \frac{k_2}{M}, \frac{\ell 1}{M'}, \frac{\ell 2}{M'}$) for any dimension (even or odd).

We associate to L(λ) the non-self-adjoint matrix operator $\hat{A}_p$ in $\mathcal{L}^2(\mathbb{R}^n) \times \mathcal{L}^2(\mathbb{R}^n)$ such that :

$$\hat{A}_{P} = \begin{pmatrix} 0 & \widehat{H}_0^{\frac{1}{2}} \\ -\widehat{H}_0^{\frac{1}{2}} & -\widehat{H}_1 \end{pmatrix}$$

Where

$$\widehat{H}_0 = a(D_x, D_y) + P^2(x,y). \tag{7}$$

$$\widehat{H}_1 = -2P(x,y).$$

The symbol of $\hat{A}_P$ is

$$A_{P=}\begin{pmatrix} 0 & \widehat{H}_0^{\frac{1}{2}} \\ -H_0^{\frac{1}{2}} & -H_1 \end{pmatrix},$$

Where

$$H_0 = \mathrm{a}(\xi\ ,\eta) + P^2(\mathrm{x},\ \mathrm{y}),$$

$$H_1 = -2\mathrm{P}\ (\mathrm{x},\ \mathrm{y}),$$

so the symbol A_P has two complex eigenvalues:

$$\mu\pm = \mathrm{P}\ (\mathrm{x},\ \mathrm{y}) \pm \sqrt{\mathrm{a}(\xi\ ,\eta).} \tag{8}$$

By using the definition (2.2) we have:

$$A_P \in S^1_{M,M'}(\mathcal{L}^2(\mathbb{R}^n) \times (\mathcal{L}^2(\mathbb{R}^2)).$$

where

$$\underline{M} = \left(\frac{k_1}{M}, \frac{k_2}{M}\right),$$

$$\underline{M'} = \left(\frac{2\ell_1}{M'}, \frac{2\ell_2}{M'}\right).$$

We find

$$Tr(A_P + \lambda)^{-m}, \lambda \to \infty,$$

i.e. we find the trace of the parametrix of the operator $Tr(A_P + \lambda)^{-m}$for m is a sufficiently big real number and for a big λ. We note that:

$$(A_P + \lambda)^m \in S^m_{M,M'}(\mathcal{L}^2(\mathbb{R}^n) \times (\mathcal{L}^2(\mathbb{R}^2)).$$

So we have the following proposition:

Proposition 3.1 [1] *For the operator $\hat{A}_P$with the symbol in (6) with P a positive quasi-homogenous operator of order M and of type (k_1, k_2) and a is a pseudo-differential, positive, quasi-homogenous operator of order M' and of type (ℓ_1, ℓ_2) if there exist j, j ≥1 such that:*

$$n_1\ell_1 + n_2\ell_2 = M'j.$$

So for

$$m > Max\{\frac{n_1k_1+n_2k_2}{M} + \frac{2(n_1\ell_1+n_2\ell_2)}{M\prime} + n, \frac{2\ell_1 n_1}{M\prime} + \frac{2\ell_2 n_2}{M\prime} + M\},$$

we have

$$T_r\left(\hat{A}_P + \lambda\right)^{-m} \asymp c_0\lambda^{\alpha 0} + c_\varsigma\lambda^{\alpha 0-\delta} + \cdots,$$

where

$$c_0 = (-1)^{\frac{\varrho}{2}}\int_{\mathbb{R}^{2n}} (1 + P(x,y))^{-m+\varrho}dxdy\int_{\mathbb{R}^{2n}} (1 + \sqrt{a(\xi,\eta)})^{-m}d\xi d\eta,$$

$$\alpha_0 = -m + \frac{k_1n_1+k_2n_2}{M} + \frac{2(\ell_1 n\ell_1+\ell_2 n_2)}{M\prime}$$

$$\delta \geq MM'.min\left\{\frac{M\prime k_1+2M\ell_1}{MM\prime}, \frac{M\prime k_2+2m\ell_2}{MM\prime}\right\},$$

$$\varrho = \frac{2\ell_1 n_1}{M'} + \frac{2\ell_2 n_2}{M'},$$

and $c_0 \neq 0$.

