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Three Modes Bifurcation Solutions of Nonlinear Fourth Order Differential Equation with Symmetry and Non- Symmetry

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Abstract

In this paper, we will discuss Bifurcation solutions of fourth order nonlinear equation using local method of Lyapunov-Schmidt. In the case when the system is non-homogenous with symmetric, it was shown that Discriminant set (bifurcation diagram) is the union of two surfaces. In the case when the system is non-homogenous with non- symmetric, geometric representation of the discriminant set (Caustic) was discovered, replete with bifurcation distributions of critical points in every region of the plane of parameters.

Key Words: local method of Lyapunov-Schmidt, Nonlinear systems, Dynamic systems.

1. Introduction

Many problems in Physics and Mathematics, equations depending on a parameter arise. The equations may be written as with parameters,can be written in the form of operator equation,

$$l(u,\lambda) = \psi, \qquad u \in O \subset \mathrm{M}, \qquad ,\psi \in \mathrm{E}, \qquad \lambda \in R^n \qquad \dots (1.1)$$

where $\psi(x) = \psi(\pi - x)$ and where l is a smoothing Fredholm map for index zero, M and E are banach spaces and O is an open subset in E. The solutions of this problem can be found by solving the equivalent equation,

$$\Phi_1(\tau,\lambda) = \rho, \quad \tau \in \chi, \qquad \rho \in Y, \qquad \dots (1.2)$$

where χ, y are a smoothing finite dimensional manifold. The reduction of equation (1.1) to the equation (1.2) by local method of Lyapunov-Schmidt (MLS). Has all properties topological and the analytical of equation (1.1) (multiplicity, bifurcation diagram, *etc*) [3,11, 13]

On the method of finite dimensional reducing the solutions of infinite-dimensional spaces coinciding with the solutions of finite-dimensional spaces, this became a major factor in modern mathematics. Nonlinear algebraic equations have a crucial role in physics and engineering, especially mechanical engineering , electrical and computer engineering. It's crucial in numerous other factors such as population growth, economy, etc. Many studies have been conducted on different sorts of bifurcation solutions.

In [6] M. A. Abdul Hussain (2007), studied bifurcation of the periodic solutions of the nonlinear system of algebraic equations of the following:

$$\xi_1(\xi_1^2+\xi_2^2)+2\xi_1(\xi_3^2+\xi_4^2)+q_1(\xi_2\xi_3-\xi_1\xi_4)+q_2\xi_1=0,$$

$$\xi_2(\xi_1^2+\xi_2^2)+2\xi_2(\xi_3^2+\xi_4^2)+q_1(\xi_1\xi_3-\xi_2\xi_4)+q_2\xi_2=0,$$

$$2\xi_3(\xi_1^2+\xi_2^2)+\xi_3(\xi_3^2+\xi_4^2)+\alpha_1\xi_1\xi_2+\alpha_2\xi_3=0,$$

$$2\xi_4(\xi_1^2+\xi_2^2)+\xi_4(\xi_3^2+\xi_4^2)+\beta_1(\xi_1^2-\xi_2^2)+\alpha_2\xi_4=0.$$

And taking the equation of the following :

$$u^4-\frac{12}{r^2}u''+\frac{12}{kr}uu''+\frac{6}{kr}(u')^2-\frac{36(r-k^2/r)}{k^2r^3}u-\frac{54}{kr^3}u^2+\frac{18}{k^2r^2}u^3=0.$$

In [7] M. A. Abdul Hussain (2009), studied bifurcation solutions of the nonlinear system of algebraic equations used the method of Lyapunov-Schmidt reduction of the following:

$$\xi_1^3+2\xi_1\xi_2^2+\lambda_1\xi_1+q_1=0,$$

$$\xi_2^3+2\xi_1^2\xi_2+\lambda_2\xi_2+q_2=0.$$

And taking the equation without symmetry of $\Psi(x)$ as following :

$$\frac{d^4u}{dx^4}+\alpha\frac{d^2u}{dx^2}+\beta u+u^3=\Psi,$$

with the boundary condition: $u(0)=u(\pi)=u''(0)=u''(\pi)=0.$

In [2] B.M. Darinskii et al (2007), studied bifurcation solutions and they found bif-spreadings of the nonlinear system of algebraic equations of the following :

$$\lambda_1\xi_1+\xi_1^3+2\xi_1\xi_2^2=0,$$

$$\lambda_2\xi_2+2\xi_1^2\xi_2+\xi_1^3=0.$$

And taking the equation of the following:

$$\frac{d^4p}{dx^4}+\alpha\frac{d^2p}{dx^2}+\beta p+p^3=0,$$

with the boundary condition: $p(0)=p(\pi)=p''(0)=p''(\pi)=0.$

In [10] M. J. Mohammed (2011), studied the bifurcation solutions using the method of Lyapunov-Schmidt reduction and he founded the bifurcation diagram of the following nonlinear system,

$$\xi_1^3 + 2\xi_1\xi_2^2 + 5r\xi_1^2 + 4r\xi_2^2 + \lambda_1\xi_1 + q_1 = 0,$$

$$\xi_2^3 + 2\xi_1^2\xi_2 + 8r\xi_1\xi_2 + \lambda_2\xi_2 = 0.$$

And taking the equation with symmetry of $\Psi(x)$ as the following:

$$\frac{d^4w}{dx^4} + \alpha\frac{d^2w}{dx^2} + \beta w + w^2 + w^3 = \Psi,$$

with the boundary condition: $w(0) = w(\pi) = w''(0) = w''(\pi) = 0.$

In [9] M. J. Mohammed (2007), he was studding the bifurcation solutions using the method of Lyapunov-Schmidt reduction and he was founding the Discriminant set. Then he calculated the topological indices of the solutions of the nonlinear system of algebraic equations of the following:

$$\eta_1^2\eta_2 + b\eta_2^3 + \widetilde{\lambda_1}\eta_1 - q_1 = 0,$$

$$\eta_1^3 + b\eta_1\eta_2^2 + \widetilde{\lambda_2}\eta_2 = 0.$$

And taking the equation of the following with symmetry of $\Psi(x)$,

$$\mathbb{Z}_{xxxx} + \alpha\mathbb{Z}_{xx} + \beta\mathbb{Z} + \mathbb{Z}^2\mathbb{Z}' = \Psi,$$

with the boundary condition: $\mathbb{Z}(0) = \mathbb{Z}(1) = \mathbb{Z}''(0) = \mathbb{Z}''(1) = 0.$

In [4] I. A. Shanan (2013), studied the bifurcation solutions using the method of Lyapunov-Schmidt reduction and he founded the bifurcation diagram of the following nonlinear system,

$$\frac{d^4u_1}{dx^4} + \lambda_1\frac{d^2u_1}{dx^2} + u_1 + u_1^3 + u_1u_2^2 + u_1^2u_3 + u_3^3 = \psi_1,$$

$$\frac{d^4u_2}{dx^4} + \lambda_2\frac{d^2u_2}{dx^2} + u_2 + u_1^3 + u_1u_3^2 + u_1^2u_2 + u_2^3 = \psi_2,$$

$$\frac{d^4u_3}{dx^4} + \lambda_3\frac{d^2u_3}{dx^2} + u_3 + u_1u_2^2 + u_1u_3^2 + u_2^3 + u_3^3 = \psi_3.$$

