On Rings Whose Principal Ideals are Generalized Pure Ideals

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الملخص

يقدم هذا البحث مفهوم الحلقات من النوع PIGP (الحلقات التي يكون فيها كـل جـزء مثالي أساسي ايمن هو من النمط GP) وإعطاء الخواص الأساسية لها . كذلك إعطاء الـشروط الضرورية والكافية للحلقة PIGP لكي تكون حلقة مقسومة وحلقة منتظمة .

ABSTRACT

This paper , introduces the notion of a right PIGP-ring (a ring in which every principal ideal of R is a GP-ideal) with some of their basic properties ; we also give necessary and sufficient conditions for PIGP-rings to be a division ring and a regular ring .

1-INTRODUCTION

Throughout this paper R denotes an associative ring with identity . Recall that: (1) A right ideal I of a ring R is said to be right (left) pure ideal if for every $a \in I$, there exists $b \in I$ such that $a=ab(a=ba)$ [4]. (2) A ring R is said to be a right PIP-ring , if every principal right ideal is pure [1] . (3) A ring R is called semi-prime ring if R has no non-zero nilpotent ideal. (4) A ring R is called regular ring if for every a in R , there exists b in R such that $a=aba$[2] . (5) A ring R is called uniform if every non-zero ideal of R is essential .(6) For any $a \in R$, r(a), (l(a)) will denote the right (left) annihilator of a . (7) Y(R), Z(R), Cent(R), N stand , respectively, for the right singular ideal, left singular ideal, center of R and nil radical of R .

2-PIGP-RING

Following [4] , an ideal I is said to be a right(left) generalized pure ideal (briefly GP-ideal) if for every $a \in I$, there exists $b \in I$ and a positive integer n such that $a^n=a^nb$ ($a^n= ba^n$) .

Definition 2.1:

A ring R is said to be a right PIGP-ring , if every principal right ideal of R is a GP-ideal .

Example:

The ring Z_{12} of integers modulo 12 is a PIGP-ring .

Lemma 2.2:

Let R be PIGP-ring. Then Z(R) is nilideal .

Proof: Let $z\in Z(R)$,then there exists a positive integer n and $w\in zR$ such that $z^n=z^nw$ where $w=zr$ for some $r\in R$. Therefore $z^n=z^nzr$. We claim $Rz^n\cap l(zr)=0$. Let $x\in Rz^n\cap l(zr)$, then $x=yz^n$ and $xzr=0$ implies that $yz^nzr=0$ and we get $x=yz^n=0$. So $Rz^n\cap l(zr)=0$ .Since $z\in Z(R)$,then $l(zr)$ is an essential left ideal . Therefore $Rz^n=0$ and we get $z^n=0$, therefore Z(R) is nilideal .

Proposition 2.3:

Let R be PIGP-ring with non-zero divisor . Then R is a division ring .

Proof: Let a be a non-zero element in R. Since R is PIGP-ring , then aR is GP-ideal and hence there exists $b\in aR$ and a positive integer n such that $a^n=a^nb$, where $b=ar$ for some $r\in R$. Therefore $a^n=a^nar$, whence$(1-ar)\in r(a^n)$. Now since $r(a)=0$, then we get $r(a^n)=0$, which implies that $ar=1$, and hence a is right invertible . Therefore R is a division ring .

Theorem 2.4:

Let R be a PIGP-ring. Then if $vu=1$ then $uv=1$ for all $u,v\in R$.

Proof: Let $u,v\in R$ such that $vu=1$, and since R is PIGP-ring ,then uR is GP-ideal and there exists a positive integer n and $r\in R$ such that $u^n=u^nur$.

Since $vu=1$ then $v^nu^n= v^{n+1}u^{n+1}=1$ and we get $u^{n+1}=u^{n+1}v^{n+1}u^{n+1}$ and $u^{n+1}r= u^{n+1}v^{n+1}u^{n+1}r$ so $u^n=u^{n+1}v^{n+1}u^n$ and implies that $u^n=u^{n+1}v(v^nu^n)=u^{n+1}v$

Now $uv=v^nu^n(uv)=v^n(u^{n+1}v)=v^nu^n=1$

Clearly every PIP-rings is PIGP-ring, however the converse is not true. We now consider a necessary and sufficient condition for PIGP-ring to be PIP-ring.

Theorem 2.5:

Let R be a PIGP-ring and $r(a^n) \subseteq r(a)$ for every $a \in R$ and a positive integer n. Then R is a PIP-ring.

Proof: Assume that R is a PIGP-ring . Then for any principal ideal I is GP-ideal ,and for any $x \in I$ there exists a positive integer n such that $x^n = x^n y$, for some $y \in I$, so $x^n(1-y)=0$ implies that $(1-y) \in r(x^n)$ and since $r(x^n) \subseteq r(x)$ so $x=xy$ .Therefore I is pure consequently R is PIP-ring .

Recall that a ring R is called that zero-commutative [3] if ab=0 implies that ba=0 for all $a,b \in R$.

Theorem 2.6:

Let R be a zero-commutative PIGP-ring . Then there exists an invertible $u \in R$ and idempotent element $e \in R$, such that $a^n = eu = ue$ and $a^n = (1-e)+u$ for some positive integer n and $a^n u = u\, a^n$

Proof: Let $0 \neq a \in R$ and since R is a PIGP-ring, then aR is GP-ideal , so there exists a positive integer n and $r \in R$ such that $a^n = a^n ar$, and since R is zero-commutative then $(a^{n-1} - a^n r) \in r(a) = l(a)$ implies that $a^n = a^n ra$ and $a^n = a^n arra$ and so we get $a^n r^n a^n = a^n b a^n$,where $r^n = b$.