By using the proposition (3.1) and the theorem of Lidskii we can prove the existence non trivial

solutions for the family of quasi-homogenous operators L_P , so this prove the following theorem:

Theorem 3.1 [1] The operator L_P in (5) is defined on $\mathcal{L}^2(\mathbb{R}^n)$ with P is a positive quasi-homogenous

polynomial of order M and of type (k_1, k_2) and a is a positive quasi-homogenous

pseudo-differential operator of order M' and of type (ℓ_1, ℓ_2), if there exists j, j ≥ 1 such that :

$$n_1\ell_1 + n_2\ell_2 = M'j,$$

where $n_1 + n_2 = n$. Then there exists $\lambda_0 \in C\ et\ u_0 \in \mathcal{L}^2(\mathbb{R}^2), u_0 \neq 0\ such\ that\ L_P(\lambda_0)u_0 = 0$.

I. Quadratic family of type $(\boldsymbol{k_1}, \boldsymbol{k_2}, \boldsymbol{\ell_1}, \boldsymbol{\ell_2})$

We consider the following quadratic family of operators:

$$\mathrm{L}(\lambda) = \mathrm{H}_0 + \mathrm{H}_1\,\lambda + \lambda^2 I, \qquad (9)$$

Where H_0 and H_1 are pseudo-differential operators with the symbols H_0 and H_1 which are of order M and $\frac{M}{2}$, respectively and of type (k, ℓ) such that:

$$k = (k_1, k_2) \ , \ \ell = (\ell_1, \ell_2)$$

i.e.:

$$H_0\big(\rho^{k_1}x, \rho^{k_2}y, \rho^{\ell_1}\xi, \rho^{\ell_2}\eta\big) = \rho^M H_0(x, y, \xi, \eta)$$

$$H_1\big(\rho^{k_1}x, \rho^{k_2}y, \rho^{\ell_1}\xi, \rho^{\ell_2}\eta\big) = \rho^{\frac{M}{2}} H_1(x, y, \xi, \eta)$$

Where $x \in \mathbb{R}^{n_1}$ $y \in \mathbb{R}^{n_2}$ $\eta \in \mathbb{R}^{n_1}$ $\xi \in \mathbb{R}^{n_1}$ and $n_1 + n_2$ = n.

H_0 and H_1 verify the hypotheses (1), (2) and (3) of the section (2.2). We assume that H_0 and $H_0 - H_1{}^2$ are positive operators. For the linearization of the problem we associate to L(λ), the following non selfadjoint matrix operator:

$$\hat{A} = \begin{pmatrix} 0 & \hat{H}_0{}^{\frac{1}{2}} \\ -\hat{H}_0{}^{\frac{1}{2}} & -\hat{H}_1 \end{pmatrix} \qquad (10)$$

with the matrix symbol A which belongs to $S^1_{k,\ell}(\mathcal{L}^2(\mathbb{R}^n) \times \mathcal{L}^2(\mathbb{R}^n))$ where:

$$k = (\tfrac{k_1}{M}, \tfrac{k_2}{M}) \ , \ \ell = (\tfrac{\ell_1}{M}, \tfrac{\ell_2}{M})$$

such that

$$\hat{A} = \begin{pmatrix} 0 & H_0{}^{\frac{1}{2}} \\ -H_0{}^{\frac{1}{2}} & -H_1 \end{pmatrix} \qquad (11)$$

and

$$\mu_{\mp} = H_1 \mp i\sqrt{H_0 - H_1{}^2}$$

We study the asymptotic form of the trace of the operator $(A+\lambda)^{-m}$, for m sufficiently big. The operator $(A+\lambda)^{-m}$ have the symbol B which belongs to $S^1_{k,\ell}(\mathcal{L}^2(\mathbb{R}^n) \times \mathcal{L}^2(\mathbb{R}^n))$ such that:

$$B(x, y, \xi, \eta, \lambda) \approx \sum\nolimits_{j \geq 0} b_{j,\lambda}(x, y, \xi, \eta)$$

$$b_{0,\lambda}(x, y, \xi, \eta) = (A(x, y, \xi, \eta) + \lambda)^{-m}$$

$$b_{j+1,\lambda}(x, y, \xi, \eta) = -b_{0,\lambda}(x, y, \xi, \eta) \sum\nolimits_{\Lambda} \Gamma(\alpha, \beta)\, \partial^{\alpha}_{\xi,\eta} D^{\beta}_{x,y}\, (A(x, y, \xi, \eta) + \lambda)^{m}$$

$$\partial^{\beta}_{\xi,\eta} D^{\alpha}_{x,y} b_l, \lambda(x, y, \xi, \eta),$$

$$\Lambda = \{|(\underline{k}, \underline{\ell}).(\alpha, \beta)| + l = j + 1. l \leq j\},$$

$$\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^{n_1} \times \mathbb{N}^{n_2}, \beta = (\beta_1, \beta_2)) \in \mathbb{N}^{n_1} \times \mathbb{N}^{n_2}.$$

So the trace have the following asymptotic form

$$\sum_{j\geq 0} c_j \int_{\mathbb{R}^{2n}} \text{tr}\,(\text{bj},\lambda)(x, y, \xi, \eta) dx dy d\xi d\eta T_r(\hat{A} + \lambda)^{-m} \asymp$$

$$\asymp \sum_{j\geq 0} c_j, -m(\lambda),$$

we set

$$c_{j,-m}(\lambda) = \int_{\mathbb{R}^{2n}} \text{tr}(\text{bj},\lambda)(x, y, \xi, \eta) dx dy d\xi d\eta,$$

and

$$b_{j,\lambda}(x, y, \xi, \eta) \in S^{-m-j}_{k,\ell}\big(\mathcal{L}^2(\mathbb{R}^n) \times \mathcal{L}^2(\mathbb{R}^n)\big).$$

Proposition 1 The operator $\hat{A}$ in (10) is defined in $\mathcal{L}^2(\mathbb{R}^n) \times \mathcal{L}^2(\mathbb{R}^n)$ where H_0 and H_1 are pseudo-differential positive operators of order M and $\frac{M}{2}$, respectively and are of type (k, ℓ), k =(k_1, k_2), ℓ = (ℓ_1, ℓ_2) such that there exists j, j ≥1 with:

$$n_1(k_1, \ell_1) + n_2(k_2, \ell_2) = M_j$$

where $n_1 + n_2$ = n. For m large enough and $\lambda \to +\infty$ we have:

$$T_r(\hat{A}_P + \lambda)^{-m} \asymp \sum_{j\geq 0} c_{j,-m}(\lambda)$$

$$\asymp c_0 \lambda^{\alpha 0} + c_\delta \lambda^{\alpha 0 - \delta} + \ldots,$$

such that

$$c_{j,-m}(\lambda) \int_{\mathbb{R}^{2n}} b_j, \lambda(x, y, \xi, \eta) dx dy d\xi d\eta,$$

where bj,λ are the terms of the symbol of the parametrix of$(\hat{A}_P + \lambda)^m$ m and

$$c_0 = (-1)^{\frac{\varrho}{2}} \int_{\mathbb{R}^{2n}} ((H_1(x, y, \xi, \eta) + \sqrt{H_0(x, y, \xi, \eta) - H_1^2(x, y, \xi, \eta)} + 1)^{-m}$$

$$c_\delta, -m = c_\delta \lambda^{\alpha 0 - \delta}$$

$$= \int_{\mathbb{R}^{2n}} b_\delta, \lambda(x, y, \xi, \eta) dx dy d\xi d\eta,$$

α_0 = -m + $\frac{2n_1\,(k_1+\ell_1)+2n_2(k_2+\ell_2)}{M}$,

$\delta = M.\min\left\{\frac{k_1+\ell_1}{M}, \frac{k_2+\ell_2}{M}\right\}$,

$\varrho = \frac{2n_1\,(k_1+\ell_1)+2n_2(k_2+\ell_2)}{M}$,

de plus $c_0 \neq 0$.