And taking the equation of the following with symmetry of $\Psi(x)$,

$$\begin{aligned}
&\tfrac{3}{2\pi}x_1^3 + \tfrac{1}{\pi}x_1x_2^2 - \tfrac{1}{2\pi}x_1^2x_3 + k_1x_1 - q_1 = 0,\\
&\tfrac{1}{\pi}x_1^2x_2 + \tfrac{3}{2\pi}x_2^3 + k_2x_2 = 0,\\
&\tfrac{3}{2\pi}x_3^3 + \tfrac{1}{2\pi}x_1x_2^2 + k_3x_3 - q_3 = 0.
\end{aligned}$$

In [1] A. H. Resan (2019) used the method of Lyapunov-Schmidt reduction to problems:

$$y'''' + \alpha\, y'' + \beta y + yy'' + (y')^2 + y^2 + y^3 = 0.$$

In [12] W. S. Sadiq (2020), studied the bifurcation solutions using the method of Lyapunov-Schmidt reduction and he founded the bifurcation diagram of the following nonlinear system,

$$y_1{}^2 + y_2{}^2 + \lambda_1 y_1 - \lambda_2 y_2 = 0,$$
$$y_1 y_2 + \lambda_3 y_1 + \lambda_4 y_2 = 0.$$

And taking the equation of the following:

$$u_t - u_{tyy} + 3uu_y + 2\mathrm{k}u_y = 2u_y u_{yy} + u u_{yyy}.$$

In [8] M. A. Jassim (2021), she studied the bifurcation solutions using the method of Lyapunov-Schmidt reduction of the following nonlinear equations,

$$\alpha x_1 + \frac{3}{2}x_1^3 + \frac{5}{2}x_1^5 = 0.$$

And taking the equation of the following:

$$\frac{d^4u}{dx^4} + \alpha\frac{d^2u}{dx^2} + u + u^3 + u^5 = 0.$$

In this paper, to find the bifurcation solutions of non-linear differential equations of the fourth order in three parameters we used the local method of Lyapunov-Schmidt,

$$l(u,\lambda) = \alpha\frac{d^4u}{dx^4} + \beta\frac{d^2u}{dx^2} + \sigma u + u\frac{du}{dx} \qquad \text{... (1.3)}$$

with the boundary conditions:

$$u(0) = u(\pi) = u''(0) = u''(\pi) = 0, \qquad \text{... (1.4)}$$

Definition 1.1 [23**]**

Suppose that(E, M are Banach spaces) and a linear continuous operator A: E ⟶ M.

The operator A is called Fredholm operator, if the kernel of A,(ker(A)) and the Cokernel of A,($Coker(A)$) is finite dimensional. The Rang of A (Im(A)), is closed in M. The number dim (Ker A) - dim (Coker A) is called Fredholm index of the operator A.

3.2 Reduction to Bifurcation with Non-Symmetric Case

We will discuss in this section when the system is non-homogenous with non-symmetric of equation (1.3), and using the boundary conditions (1.4), where $F: M \to E$ is nonlinear Fredholm operator of Banach space M to Banach space E form index zero. form Banach space M to Banach space E. $M = C^4([0,\pi], R)$ is the space of all continuous function that have derivative of order at moust four, $E = C^0([0,\pi], R)$, $x \in [0,\pi]$. Is the space of all continuous functions and $u = u(x)$, $x \in [0,\pi]$, $\lambda = (\alpha, \beta, \sigma)$. the bifurcation solutions of equation (1.3) are corresponding to the bifurcation equation (1.1).

Theorem 3.1.

The equation (1.3) has a bifurcation equation that corresponding to it, is given by the following nonlinear system of three equation

$$\Phi_1(\tau,\lambda) = \begin{pmatrix} q_1 x_1 - A_1 x_1 x_2 - A_2 x_2 x_3 - p_1 \\ q_2 x_2 + A_1 x_1^2 - A_2 x_1 x_3 - p_2 \\ q_3 x_3 + A_3 x_1 x_2 - p_3 \end{pmatrix} + o(|\tau|^3) + O(|\tau|^3) O(\gamma) = \psi, \quad \dots (1.5)$$

where $p_1, p_2, p_3, q_1, q_2, q_3 \in R$, the parameters equation $\tau = (x_1, x_2, x_3)$,

$\lambda = (p_1, p_2, p_3, q_1, q_2, q_3) \in R$, $\gamma = (\delta_1, \delta_2, \delta_3)$

Therefore, the solutions of the equation in (1.3) correspond one-to- one to the solutions of nonlinear system.

Proof:

Let $l(u,\lambda) = \psi$, $where \ \psi \in M$

So, at the point $(0,\lambda)$ the Frećhet derivative of the nonlinear operator $l(u,\lambda)$ is

$$l(u+th,\lambda) = \alpha \frac{d^4(u+th)}{dx^4} + \beta \frac{d^2(u+th)}{dx^2} + \sigma(u+th) + (u+th)\frac{d(u+th)}{dx},$$

$$df(0,\lambda)h = \alpha \frac{d^4}{dx^4} h + \beta \frac{d^2}{dx^2} h + \sigma h,$$

as a result, the equation's linearized equation (1.5). Is given by,

$$\mathcal{A}h = 0\,; \qquad h \in E,$$

$$\mathcal{A} = \frac{\partial}{\partial t} f(0,\lambda) = \alpha \frac{d^4}{dx^4} + \beta \frac{d^2}{dx^2} + \sigma, \qquad x \in [0,\pi]$$

$$u(0) = u(\pi) = u''(0) = u''(\pi) = 0.$$

By substituting in the linearized equation, we get a characteristic equation to which corresponds to this solution.

$$\alpha m^4 + \beta\, m^2 + \sigma = 0.$$

This equation gives in the plan of parameters (α, β, σ) characteristic lines l_{P}. The characteristic lines l_{P} consist of the points (α, β, σ) for which the linearized equation has non-zero solutions. The point of intersection of characteristic planes in the space of parameters (α, β, σ) is a bifurcation point, cause a bifurcation of the modes,

$$e_{\text{P}} = c_{\text{P}}\, sin\, sin\, (\text{P}x)\,, \quad \text{P} = 1,2,3, \dots$$

$$e_1 = c_1\, sin\, sin\, (x)\,, \qquad e_2 = c_2\, sin\, sin\, (2x), \quad e_3 = c_3\, sin\, sin\, (3x)$$

Where $\|e_1\| = \|e_2\| = \|e_3\| = 1.$

$$c_1 = c_2 = c_3 = \sqrt{\frac{2}{\pi}},$$

It is easy to see that for $n, m = 1,2,3$ and

$$\langle e_n, e_m \rangle = \int_0^{\pi} sin(nx)\, sin(mx)\, dx = \begin{cases} 1\; if\;\; n = m \\ 0\; if\; n \neq m \end{cases},$$

this mean that e_1, e_2 and e_3 are orthonormal basis of the null space $Ker(A)$.

