On Rings Whose Principal Ideals are Pure

Shaimaa H. Ahmad
College of Computer sciences and Mathematics
University of Mosul

Received on: 13/7/2005 **Accepted on: 9/10/2005**

الملخص

في هذا العمل درسنا الحلقات التي يكون فيها كل مثالي خاص هو مثالي نقي أيمن . كما أعطينا بعض الخواص لهذه الحلقة اليمنى من النمط PIP ثم وضحنا العلاقة بينها وبين حلقات القسمة .

ABSTRACT

In this work, we study rings whose every principal ideal is a right pure. We give some properties of right PIP – rings and the connection between such rings and division rings.

1. Introduction:

Throughout this paper , R will denote associative ring with identity . We recall that:

1) For any element a in R , we define the right annihilator of a by $r(a)=\{x\in R : ax=0\}$, and likewise the left annihilator $l(a)$.
2) Y , Z , J will denote respectively the right singular ideal , left singular ideal and Jacobson radical of R .
3) Following [2] a ring R is called reduced if R has no non – zero nilpotent elements .
4) R is called uniform if every non – zero ideal of R is essential, see [3].

2- PIP – Rings:

Following [1], an ideal I is said to be a right (left) pure if for every $a\in I$, there exists $b\in I$, such that $a=ab(ba)$.

Definition 2.1:

A ring R is said to be a right PIP – ring , if every principal ideal is pure .

Example:

The ring Z_6 is a PIP – ring.

Lemma 2.2:

Let R be a right PIP – ring. Then R is a reduced ring.

Proof:

Let $a \in R$, such that $a^2 = 0$. Since R is PIP – ring , then every principal ideal is pure and hence there exists $b \in aR$. Such $a = ab$, where $b = ar$, for some $r \in R$. Therefore $a = aar$, hence $a = a^2 r$, whence $a = 0$. Therefore R is a reduced ring .

Proposition 2.3:

Let R be PIP – ring . Then $Z(R) = 0$.

Proof:

Suppose that $\mathrm{Z}(\mathrm{R}) \neq 0$. Then by Lemma 7 [3] , there exists $0 \neq \mathrm{a} \in \mathrm{Z}(\mathrm{R})$. Such that $a^2 = 0$. Since R is a PIP – ring then $a = aar$, for some $r \in R$. In fact $Ra \cap l(ar) = 0$, since $a \in Z(R)$, then $l(ar)$ is essential left ideal . Therefore $Ra = 0$, where $a = 0$, a contradiction . Therefore $Z(R) = 0$.

Proposition 2.4:

Let R be PIP – ring, then $J(R) = 0$.

Proof:

Let $a \in J(R)$, since every principal ideal is pure then there exists $b = ar \in J(R)$ such that $a = aar$. For $u \in R$ such that $(1-b)u = 1$ then $a(1-b)u = a$ so $a = 0$ thus $J(R) = 0$.

3- The connection between PIP – rings and other rings:

In this section we study the connection between PIP – rings and division rings, semi ring – ring.

Theorem 3.1:

Let R be a right PIP – ring, without zero – divisors. Then R is a division ring.

Proof:

Let a be a non – zero element of R . Since R is PIP – ring , then every principal ideal is pure . There exists $b \in aR$, such that $a = ab$, where $b = ar$ for some $r \in R$. Therefore $a = aar$, whence $(1-ar) \in r(a) = 0$, so $1 = ar$. Now since $r(a) = l(a)$ (Lemma 2 – 2) , so $a = ara$, gives $a(1-ra) = 0$, so $1 = ra$.
Therefore a is invertible , whence R is a division ring .

The next result considers other conditions for a right PIP – ring to a division ring.

Theorem 3.2:

Let R be a right uniform, PIP – ring. Then R is a division ring.

Proof:

Let a be a non – zero element of R . As in (Theorem 3 – 1) . $a = aar$, for some $r \in R$. Since R is right uniform, so every right ideal is an essential ideal. In fact $Ra \cap l(ar) = 0$, let $x \in Ra \cap l(ar)$ implies that $x = sa$ and $xar = 0$ for some $s \in R$, whence $saar = 0$ yielding $sa = 0$, so $x = 0$. Therefore $Ra \cap l(ar) = 0$ implies that $l(ar) = 0$, on the other hand, since $a = aar$ then $ra = raar$. If we set $e = ra$, note that e is an idempotent element of R, this implies that $ar = arar$,and hence $(1-ar)ar = 0$, this implies that $1 - ar \in l(ar) = 0$. Therefore $ar = 1$ and hence a is a right inevitable. Now since $ar = 1$, we have $ara = a$, which implies that .$(1-ra) \in r(a)$ Since R is reduced, we get $1 - ra \in r(a) = r(ar) = 0$. Therefore $1 - ra = 0$. Whence $ra = 1$, so a is a left inevitable. Hence R is a division ring.

Recall that a ring R is said to be semi prime if it has 0 as the only nilpotent ideal.

Proposition 3.3:

Let R be a right PIP – ring. Then R is a semi – prime ring.

Proof:

Let I be a non – zero right ideal of R such that $I^2 = 0$. Let $a \in I$ and R is PIP – ring, this implies that every principal ideal is pure, hence there exists $b \in aR$ such that $a = ab$ where $b = ar$, for some $r \in R$, therefore $a = aar \in I^2 = 0$, a contradiction . Therefore R is a semi – prime ring.

Theorem 3.4:

Let R be a right PIP – ring with $aR = Ra$, and without zero divisors then there exists a unit element $u \in R$, and idempotent element $e \in R$, such that $a = eu = ue$ and $a = (1-e)+u$.

Proof:

Let $0 \neq a \in R$ and since R is PIP – ring then $a = aar = a^2r$ and $a^2r = ra^2$ (without zero divisors). If we set $e = ar$, that e is an idempotent element of R, this implies that:

$a = ae = ea = ara$, let $u = a + e - 1$, then :

$$eu = e(a+e-1) = ea + e^2 - e = ea = a$$

$$ue = (a+e-1)e = ae + e^2 - e = ae = a$$

Since u is a unit, then there exists $v = ar + e - 1$

$$\begin{aligned} uv &= (a+e-1)(ar+e-1) = a^2r + a(e-1) + (e-1)ar + (e-1)^2 \\ &= e + ae - a + ear - ar + 1 - e \\ &= ar - ar + 1 = 1 \end{aligned}$$

and , $vu = (ar+e-1)(a+e-1) = 1$

Therefore $a = eu = ue$

Now, $(1-e)+u = 1 - e + a + e - 1 = a$

Thus $a = (1-e)+u$.

REFERENCES

[1] Al-Ezeh H. (1989) "Pure ideals in commutative reduced Gelfand rings with unity", **Arch. Math**. Vol.53, pp. 266 – 269.

[2] Yue Chi Ming (1983) "Maximal ideals in regular rings", Hokkaldo. **Math. Jour**. Vol.12, pp. 119 – 128.

[3] Yue Chi Ming (1983) "On quasi – injectivity and Von Neumann regularity", **Manatshefte, Math**. (95) , pp 25 – 32 .

[4] Yue Chi Ming (1986) "On semi prime and reduced ring", **Riv Math. Univ. Parma**. No.4, Vol. 12, pp.167 – 175.