Proof: To find the asymptotic form of the trace we start by calculate the trace of the first term b_0, λ i.e. :

$$t_r\,(b_0, \lambda\,) = \Re(H_1 + \sqrt[i]{H_0 - H_1^2} + \lambda)^{-m}).$$

For the integral

$$\int_{\mathbb{R}^{2n}} (H_1(\mathrm{x}, \mathrm{y}, \xi, \eta) + \sqrt[i]{H_0(\mathrm{x}, \mathrm{y}, \xi, \eta) - H_1^2(\mathrm{x}, \mathrm{y}, \xi, \eta)} + \lambda)^{-m}\, dxdyd\xi d\eta,$$

we do the changement of variables:

$$\mathrm{x} = \lambda\,\frac{2k_1}{M}\,x' \Rightarrow dx = \lambda\,\frac{2k_1n_1}{M}\,\mathrm{dx'}$$

$$\mathrm{y} = \lambda\,\frac{2k_2}{M}\,y' \Rightarrow dy = \lambda\,\frac{2k_2n_2}{M}\,\mathrm{dy'}$$

$$\xi = \lambda\,\frac{2\ell_1}{M}\,\xi' \Rightarrow d\xi = \lambda\,\frac{2\ell_1n_1}{M}\,\mathrm{d}\,\xi'$$

$$\eta = \lambda\,\frac{2\ell_2}{M}\,\eta' \Rightarrow d\eta = \lambda\,\frac{2\ell_2n_2}{M}\,\mathrm{d}\,\eta'$$

we get

$$c_{0,-m}(\lambda) = (\lambda)^{\alpha_0} \int_{\mathbb{R}^{2n}} \Re\Big(\Big(H_1(\mathrm{x}', \mathrm{y}', \xi', \eta') + \sqrt[i]{H_0(\mathrm{x}', \mathrm{y}', \xi', \eta') - H_1^2(\mathrm{x}', \mathrm{y}', \xi', \eta')} + 1\Big)^{-m} dx'dy'd\xi'd\eta',$$

where

c_α = -m+ $\frac{2n_1\,(k_1+\ell_1)+2n_2(k_2+\ell_2)}{M}$,

we set

$$c_0 = \int_{\mathbb{R}^{2n}} \Re\Big(\Big(H_1(\mathrm{x}', \mathrm{y}', \xi', \eta') + \sqrt[i]{H_0(\mathrm{x}', \mathrm{y}', \xi', \eta') - H_1^2(\mathrm{x}', \mathrm{y}', \xi', \eta')} + 1\Big)^{-m} dx'dy'd\xi'd\eta',$$

i.e.

$$c_{0,-m}(\lambda) = (\lambda)^{\alpha_0}$$

To calculate c_0 we follow the same method of previous sections. We assume that

$\mathrm{f}(\alpha) = \int_{\mathbb{R}^{2n}}(\left(H_1(\mathrm{x}',\mathrm{y}',\xi',\eta') + \sqrt[\alpha]{H_0(\mathrm{x}',\mathrm{y}',\xi',\eta') - H_1^2(\mathrm{x}',\mathrm{y}',\xi',\eta')} + 1\right)^{-m} dx'dy'd\xi'd\eta'$,

by doing the changement of variables

$$\mathrm{x}' = (\tfrac{1}{\alpha})^{\frac{2k_1}{M}}\,\mathrm{x}'' \Rightarrow dx' = (\tfrac{1}{\alpha})^{\frac{2k_1n_1}{M}}\, dx''$$