Let $N = Ker(A) = span\{e_1, e_2, e_3\}$, then the space M can be decomposed in direct sum of two subspaces, N and the orthogonal complement to N,

$$M = N \oplus N^{\perp}, \qquad N^{\perp} = \left\{ Z\, \in E\colon \int_0^1 Z\, e_{\text{P}}\; dx = 0, \text{P} = 1, 2, 3 \right\}.$$

In the same way, the space M can be decomposed into the direct sum of two subspaces $\breve{N}$ and orthogonal complement $\breve{N}$,

$$E = \breve{N} \oplus \breve{N}^{\perp},$$

$$\breve{N}^{\perp} = \left\{ \acute{\omega}\, \in M\colon \int_0^{\pi} \acute{\omega}\, e_{\text{P}}\; dx = 0,\;\; wherever, \text{P} = 1,2,3 \right\}.$$

There exist two projections $P\colon E\; \to N$ also, $(I - P)\colon E\; \to N^{\perp}$, such that

$$Pu = \breve{V} \quad \text{and } (I - P)\, u = \breve{Z}.$$

Hence every vector $u \in E$ can be written in the unique form,

$$u = \breve{V} \;\; + \breve{Z},\; \breve{V} \;\; = \sum_{i=1}^{3} x_i e_i \in N,\; \breve{Z} \in N^{\perp},\; x_i = \langle u\,, e_i \rangle.$$

Similarly, there exist two projections

$$Q: M \to \breve{N} \text{ and } (I - Q): M \to \breve{N}^{\perp},$$

such that

$$L_1(u, \lambda) = QL(u, \lambda),$$

$$L_2(u, \lambda) = (I - Q)L(u, \lambda).$$

And hence,

$$L(u, \lambda) = L_1(u, \lambda) + L_2(u, \lambda)$$

$$= QL(u, \lambda) + (I - Q)F(u, \lambda),$$

$$L_1(u, \lambda) = QF(u, \lambda) = \sum_{i=1}^{3} v_i(u, \lambda)e_i \in \breve{N}, \ \ when, \ \ v_i(u, \lambda) = \langle L(u, \lambda), e_i \rangle,$$

$$L_2(u, \lambda) = (I - Q)L(u, \lambda) \in \breve{N}^{\perp}.$$

Accordingly, equation (1.5) can be written in the form,

$$QL(u, \lambda) = \psi_1,$$

$$(I - Q)L(u, \lambda) = \psi_2.$$

Whereas $\psi = \psi_1 + \psi_2$

Or
$$QL(\breve{V} + \breve{Z}, \lambda) = \psi_1,$$

$$(I - Q)L(\breve{V} + \breve{Z}, \lambda) = \psi_2.$$

By using Implicit Function Theorem (IFT)[14], there exists a smooth map $\Theta: N \to N^{\perp}$ (relies on λ) such that $\Theta\,(u, \lambda) = \breve{Z}$ and,

$$L_2(u, \lambda)\left(\breve{V} + \Theta(\breve{V}, \lambda), \lambda\right) = \psi.$$

To find the solution for (1.3) in the neighbourhood of the point $u = 0$, it is sufficient to find the solutions of the equation

$$L_1(u, \lambda)\left(\breve{V} + \Theta(\breve{V}, \lambda), \lambda\right) = \psi. \qquad \dots (1.6)$$

Equation (1.6) is called (bifurcation equation) for equation (1.3). So, the bifurcation equation is $\Phi(\tau, \lambda) = \psi$,

$\tau = (x_1, x_2, x_3, , p_1, p_2, p_3)$, $\lambda = (\alpha, \beta, \sigma)$

$$t_1\, e_1 = \langle \psi^*, e_1 \rangle = \langle \psi^*, \left(\sqrt{\frac{1}{2}} \sin x\right) \rangle = \int_0^{\pi} \psi^* \sqrt{\frac{1}{2}} \sin x \quad dx = p_1,$$

$$t_2 e_2 = \langle \psi^*, e_2 \rangle = \langle \psi^*, \sqrt{\frac{2}{\pi}} \sin(x) \rangle = \int_0^{\pi} \psi^* \sqrt{\frac{2}{\pi}} \sin(x) = p_2,$$

$$t_3\, e_3 = \langle \psi^*, e_3 \rangle = \langle \psi^*, \left(\sqrt{\frac{1}{2}} \sin x \right) \rangle = \int_0^{\pi} \psi^* \sqrt{\frac{1}{2}} \sin x \quad dx = p_3,$$

where

$$\Phi(\tau, \lambda) = L_1\big(\breve{V} + \Theta(\breve{V}, \lambda), \lambda\big),$$

can be written Equation (1.5) in the form,

$$F\big(\breve{V} + \breve{Z}, \lambda\big) = A\big(\breve{V} + \breve{Z}\big) + K\big(\breve{V} + \breve{Z}\big).$$

Simple calculations show that

$$K\big(\breve{V} + \breve{Z}\big) = K\breve{V} + terms\ consists\ elements\ of\ \breve{Z}.$$

Hence,

$$L\big(\breve{V} + \breve{Z}, \lambda\big) = A\breve{V} + K\breve{V} + \cdots$$

Where the dots denote the terms that consist of the element Z . Hence,

$$\Phi\ (\tau, \lambda) = QL\big(\breve{V} + \breve{Z}, \lambda\big) = \sum_{i=1}^{3} \langle A\breve{V} + K\breve{V}, e_i \rangle + \cdots = \psi. \qquad \text{... (1.7)}$$

By using the inner product $\langle .,. \rangle_H$ in Hilbert space $\mathcal{L}_2([0, \pi], R)$ and by some calculations of equation (1.7), implies that,

$$\langle A\breve{V} + K\,\breve{V}, e_1 \rangle\, e_1 + \langle A\,\breve{V} + K\,\breve{V}, e_2 \rangle e_2 + \langle A\,\breve{V} + K\,\breve{V}, e_3 \rangle\, e_3 = \psi \qquad \text{... (1.8)}$$

By using the properties of the inner product $\langle .,. \rangle_H$, Simple calculations of equation (1.8) implies that,

$$\langle A\breve{V} + K\,\breve{V}, e_1 \rangle = -A_1 x_1 x_2 - A_2 x_2 x_3 + \mathrm{q}_1 x_1 - p_1,$$

$$\langle A\breve{V} + K\,\breve{V}, e_2 \rangle = A_1 x_1^2 - A_2 x_1 x_3 + \mathrm{q}_2 x_2 - p_2,$$

And,

$$\langle A\breve{V} + K\,\breve{V}, e_3 \rangle = A_3 x_1 x_2 + \mathrm{q}_3 x_3 - p_3.$$

Where,

$$A_1 = \frac{1}{\sqrt{2\pi}}, A_2 = \sqrt{\frac{2}{\pi}}, A_3 = \frac{5}{\sqrt{2\pi}},$$

$$\mathrm{q}_1 = Ae_1 = \xi_1(\lambda) e_1, \quad \mathrm{q}_2 = Ae_2 = \xi_2(\lambda) e_2, \quad \mathrm{q}_3 = Ae_3 = \xi_3(\lambda) e_3,$$

$$(-A_1x_1x_2 - A_2x_2x_3 + q_1x_1 - p_1)e_1 + (A_1x_1^2 - A_2x_1x_3 + q_2x_2 - p_2)e_2 + (A_3x_1x_2 + q_3x_3 - p_3)e_3 + \cdots = 0.$$

By equating coefficients of e_1, e_2 *and* e_3 we have,

$$\Phi_1(\tau,\lambda) = \begin{pmatrix} q_1x_1 - A_1x_1x_2 - A_2x_2x_{3-}p_1 \\ q_2x_2 + B_1x_1^2 - B_2x_1x_3 - p_2 \\ q_3x_3 + Cx_1x_2 - p_3 \end{pmatrix} + o(|\tau|^3) + O(|\tau|^3)O(\gamma) = 0. \qquad \dots (1.9)$$

From theorem (1.1) we found Discriminate set (bifurcation diagram) of bifurcation set of equation (1.9) and we draw the p_1p_3 -plane of q_1, q_2, q_3. As shown in (the fig.1.1), (the fig.1.2) and (the fig.1.3). (all these figures were found by using the Maple program.

