

## On Rings whose Maximal Ideals are GP-Ideals

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### الملخص

يقدم هذا البحث تحديدا لمفهوم المثاليات المعممة العظمى وتمت دراسة الحلقات التي تكون فيها المثاليات العظمى معممة. كما تم تعريف هذه الحلقات على أنها من النمط MRGP وتم عرض بعض خواصها وعلاقتها مع الحلقة المنتظمة بقوة ، حلقة كاش وحلقة القسمة.

### ABSTRACT

This paper introduces the notion of maximal GP-ideal .We studied the class of rings whose maximal left ideal are right GP-ideal. We call such ring MRGP-rings. We consider a necessary and sufficient condition for MRGP-rings to be MRCP-rings. We also study the connection between MRGP-ring, kasch ring, division ring and the strongly regular ring.

## 1- Introduction:

Throughout this paper,  $R$  will denote an associative ring with identity. For any element  $a$  in  $R$ , we define the right annihilator of  $a$  by  $r(a) = \{x \in R : ax = 0\}$ , and likewise the left annihilator  $l(a)$ .  $Y, Z, J$  will denote respectively the right singular ideal, the left singular ideal and the Jacobson radical of  $R$ . Recall that 1) An ideal  $I$  is said to be a right (left) pure if for every  $a \in I$ , there exists  $b \in I$  such that  $a = ab(ba)$  [1] 2)  $R$  is called a uniform ring if for every non-zero ideal of  $R$  is essential, [4] .3)

A ring  $R$  is said to be left Kasch ring, if every maximal right ideal is a right annihilator [3] 4)  $R$  is said to be strongly regular if for each  $a \in R$ , there exists  $x \in R$  such that  $a = a^2x$ . Following [1] . 5) A ring  $R$  is called reduced if  $R$  has no non-zero nilpotent element and an ideal  $I$  of a ring  $R$  is said to be right (left) GP-ideals if for every  $a \in I$ , there exists  $b \in I$  and a positive integer  $n$  such that  $a^n = a^n b(ba^n)$  [5].

## 2- MRGP-Rings:

Following [6] a maximal left ideal  $M$  of the ring  $R$  is said to be a right Co-pure if for every  $a \in M$ ,  $Ma$  is a right pure.

**Definition 2-1:** A ring  $R$  is called MRCP-ring, if all maximal left ideals are right Co-pure, .see[6]

**Definition 2-2:**  $R$  is called MRGP-ring, if for any maximal left ideal  $M$  of  $R$ , any  $a \in M$ ,  $Ma$  is a right GP-ideal.

Clearly every MRCP-ring is an MRGP-ring, however the converse is not true as the following example shows:

**Example:** The ring  $Z_{12}$  of integers modulo 12 is an MRGP-ring but not an MRCP-ring.

We now consider a necessary and sufficient condition for MRGP-ring to be an MRCP-ring.

**Theorem 2-3:** Let  $R$  be a reduced MRGP-ring. Then  $R$  is MRCP-ring

**Proof:** Let  $M$  be any maximal left ideal of  $R$ . Since  $R$  is MRGP-ring, then  $Ma$  is a right GP-ideal of  $R$  and there exists  $c, b \in M$  and a positive integer  $n$  such that  $(ba)^n = (ba)^n(ca)$  this implies that  $(ba)^n(1-ca) = 0$  and hence  $(1-ca) \in r(ba)^n \subseteq r(ba)$ .

Therefore  $(1-ca) \in r(ba)$ , whence  $(ba)(1-ca) = 0$ . Thus  $ba = ba ca$  ( $Ma$  is a right pure) and hence  $R$  is MRCP-ring.

**Lemma 2-4:** let  $a$  be a non-zero element of the ring  $R$  and let  $r(a)=0$ . Then  $r(a^{2^n})=0$

**Theorem 2-5:** let  $R$  be an MRGP-ring and  $r(a)=0$ . Then  $a$  is a left invertible.

**Proof:** let  $a \in R$  with  $r(a)=0$ . If  $Ra \neq R$ , there exists a maximal left ideal  $M$  containing  $Ra$ . Since  $R$  is an MRGP- ring, there exists  $b \in M$  and a positive integer  $n$  such that  $a^{2^n} = a^{2^n}(ba)$ . Whence  $(1-ba) \in r(a^{2^n}) = 0$ , yielding  $1 \in M$ , which contradicts  $M \neq R$ . Therefore  $Ra = R$ . In particular  $ra = 1$  for some  $r \in R$ . Hence  $a$  is a left invertible.