$$\mathrm{y}' = (\tfrac{1}{\alpha})^{\frac{2k_2}{M}}\,\mathrm{y}'' \Rightarrow dy' = (\tfrac{1}{\alpha})^{\frac{2k_2n_2}{M}}\, dy''$$

$$\xi' = (\tfrac{1}{\alpha})^{\frac{2\ell_1}{M}}\,\xi'' \Rightarrow d\xi' = (\tfrac{1}{\alpha})^{\frac{2\ell_1n_1}{M}}\, d\xi''$$

$$\eta' = (\tfrac{1}{\alpha})^{\frac{2\ell_2}{M}}\,\eta'' \Rightarrow d\eta' = (\tfrac{1}{\alpha})^{\frac{2\ell_2n_2}{M}}\, d\eta''$$

we get

$\mathrm{f}(\alpha) = \left(\frac{1}{\alpha}\right)^{\varrho} \int_{\mathbb{R}^{2n}} \left(H_1(\mathrm{x}'',\mathrm{y}'',\xi'',\eta'') + \sqrt{H_0(\mathrm{x}'',\mathrm{y}'',\xi'',\eta'') - H_1^2(\mathrm{x}'',\mathrm{y}'',\xi'',\eta'')} + 1\right)^{-m} dx''dy''d\xi''d\eta''$,

where $\varrho = \frac{2n_1\,(k_1+\ell_1)+2n_2(k_2+\ell_2)}{M}$,

By considering an analytic prolongation of $\mathrm{f}(\alpha)$, we can suppose that $\alpha = i$, and we obtain :

$c_0 = \Re(i)^{-\varrho}\int_{\mathbb{R}^{2n}} \left(H_1(\mathrm{x}'',\mathrm{y}'',\xi'',\eta'') + \sqrt{H_0(\mathrm{x}'',\mathrm{y}'',\xi'',\eta'') - H_1^2(\mathrm{x}'',\mathrm{y}'',\xi'',\eta'')} + 1\right)^{-m} dx''dy''d\xi''d\eta''$,

so $c_0 \neq 0$, there exists j, j ≥ 1 such that:

$$n_1\,(k_1+\ell_1) + n_2(k_2+\ell_2) = M_j$$

we have either n is even or n is odd.

<u>When n is even</u>, we have n1 and n2 are even (n_1 and n_2 are odd), then for M even we must have (k_1, k_2) and (ℓ_1, ℓ_2) such that (13) is verified (the sums $k_1 + \ell_1\ et\ k_2 + \ell_2$ are even such that (13) is verified).

<u>When n is odd</u>, we have n1 is even and n_2 is odd (n_1 is odd and n_2 is even) and M even so the sum must be $k_2 + \ell_2$ either even and (13) is verified (the sum $k_1 + \ell_1$ either even and (13) is

verified). For example, the case n = 5, n_1 = 2, n_2 = 3, (k_1, k_2) = (2, 1), (ℓ_1, ℓ_2) = (1, 1) et M = 4. So, we obtain

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c_0=$(-1)^{-\frac{\varrho}{2}}\int_{\mathbb{R}^{2n}}\Big(H_1(\mathrm{x}'',\mathrm{y}'',\xi'',\eta'')+\sqrt{H_0(\mathrm{x}'',\mathrm{y}'',\xi'',\eta'')-H_1^2(\mathrm{x}'',\mathrm{y}'',\xi'',\eta'')}+1\Big)^{-m}dx''dy''d\xi''d\eta''$,

and $c_0\neq 0$, since $H_1(\mathrm{x}'',\mathrm{y}'',\xi'',\eta'')+\sqrt{H_0(\mathrm{x}'',\mathrm{y}'',\xi'',\eta'')-H_1^2(\mathrm{x}'',\mathrm{y}'',\xi'',\eta'')}+1$ is positive. For finding the second term of the trace de$(\hat{A}+\lambda)^{-m}$ we set:

$$\mathrm{k} = \min\left\{\frac{k_1+\ell_1}{M},\frac{k_2+\ell_2}{M}\right\},$$

we assume that

$$\mathrm{k} = \frac{k_1+\ell_1}{M},$$

so we take in consideration the type of homogeneity we obtain $b_{kM,\lambda}$ is the second term when l= 0 and $\alpha_1+\beta_1$ = ,M , $\alpha_2+\beta_2$=0 i.e.

