

On Weakly Regular Rings and SSF-rings

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الملخص

في هذا البحث درسنا الحلقات المنتظمة الضعيفة، التي تكون مقاساتها الشاذة اليمنى البسيطة مسطحة. وكذلك درسنا الشرط (*) على الحلقة R والذي يحقق $L(a) \subseteq r(a)$ لكل a في R . وأثبتنا أن الحلقة التي تحقق الشرط (*) والتي تكون مقاساتها الشاذة اليمنى البسيطة مسطحة، فإن مركز الحلقة يكون حلقة منتظمة. وأثبتنا أيضا إن الحلقة R التي تكون مقاساتها الشاذة اليمنى البسيطة مسطحة و تحقق الشرط (*) أو مقيدة يمنى قوية فإن R حلقة مختزلة ومنتظمة ضعيفة.

ABSTRACT

In this work we consider weakly regular rings whose simple singular right R -Modules are flat. We also consider the condition (*): R satisfies $L(a) \subseteq r(a)$ for any $a \in R$. We prove that if R satisfies (*) and whose simple singular right R -module are flat, then $Z(R)$ the center of R is a von Neumann regular ring. We also show that a ring R either satisfies (*) or a strongly right bounded ring in which every simple singular right R -module is flat, then R is reduced weakly regular rings.

1- Introduction

Through out this paper, R denotes an associative ring with identity, and all modules are unitary ring R -modules. For any non-empty subset X of ring R , $r(X)$ and $L(X)$ denote the right annihilator of X and the left annihilator of X , respectively. $Z(R)$, $J(R)$ will denote respectively the center of R and Jacobson radical of R . Recall that :- 1- R is called reduced if R has no non-zero nilpotent elements . 2- R is said to be Von Neumann regular (or just regular) if , $a \in aRa$ for every $a \in R$, and R is called strongly regular if $a \in a^2R$. Clearly every strongly regular ring is a regular reduced ring. 3- A ring R is said to be right (left) quasi duo ring [3], if every maximal right(left)ideal is a two-sided ideal of R . 4- Following [6], for any ideal of R , R/I is flat if and only if for each $a \in I$, there exists $b \in I$ such that $a=ba$. 5- A ring R is called weakly right duo[4] if for any $a \in R$, there exists a positive integer n such that a^nR is a two-sided ideal of R .

2- Rings Whose Simple Singular Modules are Flat

Definition 2-1:-

A ring R is called a right (left) SSF-ring ,if every simple singular right (left) R -module is flat.

The following lemma is well-known, so we omit its proof.

Lemma 2-2:-

For any $a \in Z(R)$, if $ara=a$ for some $r \in R$, then there exists $b \in Z(R)$ such that $a=aba$.

Proof:- see[7]

We consider the condition(*) : R satisfies $L(a) \subseteq r(a)$ for any $a \in R$.

Proposition 2-3:-

If R satisfies (*), SSF-ring, then the center $Z(R)$ of R is a Von Neumann regular ring.

Proof:- First we will show that $aR+r(a)=R$ for any $a \in Z(R)$.

If not, there exists a maximal right ideal M of R such that $aR+r(a) \subseteq M$. Since $a \in Z(R)$, $aR+r(a)$ is an essential right ideal and so M must be an essential right ideal of R . Since R/M is flat and $a \in M$, then there exists $b \in M$ such that $a=ba$ and hence $(1-b) \in L(a) \subseteq r(a) \subseteq M$ implies $1 \in M$, which is a contradiction. Therefore $aR+r(a)=R$ for any $a \in Z(R)$ and so we have $a=ara$ for some $r \in R$. Applying Lemma (2-2) $Z(R)$ is a Von Neumann regular ring.

Recall that a ring R is right (left) weakly regular if $I^2=I$ for each right (left) ideal I of R ; equivalently. $a \in aRaR$ ($a \in RaRa$) for every $a \in R$. R is weakly regular if it is both right and left weakly regular[5].

Lemma 2-4: -

If R satisfies (*), then $RaR+r(a)$ is an essential right ideal.

Proof :- see[7]

Theorem 2-5: -

If R satisfies (*), and SSF-ring, then R is a reduced weakly regular ring.

Proof:- Let $a^2=0$. Suppose that $a \neq 0$. By Lemma (2-4), $r(a)$ is essential right ideal of R . Since $a \neq 0$, $r(a) \neq R$. Thus there exists a maximal essential right ideal M of R containing $r(a)$. Since R/M is flat and $a \in M$ there exists $b \in M$ such that $a=ba$ and hence $(1-b) \in L(a) \subseteq r(a) \subseteq M$ and so $1 \in M$, which is a contradiction. Hence $a=0$ and so R is reduced.