**Proposition 2-6:** Let  $R$  be a right uniform reduced MRGP-ring. Then  $R$  is a division ring.

**Proof:** If  $0 \neq a \in R$  and  $Ra \neq R$ . Let  $M$  be a maximal left ideal containing  $Ra$ . Since  $R$  is MRGP-rings, then for every  $a \in M$ ,  $Ma$  is a right GP-ideal, so  $(ca)^n = (ca)^n ba$ , for some  $c, b \in M$  and a positive integer  $n$ . Since  $R$  is a right uniform then every right ideal is essential ideal.

Consider  $l(ba) \cap R(ca)^n$ , let  $x \in l(ba) \cap R(ca)^n$  implies that  $xba = 0$  and  $(ca)^n = x$ , so  $(ca)^n ba = 0$ , then  $(ca)^n ba = (ca)^n = 0$ . Therefore  $l(ba) \cap R(ca)^n = 0$  implies  $l(ba) = 0$ , since  $R$  is reduced,  $r(ba) = 0$ . By Theorem (2-5),  $ba$  is a left invertible, there exists  $y \in R$  such that  $v(ba) = 1$ , so  $(yb)a = 1 \in M$ , a contradiction. Therefore  $Ra = R$ . So  $R$  is division ring.

A ring  $R$  is called zero commutative (briefly ZC) [2] if for  $a, b \in R$   $ab = 0$  implies  $ba = 0$

**Proposition(2-7):** Let  $R$  be a zero commutative, MRGP-ring. Then  $R$  is a kasch ring.

**Proof:** Let  $M$  be any maximal left ideal of  $R$ , and let  $Z$  be the left singular ideal of  $R$ , if  $M \cap Z = 0$ , then for any  $y \in Z$ ,  $y \notin M$ , this implies that  $l(y)$  is an essential left ideal of  $R$ .

Let  $x \in l(y) \cap l(1-y)$ , then  $xy = 0$  and  $x(1-y) = 0$ , yields  $x = xy = 0$ . Therefore  $l(y) \cap l(1-y) = 0$ , whence  $l(1-y) = 0$ .

Since  $R$  is a zero commutative, then we have  $r(1-y) = 0$ . By Theorem (2-5)  $1-y$  is an invertible element of  $R$ , hence  $y \in J \subset M$  a contradiction. Thus  $M \cap Z \neq 0$ .

Let  $a \in M \cap Z$ , since  $R$  is an MRGP-ring, then  $Ma$  is a right GP-ideal of  $R$  and there exists  $c, b \in M$  and a positive integer  $n$  such that

$(ba)^n = (ba)^n(ca)$ . We claim that  $l(ca) \cap R(ba)^n = 0$ , if not let  $d \in l(ca) \cap R(ba)^n$ , then  $dca = 0$  and  $d = r(ba)^n$  for some  $r$  in  $R$ , so  $r(ba)^nca = 0$ , implies  $r(ba)^n = 0$ , whence  $d = 0$ . Therefore  $l(ca) \cap R(ba)^n = 0$ , but  $r(ca)$  is essential then  $R(ba)^n = 0$  and hence  $b^n a^n = 0$  implies that  $b^n \in l(a^n)$  and  $a^n \in r(b^n)$ . Therefore  $M = l(a^n)$  and  $M = r(b^n)$ .

The following theorem gives the condition of being MRGP-ring is strongly regular.

**Theorem 2-8:** Let  $R$  be a reduced MRGP-ring. Then  $R$  is strongly regular.

**Proof:** Let  $z$  be a non-zero element in  $R$ . We claim that  $Rz + l(z) = R$ . If  $Rz + l(z) \neq R$ , let  $M$  be a maximal left ideal containing  $Rz + l(z)$ . Since  $R$  is an MRGP- ring, then  $Mz$  is a right GP-ideal and there exists  $c \in M$  and a positive integer  $n$  such that  $(z)^{2n} = (z)^{2n}(cz)$ . Whence  $(1-cz) \in r(z)^{2n}$ . Since  $R$  is reduced, we have  $(1-cz) \in r(z)^{2n} = l(z^{2n}) = l(z) \subseteq M$  this implies that  $1 \in M$ , a contradiction. Therefore  $Rz + l(z) = R$ , in particular  $xz + y = 1$ ,  $x \in R$ ,  $y \in l(z)$ .

Thus  $z = xz^2$ , and therefore  $R$  is strongly regular.

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