$$\begin{aligned}b_{kM,\lambda}(\mathrm{x,y},\xi,\eta)= &-\Gamma(M,0)b_{0,\lambda}(\mathrm{x,y},\xi,\eta)\partial_\xi^M(A_P((\mathrm{x,y},\xi,\eta)+\lambda)^m D_x^M b_{0,\lambda}(\mathrm{x,y},\xi,\eta)\\ &-\Gamma(0,M)b_{0,\lambda}(\mathrm{x,y},\xi,\eta)D_x^M(A_P((\mathrm{x,y},\xi,\eta)+\lambda)^m\partial_\xi^M b_{0,\lambda}(\mathrm{x,y},\xi,\eta)\\ &-\Gamma(M-1,1)b_{0,\lambda}(\mathrm{x,y},\xi,\eta)\partial_\xi^{M-1}D_x^1(A_P((\mathrm{x,y},\xi,\eta)+\lambda)^m\partial_\xi^1 D_x^{M-1}b_{0,\lambda}(\mathrm{x,y},\xi,\eta)\\ &-\Gamma(M-2,2)b_{0,\lambda}(\mathrm{x,y},\xi,\eta)\partial_\xi^{M-2}D_x^2(A_P((\mathrm{x,y},\xi,\eta)+\lambda)^m\partial_\xi^2 D_x^{M-2}b_{0,\lambda}(\mathrm{x,y},\xi,\eta)\\ &-\Gamma(1,M-1)b_{0,\lambda}(\mathrm{x,y},\xi,\eta)\partial_\xi^1 D_x^{M-1}(A_P((\mathrm{x,y},\xi,\eta)+\lambda)^m\partial_\xi^{M-1}D_x^1 b_{0,\lambda}(\mathrm{x,y},\xi,\eta).\end{aligned}$$

We have $b_{k,M,\lambda}\in S_{\underline{k},\underline{\ell}}^{-m-(kM)}$ so by doing the previous changement of variables we obtain

$$\int_{\mathbb{R}^{2n}}t_r(b_{kM,\lambda}(\mathrm{x,y},\xi,\eta))\,dxdyd\xi d\eta=\lambda^{\alpha_1}\int_{\mathbb{R}^{2n}}t_r(b'_{kM}(\mathrm{x',y'},\xi',\eta'))\,dx'dy'd\xi'd\eta',$$

where

$$\alpha_1=\alpha_0-\mathrm{kM}$$

and $b'_{kM}(\mathrm{x',y'},\xi',\eta')$is the rest of $b_{kM,\lambda}(\mathrm{x,y},\xi,\eta)$ after the application of changement of variables (12) and taking λ as a factor.

□

By using the proposition 1 and the theorem of Lidskii we can prove the following theorem:

Theorem 1 The operator LP in (9) is defined on $\mathcal{L}^2(\mathbb{R}^n)$with H_0 and H_1 are positive pseudo-differential operators of order M and $\frac{M}{2}$, respectively and are of types (k, ℓ), k = (k_1, k_2), ℓ = (ℓ_1, ℓ_2), if there exists j, j ≥ 1 such that :

$$n_1(k_1 + \ell_1) + n_2(k_2 + \ell_2) = M_j$$

Where $n_1 + n_2 = n.$ So there exists $\lambda_0 \in \mathbb{C}\ et\ u_0 \in \mathcal{L}^2(\mathbb{R}^n),\ u_0 \neq 0$ such that , $L_P(\lambda_0)u_0$=0