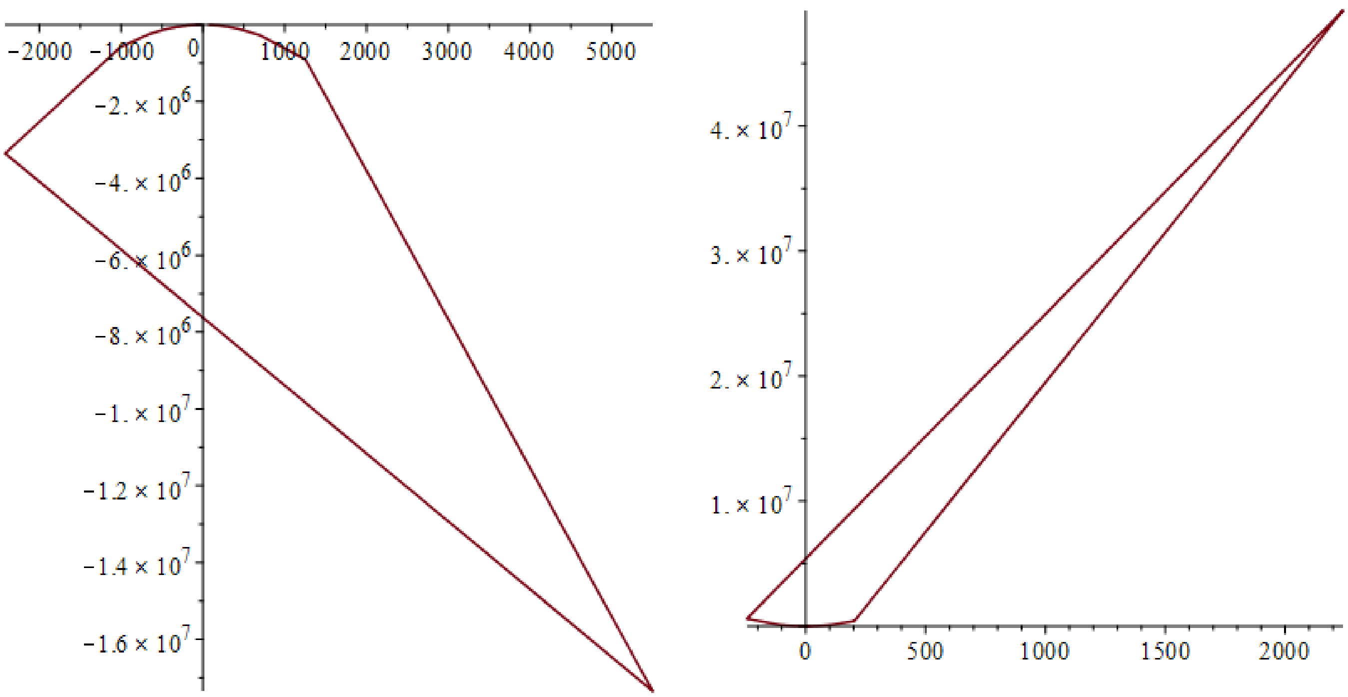

Fig. 1.1 Describe the Discriminate set of equation $\Phi_1(\tau,\lambda) = 0$ when $q_1 = 3, q_2 = 8, q_3 = 9$

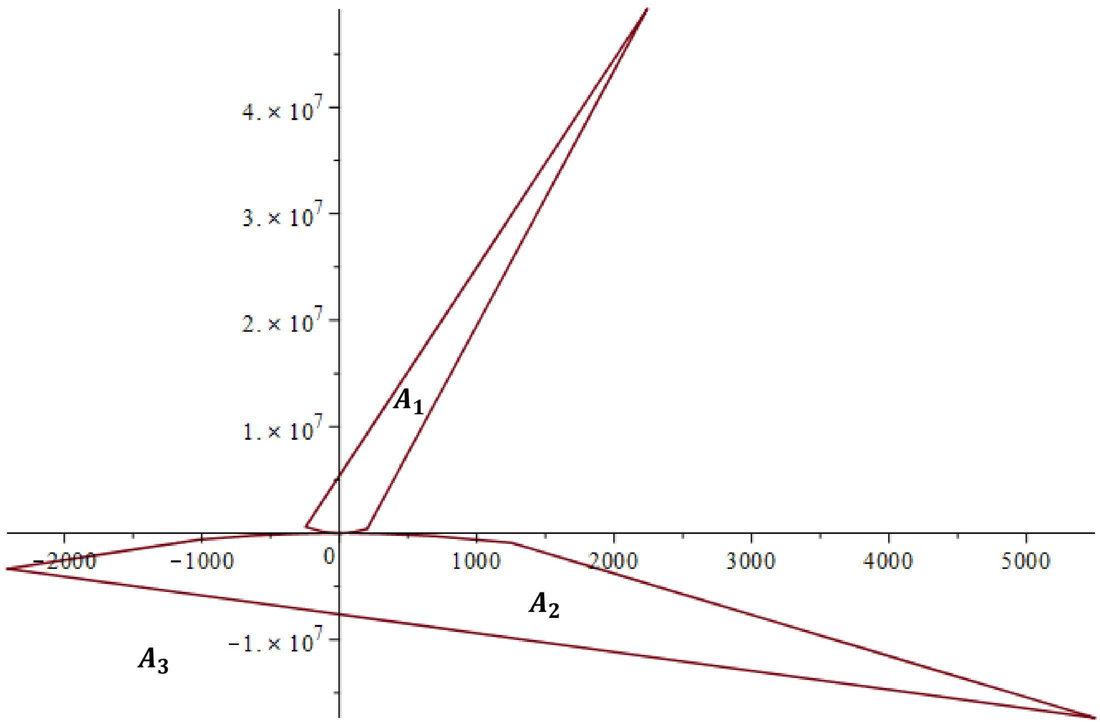

Fig. 1.2 Describe the union Discriminate set of equation $\Phi_1(\tau, \lambda) = 0$ when $q_1 = 3, q_2 = 8, q_3 = 9$