Now , let $e = a^n b$ and $e^2 = a^n b a^n b = a^n b = e$. So e is idempotent and $a^n e = e a^n = a^n$. Let $u = 1-e+a^n$ and $v = 1-e+eb$

$uv = (1-e+a^n)(1-e+eb) = 1-e+eb-e^2 b + a^n eb = 1-e+e\, a^n b = e+e^2 = 1$

and applying theorem 2.4 we have vu=1. Now $eu = e(1-e+a^n) = e\, a^n = a^n$. Therefore $a^n = eu = ue$ and $(1-e)+u = 1-e+a^n+e-1 = a^n$ .Thus $a^n = (1-e)+u$.

The following theorem gives a condition for PIGP-ring be a regular ring.

Theorem 2.7:

Let R be a PIGP-ring and $r(a^n) \subseteq r(a)$ for all $a \in R$. Then R is a regular ring .

Proof: For any $a \in R$, we have to prove that $aR+r(a)=R$,if not , then there exists a maximal right ideal M such that $aR+r(a) \subseteq M$ .Since R is a PIGP-ring, then aR is GP-ideal and there exists a positive integer n and $r \in R$ such that $a^n = a^n ar$. We get $(1-ar) \in r(a^n)$. Since $r(a^n) \subseteq r(a)$ then $(1-ar) \in r(a) \subseteq M$ implies that $1 \in M$ which is a contradiction . So $aR+r(a)=R$, and $a = aba$ for some $b \in R$. Then R is a regular ring .

Corollary 2.8:

Let R be a PIGP-ring , then any reduced ideal is regular .

Lemma 2.9:

If R is a semi-prime ring then $r(a^n)=r(a)$ for any $a\in Cent(R)$.

Proof: See [5] .

Corollary 2.10:

Let R be a semi-prime PIGP-ring .Then Cent(R) is a regular ring .

Lemma 2.11:

Let R be a zero-commutative ring , then $N\subseteq Y(R)$.

Proof: See [6] .

Our next result characterizes a uniform PIGP-ring in terms of nilpotent and non-zero divisor

Theorem 2.12:

Let R a be zero-commutative uniform ring . Then R is a PIGP-ring if and only if :

1- For every element of R is either non-zero divisor or nilpotent

2- Every non-zero divisor of R is an invertible

3- N is a right ideal in R .

Proof: Let R be a zero-commutative uniform PIGP-ring .Let a be a non-zero element in R . Then aR is a GP-ideal and hence there exists a positive integer n and $r\in R$ such that $a^n= a^nar$. We claim that $a^nR\cap r(a)=0$, let $x\in a^nR\cap r(a)$. Then $x=a^ns$ and $ax=0$ for some $s\in R$ and $ax=a(a^ns)=0$ implies that $sa^{n+1}=0$ [since R is a zero-commutative] and we get $sa^{n+1}r=0$ implies that $sa^n=0$ and $x=a^ns=0$. So $a^nR\cap r(a)=0$. Since R is a uniform ring , then either r(a) is essential then $a^nR=0$. This implies that $a^n=0$, so a is nilpotent , or a^nR is essential then r(a)=0. This implies that a is a non-zero divisor and applying proposition 2.3 then any non-zero divisor is invertible and applying lemma 2.2 and 2.11 , then N=Y(R), so N is a right ideal in R.

Conversely: For any $a\in R$. If a is a non-zero divisor ,then a is invertible and aR=R which is a GP-ideal . On the other hand, if a is nilpotent then $a^m=0$ for some positive integer m and since N is a right ideal then aR is a GP-ideal.

Theorem 2.13:

Let R be a commutative PIGP-ring .Then Rad(aR)=aR+N . for all $a \in R$

Proof: Since $aR \subseteq Rad(aR)$ and $N \subseteq Rad(aR)$, then $aR+N \subseteq Rad(aR)$. Now let $x \in Rad(aR)$, then there exists a positive integer m such that $x^m \in aR$ and since R is a PIGP-ring ,then there exists a positive integer n and $b \in aR$ such that $x^{mn}=x^{mn}b$ and we get $x^{mn}(1-b)=0$.

Then $[x(1-b)]^{mn}=x^{mn}(1-b)^{mn}=0$,

so $x(1-b) \in N$, and because $bx \in aR$, we have

$x=bx+x(1-b) \in aR+N$. Thus Rad(aR)=aR+N.

REFERENCES

[1] Ahmad, Sh. H. (2006) " On Rings Whose Principal ideal are Pure", **RAf. Jour. Of Comp. & Math's**.Vol.3,No.2,PP.53-57

[2] Goodearl, K.R. (1979) **Von Neumann regular rings** , London, Pitman.

[3] Kim, N.K.; S.B. Nam, and J.Y. Kim (1999) "On Simple Singular GP-Injective Modules", **Communications in Algebra**, Vol.27,No.5, PP.215-218 .

[4] Mahmod, R. D. (2000) "ON Pure Ideals and Pure Sub-Modules" , Ph.D. , Thesis , **Mosul University** .

[5] Nam, S.B. (1999) "A note on Simple Singular GP-Injective Modules" ,Kangweon-Kyungki Math. Jour., Vol.7,No.2,PP.215-218.

[6] Shuker, N.H. and A.S. Younis (2005) " A note on non-Singular ring" **RAf. Jour. of Comp. and Math's** , Vol. 2, No. 2 , PP.21-26 .