1. Example with n = 3

We consider the operator:

$$L_P(\lambda) = D_x^4 - D_y^2 + (P(x,y) - \lambda)^2,$$

with P is a positive quasi-homogenous of degree M and of type (k_1, k_2) i.e. :

$$P(\rho^{k1}x, \rho^{k2}y) = \rho^M \mathrm{P(x,y)},\ x \in \mathbb{R}^2,\ y \in \mathbb{R}.$$

By using the theorem of Lidskii and the proposition (3.1), we can prove the following theorem: **Theorem 2** *The operator L_P of the form*

$L_P(\lambda) = D_x^4 - D_y^2 + (P(x,y) - \lambda)^2, x \in \mathbb{R}^2, y \in \mathbb{R}$,

is defined on $\mathcal{L}^2(\mathcal{R}^3)$ with P is a positive quasi homogenous polynomial of degree M and of type

(k_1, k_2). So there exists $\lambda_0 \in \mathbb{C}$ and $u_0 \neq 0$ such that $L_P(\lambda_0)u_0 = 0$.

2. Example with n = 5

We consider the operator:

$$\mathrm{L}(\lambda) = D_x^4 - D_y^4 + |x|^2 + |y|^4 + \lambda(\sqrt{|x|^2 + |y|^4}) + \lambda^2,$$

where x ∈ $\mathbb{R}^2$, y ∈ $\mathbb{R}^3$. The operator L is associated to the matrix operator $\widehat{A}$ in $\mathcal{L}^2(\mathcal{R}^5) \times \mathcal{L}^2(\mathcal{R}^5)$ such that

$$\hat{A} = \begin{pmatrix} 0 & \widehat{H}_0^{\frac{1}{2}} \\ -\widehat{H}_0^{\frac{1}{2}} & -\widehat{H}_1 \end{pmatrix}$$

where

$$\widehat{H}_0(\mathrm{x,y},D_x, D_y) = D_x^4 - D_y^4 + |x|^2 + |y|^4,$$

$$\widehat{H}_1(\mathrm{x,y},D_x, D_y) = (\sqrt{|x|^2 + |y|^4}.$$

Hence we obtain the matricial operator $\hat{A}$ have the symbol A with

$$H_0\,(x,y,\xi,\eta) = |\xi|^4 + |\eta|^4 + |x|^2 + |y|^4,$$

$$H_1\,(x,y,\xi,\eta) = (\sqrt{|x|^2 + |y|^4},$$

H_0 and H_1 are of order 4 and 2 respectively of the type (k,ℓ) where

$$k = (2, 1)\,,\ \ell = (1, 1).$$

For m > 9 and $\lambda \to +\infty$, the trace of $(\hat{A} + \lambda)^{-m}$ have the following asymptotic form :

$$T_r\left(\hat{A} + \lambda\right)^{-m} \asymp \sum_{j\geq 0} c_{j,-m}(\lambda),$$

$$\asymp \lambda^{\alpha_0} c_0 + \lambda^{\alpha_2} c_2 + \ldots,$$

Where

$$c_0 = -\int_{\mathbb{R}^5} \left(\sqrt{|x|^2 + |y|^4} + 1\right)^{-m+5} dxdy - \int_{\mathbb{R}^5} (\sqrt{|\xi|^2 + |\eta|^4} + 1)^{-m} d\xi d\eta$$

$$c_{2,-m}(\lambda) = \int_{\mathbb{R}^{10}} b_{2,\lambda}(x,y,\xi,\eta) dxdyd\xi d\eta$$

$$\alpha_0 = -m + 6$$

$$\alpha_2 = -m + 4$$

And $c_0 \neq 0$ car $\left(\sqrt{|x|^2 + |y|^4} + 1\right)^{-m+5}$ and $(\sqrt{|\xi|^4 + |\eta|^4} + 1)^{-m}$ are positives. So we use the

proposition 1 and theorem 1 which give the existence of the non trivial solutions for the operator L .

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