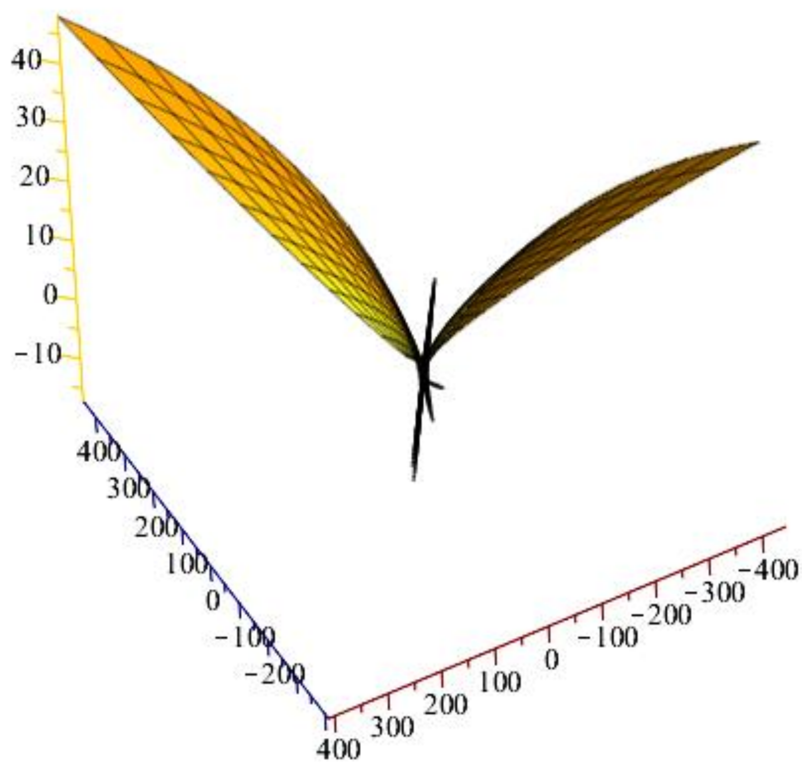

Fig. 1.3 Describe the Discriminate set of equation $\Phi_1(\tau, \lambda) = 0$ when $q_1 = -0.1, q_2 = -0.2, q_3 = 0.3$

The Discriminate set (bifurcation set) deteriorate the solution set in regions, A_1 , A_2 , $A_{3,}$ we have only one regular solutions, where 1,1,1, topological indices.

3.2. Reduction to Bifurcation with Symmetric Case

In this section, we will discuss when the system is non-homogeneous and symmetric,

$$\alpha\frac{d^4u}{dx^4}+\beta\frac{d^2u}{dx^2}+\sigma u+u\frac{du}{dx}=\psi,$$

with boundary condition

$$u(0)=u(\pi)=u''(0)=u''(\pi)=0.$$

Using the same method as in section (3.1) in the following from, we have the bifurcation equation corresponding to the above system,

$$(\mathrm{q}_1x_1-\frac{1}{\sqrt{2\pi}}x_1x_2-\sqrt{\frac{2}{\pi}}x_2x_3)e_1=t_1e_1,$$

$$\left(\mathrm{q}_2x_2+\frac{1}{\sqrt{2\pi}}x_1^2-\sqrt{\frac{2}{\pi}}x_1x_3\right)e_2=t_2e_2,$$

$$\left(\mathrm{q}_3x_3+\frac{5}{\sqrt{2\pi}}x_1x_2\right)e_3=t_3e_3.$$

The symmetry of the function $\psi(x)$ with respect of the in volition,

I: $\psi(x)\to\psi(1-\pi)$ implist that $t_2=0$,

$$t_1e_1=\langle\psi^*,e_1\rangle=\langle\psi^*,\sqrt{\frac{2}{\pi}}\sin(x)\rangle=\int_0^\pi\psi^*\sqrt{\frac{2}{\pi}}\sin(x)=p_1,$$

$$t_3e_3=\langle\psi^*,e_3\rangle=\langle\psi^*,\sqrt{\frac{2}{\pi}}\sin(x)\rangle=\int_0^\pi\psi^*\sqrt{\frac{2}{\pi}}\sin(x)=p_3.$$

Following that, we get:

$$\left.\begin{array}{c} \dfrac{-x_1x_2}{\sqrt{2\pi}} - \sqrt{\dfrac{2}{\pi}}x_2x_3 + q_1x_1 = p_1 \\ \dfrac{x_1^2}{\sqrt{2\pi}} - \sqrt{\dfrac{2}{\pi}}x_1x_3 + q_2x_2 = 0 \\ \dfrac{5x_1x_2}{\sqrt{2\pi}} + q_3x_3 = p_3 \end{array}\right\}. \qquad \dots (1.10)$$

The bifurcation equation of the equation (1.10) is equivalent in the neighborhood of point zero to the equation,

$$\Phi_2(\tau,\lambda) = \begin{pmatrix} q_1x_1 - A_1x_1x_2 - A_2x_2x_{3-}p_1 \\ q_2x_2 + B_1x_1^2 - B_2x_1x_3 \\ q_3x_3 + Cx_1x_2 - p_3 \end{pmatrix}. \qquad \dots (1.11)$$

Where $q_1, q_2, q_3, p_1, p_3 \in R$, the parameters equation can be found by solving the equation (1.11) in terms of x_1, x_2, x_3 and then substituting the results in the equation

$$det\left(\frac{\partial\phi_2(\tau,\lambda)}{\partial x_i}\right) = 0, \qquad i = 1,2,3. \qquad \dots (1.12)$$

We found Discriminate set (bifurcation set) of equation (1.11). As shown in (the fig.1.4), (the fig.1.5) and (the fig.1.6). (all these figures were found by using the Maple program to draw the p_1p_3 -plane for some values of q_1, q_2, q_3.