Now, we will show that $RaR+r(a)=R$ for any $a \in R$. Suppose that there exists $b \in R$ such that $RbR+r(b) \neq R$. Then there exists a maximal right ideal M of R containing $RbR+r(b)$. By Lemma(2-4), M must be essential in

R. Therefore R/M is flat. Then there exists $c \in M$ such that $b = cb$ and hence $(1-c) \in L(b) \subseteq r(b) \subseteq M$ and so $1 \in M$, which is a contradiction. Therefore $RaR + r(a) = R$ for any $a \in R$. Hence R is a right weakly regular ring. Since R is reduced, it also can be easily verified that R is a weakly regular ring.

Corollary 2-6: -

If R is a reduced and SSF-ring, then R is a weakly regular ring.

Lemma 2-7: -

Let R be a right quasi duo ring. Then $R/J(R)$ is a reduced ring.

Proof:- see[6]

Proposition 2-8 :-

Let R be a right quasi duo ring. The following statements are equivalent.

- a) R is a right weakly regular ring.
- b) R is a strongly regular ring.

Proof: - see[6]

Proposition 2-9:-

Every weakly right (left) duo ring is right (left) quasi-duo.

Proof:- see[1]

Proposition 2-10: -

Let R be a right(left) quasi duo ring. If every simple singular right R -module is flat, then $R/J(R)$ is strongly regular.

Proof: - Let $0 \neq \bar{a} \in \bar{R} = R/J(R)$. We will show that $\bar{R}\bar{a}\bar{R} + r(\bar{a}) = \bar{R}$.

Suppose not. Then there exists a maximal right ideal M of R such that $\bar{R}\bar{a}\bar{R} + r(\bar{a}) \subseteq M/J(R)$. From Lemma (2-7), $\bar{R}$ is reduced we have $r(\bar{a}) = L(\bar{a})$ for any $\bar{a} \in \bar{R}$. Then by Lemma(2-4) $\bar{R}\bar{a}\bar{R} + r(\bar{a})$ is an essential right ideal of $\bar{R}$. Thus $R/J(R)$ must be right essential in $\bar{R}$. Therefore R/M is a simple singular right R -module and so R/M is flat, then there exists $c \in M$ such that $a = ca$ and hence $(1-c) \in L(a) \subseteq r(a) \subseteq M$ and so $1 \in M$, which is a contradiction. Therefore $R/J(R)$ is right weakly regular since $R/J(R)$ is reduced it also can be easily verified that $R/J(R)$ is a weakly regular ring. By proposition(2-8), R is a strongly regular ring.

Corollary 2-11: -

Let R be a weakly right duo, SSF-ring. Then $R/J(R)$ is a right weakly regular ring.

Proof:- By Proposition (2-9) R is a right quasi duo ring. Also by Proposition (2-10) $R/J(R)$ is a right weakly regular ring.

A ring R is called strongly right bounded (briefly SRB) [2] if every non-zero right ideal contains a non-zero two-sided ideal of R .

Lemma 2-12:-

If R is a semi prime SRB ring, then R is reduced.

Proof:- see[2]

Theorem 2-13: -

Let R be a SRB and SSF-ring. Then R is a reduced weakly regular.

Proof: - By Corollary(2-6) and Lemma (2-12), it is enough to show that R is semi prime. Suppose that there exists a non-zero right ideal A of R such that $A^2=0$. Then there exists $0 \neq a \in A$ such that $a^2=0$. First observe that $r(a)$ is an essential right ideal of R . If not there exists a non-zero right ideal K such that $r(a) \oplus K$ is right essential in R . Since R is SRB, there is a non-zero ideal I of R such that $I \subseteq K$.

Now $aI \subseteq aR \cap I \subseteq r(a) \cap K = 0$. Hence $I \subseteq r(a) \cap K = 0$ ($aI \subseteq I$).

This is a contradiction. Thus $r(a)$ must be a proper essential right ideal of R . Hence there exists a maximal right ideal M of R containing $r(a)$. Clearly M is an essential right ideal of R , R/M is flat, then there exists $c \in M$ such that $a=ca$. Now $aca \in aRa \subseteq A^2=0$ and so $1 \in M$, which is a contradiction. Therefore R must be semi-prime, hence R is a reduced weakly regular.

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