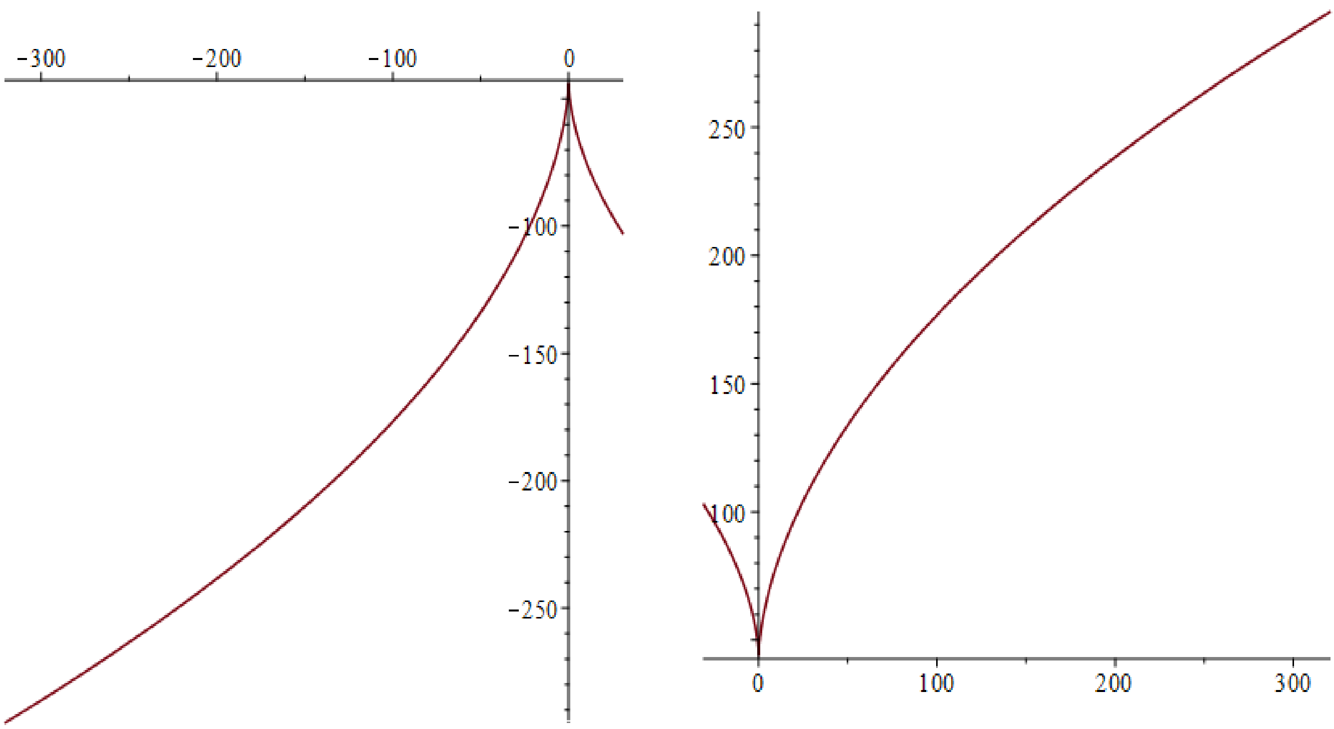

Fig. 1.4 Describe the Discriminate set of equation $\Phi_2(\tau,\lambda) = 0$ when $q_1 = 3, q_2 = 8, q_3 = 9$

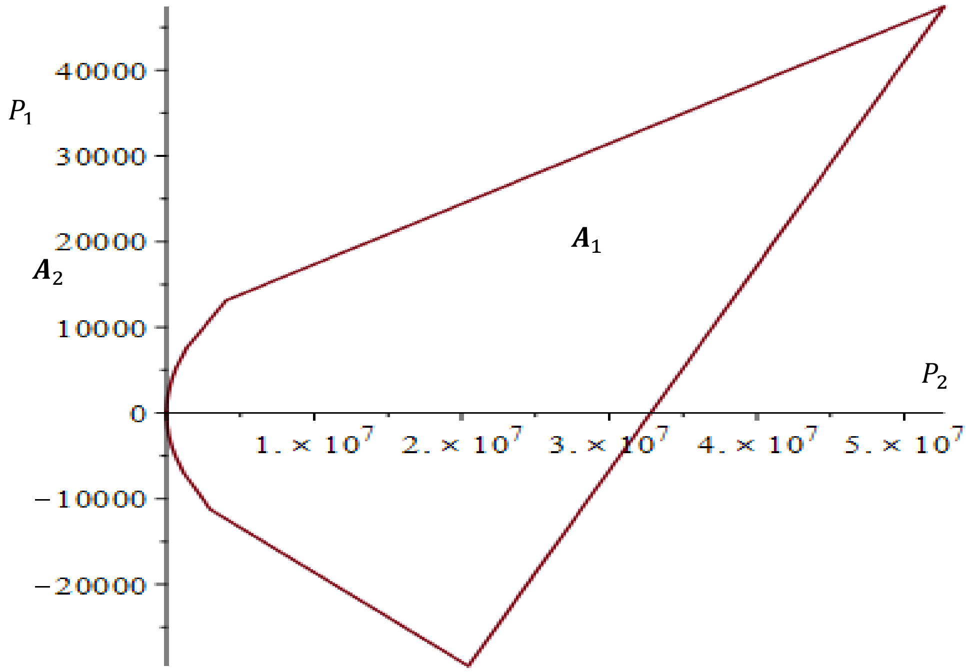

Fig. 1.5 Describe the union Discriminate set of equation $\Phi_2(\tau,\lambda)=0$

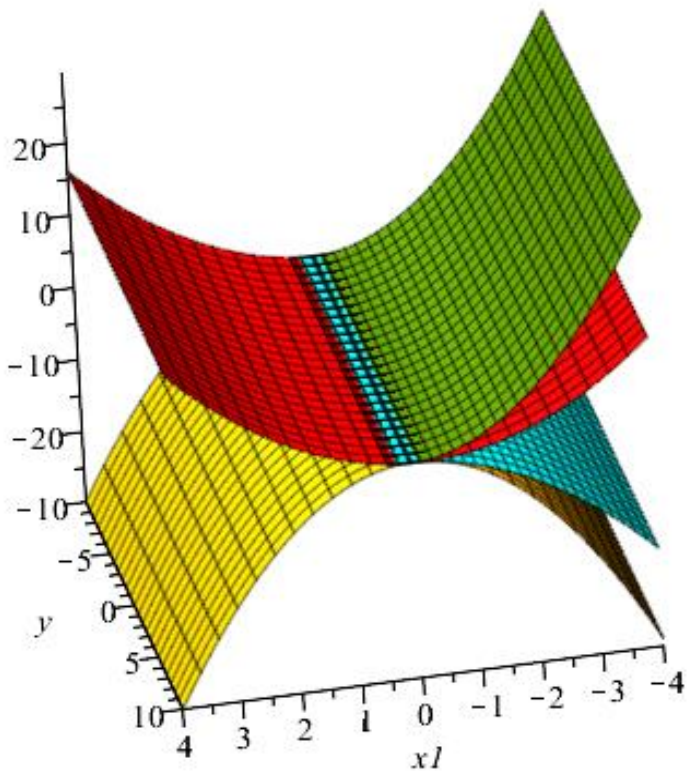

Fig. 1.6 Describe the Discriminate set of equation $\Phi_2(\tau,\lambda)=0$ p_1p_2-plan when $q_1=-0.1, q_2=-0.2, q_3=0.3, p_3=0.5$

Fig. (1.5) the Discriminate set decomposes the solution set into regions, A_1, A_2 we have two regular solutions with topological indices: +1, +1.

3.3 Application to dynamical system: Phase Portrait

3.3.1. Phase portrait In the case non-symmetry

The solutions of (1.9) represent equilibrium points for the below system:

$$\left.\begin{aligned} \dot{x}_1 &= \frac{-x_1x_2}{\sqrt{2\pi}} - \sqrt{\frac{2}{\pi}}x_2x_3 + q_1x_1 - p_1 \\ \dot{x}_2 &= \frac{x_1^2}{\sqrt{2\pi}} - \sqrt{\frac{2}{\pi}}x_1x_3 + q_2x_2 - p_2 \\ \dot{x}_3 &= \frac{5x_1x_2}{\sqrt{2\pi}} + q_3x_3 - p_3 \end{aligned}\right\}. \qquad \dots (1.13)$$

We can classify the equilibrium points of system (1.13), in the fig. (1.2). corresponding to the conduct of (1.9), at the real results. By Fixing the values of $q_1 = 3, q_2 = 8, q_3 = 9\ and\ p_3 = 0.5$, in every set A_i: $i = 1,2,3$ and by testing the eigenvalues of the jacobian matrix of system (1.13) at every equilibrium point in A_i we can to determine the stability and the type of every equilibrium point in these sets.

In the set A_1 we choose $q_1 = 1.763$ and $q_2 = 12.723$. for these values we have the following one real solution (equilibrium point) is one real solution (equilibrium point) of system (1.13). $p = \{0.8824773334, 1.060396551\}$. For these value we have one equilibrium point is (Node /unstable).

After testing in the set A_2 we choose $q_1 = -0.512$ and $q_2 = 7.043$. for these values we have the following one real solution (equilibrium point) is one real solution (equilibrium point) of system (1.13). $p = \{-0.2023046841, 0.7279458457\}$. For these value we have one equilibrium point is (Saddle /unstable).

After testing in the set A_3 we choose $q_1 = -5.483$ and $q_2 = -8.575$, for these values we have the following one real solution (equilibrium point) of system (1.13).

$p = \{1.763803048, -0.1041386250\}$. For these value we have one equilibrium point is (Node /unstable). In each set A_i: $i = 1,2,3$ is given below. All figures has been drawn by the Mathematica program.

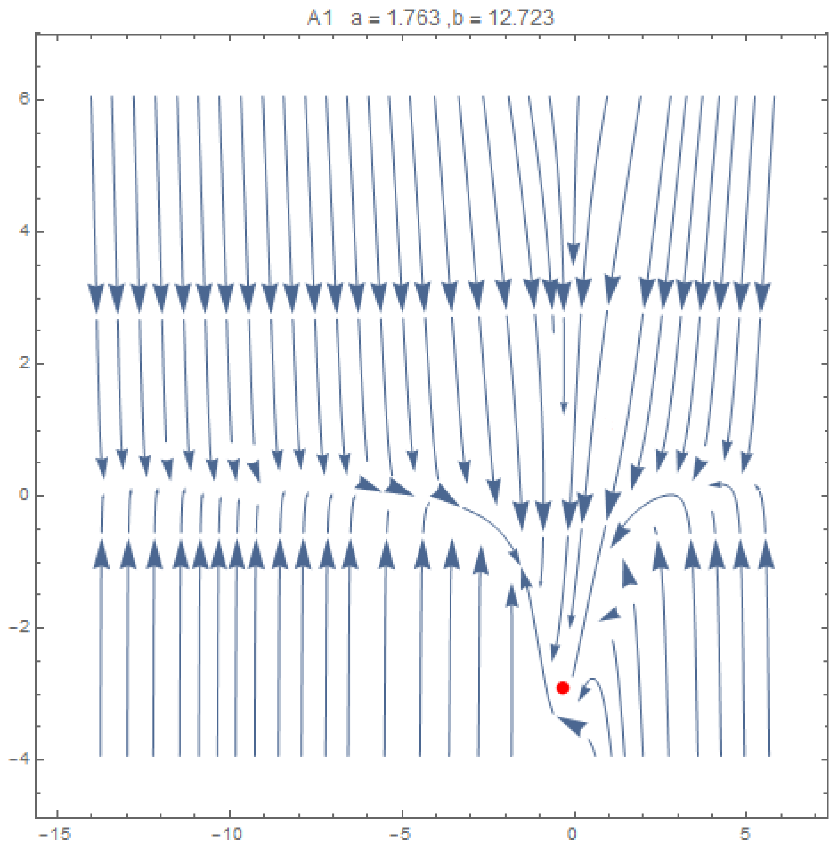

Fig.1.7. The phase portrait for set $\boldsymbol{A_1}$

$A_2\ a = -0.512, b = 7.04$

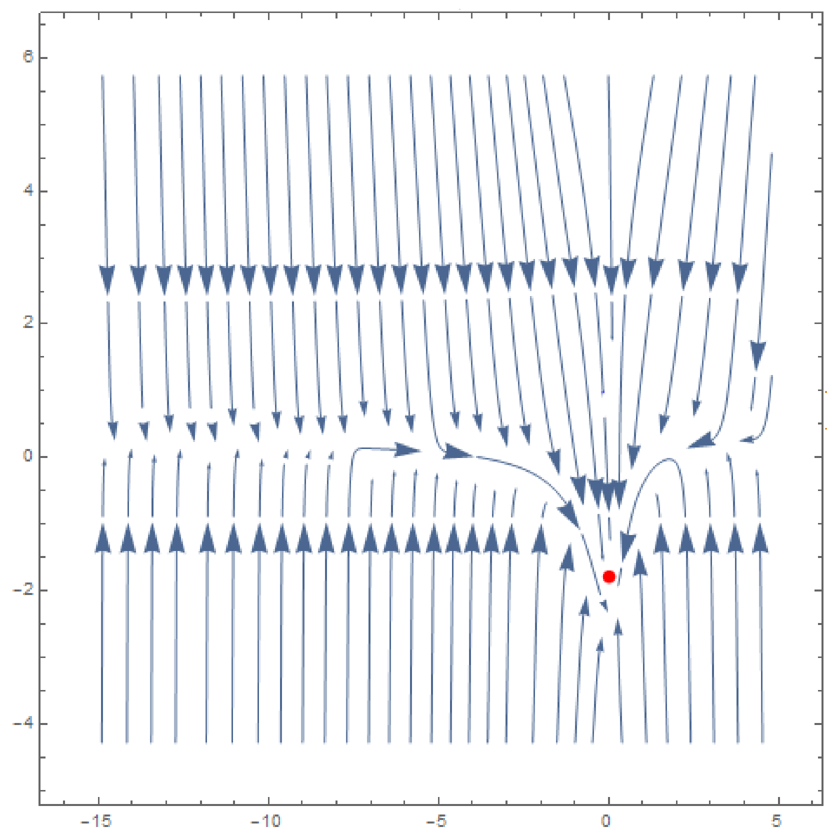

Fig.1.8. The phase portrait for set $\boldsymbol{A_2}$

$A_3\ a = -5.483, b = \ -8.575$

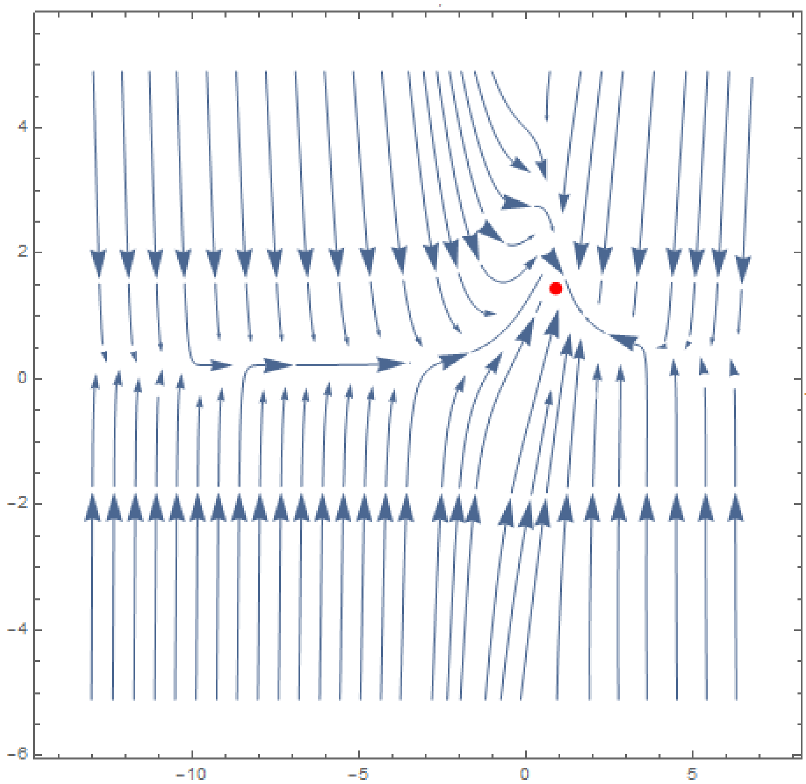

Fig.1.9. The phase portrait for set $\boldsymbol{A_3}$

3.3.2. Phase portrait In the symmetry case

The solutions of (1.10) represent equilibrium points for the below system:

$$\left.\begin{aligned} \dot{x}_1 &= \frac{-x_1x_2}{\sqrt{2\pi}} - \sqrt{\frac{2}{\pi}}x_2x_3 + q_1x_1 - p_1 \\ \dot{x}_2 &= \frac{x_1^2}{\sqrt{2\pi}} - \sqrt{\frac{2}{\pi}}x_1x_3 + q_2x_2 \\ \dot{x}_3 &= \frac{5x_1x_2}{\sqrt{2\pi}} + q_3x_3 - p_3 \end{aligned}\right\}. \qquad \ldots(1.14)$$

We can classify the equilibrium points of system (1.14), in the fig. 1.5. corresponding to the conduct of (1.12), at the real results. By Fixing the values of $q_1 = 3, q_2 = 8, q_3 = 9\ and\ p_3 = 0.5$, in every set $A_1\ and\ A_2$ and by testing the eigenvalues of the jacobian matrix of system (1.14) at every equilibrium point in A_1 *and* A_2 we can to determine the stability and the type of every equilibrium point in these sets.

In the set A_1 we choose $q_1 = 4.844$ and $q_2 = 0.005$. for these values we have the following one real solution (equilibrium point) is one real solution (equilibrium point) of system (1.14). $p = \{1.47977432, 0.0442151379\}$. For these value we have one equilibrium point is (Node /unstable).

After testing in the set A_2 we choose $q_1 = -8.123$ and $q_2 = 0.002$. for these values we have the following one real solution (equilibrium point) is one real solution (equilibrium point) of system (1.14). $p = \{-2.240826018, -0.1458799357\}$. For these value we have one equilibrium point is (Saddle /unstable).

In each set A_1 *and* A_2 is given below. All figures has been drawn by the Mathematica program

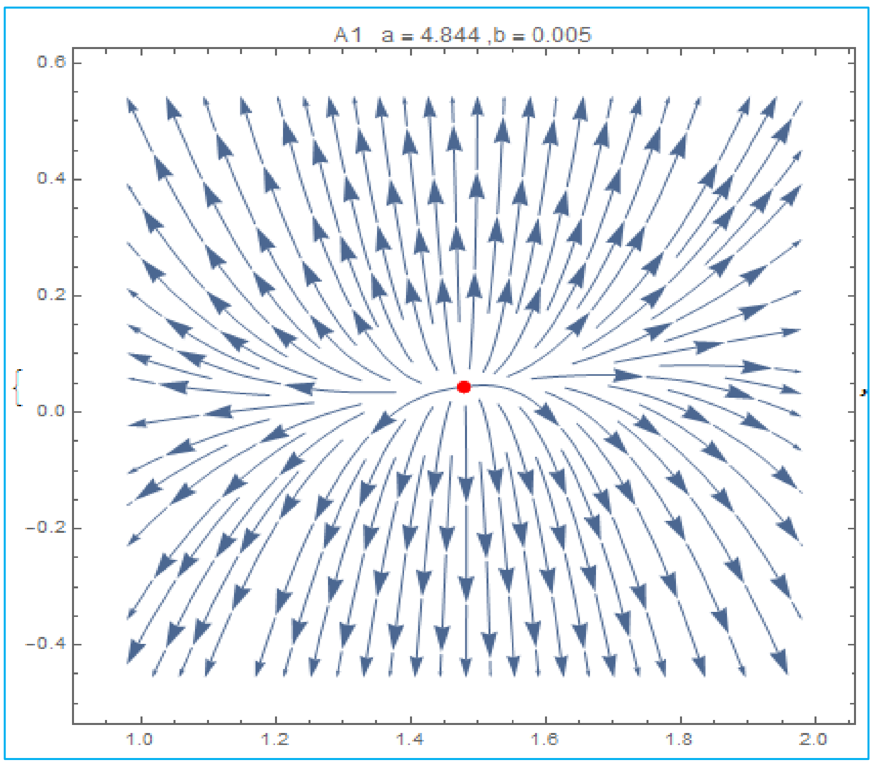

Fig.3.10. The phase portrait for set $\boldsymbol{A_1}$

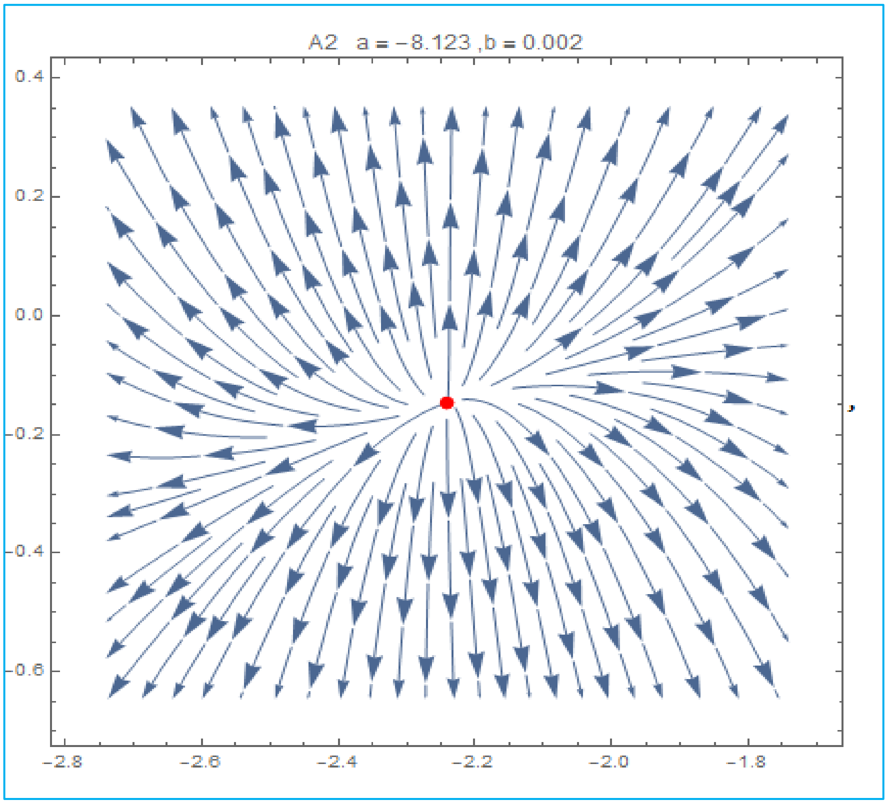

Fig.3.11. The phase portrait for set A_2

المستخلص

درسنا في هذا البحث حلول التفرع للمعادلة التفاضلية اللاخطية باستخدام طريقة ليبانوف – شمدت المحلية في حالتي التناظر وعدم التناظر. عند تخفيض النظام حصلنا على نظام من ثلاث معادلات جبرية لاخطية ، كذلك تم ايجاد وصف هندسي للمجموعة المميزة مع انتشار الحلول المنتظمة للنظام ورسمنا صورة الطور(The phase portrait) لكلا الحالتين.

زهراء عبدالله شاوي مظهر عبدالواحد عبد الحسين

جامعة ألبصرة- كلية التربية للعلوم ألصرفة- قسم الرياضيات

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