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Another type of separation axioms

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Abstract:

The main purpose of this paper is to study properties of semi-closed(open) sets s-closed (s-open)sets and generlized -closed(open)sets g-closed(g-open)sets and study the relationship between them.And also inroduce another type of separation axioms called it S- separation axiom and the characteristics that can be prserved of some type of functions on many spaces as S-T_0 , S-T_1 , S-T_2 and we explain that the property of S-T_0 , S-T_1 , S-T_2 are toplological property if the function is injactive and s^*-open .

1- **Intoduction**:

The concepts of semi closed(open) sets s-closed(s-open)sets are intoduction by Levine.N in 1963 [7]. He definned a set A in a topological space X to be s-open set if for some set G, $G \subseteq A \subseteq cl(A)$, wher *cl(A)* denoted to the cloure of a in X . A set F is s-closed if it's complment is s-open set.

In 1970 Levine .N [8] introduced another concepts called it generlized closed (open) sets g-closed(g-open) sets in order to extend many of the important properties of closed set to larger family.

In 1971 Crossieg . S.G and Hildebrand .S.K , introduced the the concept of semi cloure ane they define it, the semi closure of a set A in a topological space X is the smallest semi closed (s-closed) set containing A [4],and denoted it by *scl(A).*

In 1982 Malgahan introduce g-closed function s concept and give some theorems of preservation of normality and regularity [9].

In this paper we continue to study of s-closed set and study the relation between s-closed set and g-closed set ,and proved they are independent conceptes, and give a new notation of semi separation axioms (S-separation axiom) and characteristic that can be preserved of some type of function on many spaces , as S-T_0 , S-T_1 , S-T_2 .

Also we will study the relation between the usual sepration axiom and

S-separation axiom .

2- **Preliminaries:**

In this section we give definitions, remarks, examples and also we prove some results on s-closed(s-open) sets .

Definition (2-1): [7]

Let X be a topological space , $A \subseteq X$, A is called semi closed set (s-closed set) if there exist F closed set in X such that

$Int(F) \subseteq A \subseteq F$ Where *Int(F)* denoted to interior of F.

Definition (2-2): [4]

Let X be a topological space , $A \subseteq X$, A is called semi closed set (s-closed set) if $Int(cl(A)) \subseteq A$.

Remark (2-3) : [5]

The two above difintion (2-1),(2-2) are equivalence .

Remark (2-4) : [1]

The complemant of semi closed(s- closed) set is called semi open (s-open) set.

Remark (2-5) : [1]

Every open set is s-open set but the converse may be not true as following example .

Example (2-5):

Let X={1,2,3} , $T = \{\phi, X, \{1\}, \{1,2\}\}$ is topology on X .

Clear the s-open set in X are $\{\phi, X, \{1\}, \{1,2\}, \{1,3\}\}$.

Hence {1,3} is s-open set but it is not open set.

Definition (2-6): [8]

Let (X,T) be a topological space , $A \subseteq X$ is called g-closed set if *cl(A)* $\subseteq O$ whenever $A \subseteq O$, *O* is open set in X . The complemant of g-closed set is g-open set.

Remark (2-7) :

Every closed set is g-colsed set but the convers my not be true as the following example .

Example (2-8):

Let X={a,b,c,d} , $T=\{\phi, X, \{a\}, \{b\}, \{a,b\}, \{b,c,d\}\}$, is topology on X. Let A={c} , *cl(A)* ={c,d} , hence A is g-closed set but it is not closed set .

Remark (2-9) :

The intersection of two s-open set not may be s-open set as the following example:

Example (2-10):

Let (R , T_u) is usuale topology .
Clear that (0,5] and [2,7) are two s-open sets in (R , T_u) ,
but it's intersection [2,5] is not s-open set in (R , T_u) .

Theorem (2-11): [6]

If A is s-open set in X and U is open in X then U$\cap$A is s-open in U.

Theorem (2-12):

IF F is closed set and B is s-closed set in X ,then F$\cup$B is s-closed set.

Proof:

Let (X,T) be a topological space ,and let F is closed set and B is s-closed set in X . $\therefore$ F^c is open set in X , B^c is s-open set in X . then $F^c \cap B^c$ is s-open set in X [theorem 2-11]

But $F^c \cap B^c = (F \cup B)^c$ Demorgan's law. $\therefore$ $(F \cup B)^c$ is s-open in X .

Hence F$\cup$B is s-closed set in X (**Remark (2-4)) .**

Definition(2-13):[1]

A function $f: X \to Y$ is called :

a- S-open (S-closed) function if $\forall G \subseteq X$ is open (closed) then *f(G)* $\subseteq$ *Y* is s-open (s-closed).

b- S^*-open (S^*-closed) function if $\forall U \subseteq X$ is s-open (s-closed) then *f(U)* $\subseteq$ *Y* is open (closed).

c- S^{**}-open (S^{**}-closed) function if $\forall U \subseteq X$ is s-open (s-closed) then

f(U) $\subseteq$ *Y* is s-open (s-closed).

3- The Main Result:

In this section we will study the relation between the concepts s-closed and g-closed sets and we will show that there are two independent concepts completely through the following four examples:

The following example show that there exist g-closed set and s-closed set at the same time.

Example (3-1) :

Show in example (2-8) the set A={c} , $cl(A)$ ={c,d} , hence A is g-closed set , and it's s-closed set because if F={c,d} is closed set , then $int(F) = \{\phi\} \Rightarrow \phi \subseteq \{c\} \subseteq \{c,d\}$ $\therefore \ \operatorname{int}(F) \subseteq A \subseteq F$

Hence A ={c} is g-closed set and s-closed set at the same time.

The following examlpe show that there exist some sets are g-closed set but it is not s-closed set.

Example (3-2) :

Let X={a,b,c} , $T = \{\phi, X, \{a\}, \{a,b\}\}$ be topology on X .

Let A={a,c} , A is g-closed set but it is not s-closed set.

The following examlpe explain that there exist some sets are s-closed set but it is not g-closed set.

Example (3-3) :

Let (R,T_u) be the usual topology on R , let A=[0,1) clearly A is s-closed set but it is not g-closed set because

If O = (-1,1) open interval in R , show that $A \subseteq O$ but $cl(A) \not\subset O$

Hence A is not g-closed set .

The following examlpe show that there exist some sets that are not s-closed and are not g-closed set.

Example (3-4) :

In the example (2-8) ,let A={a,b} , clear that $cl(A)=X$, $int(cl(A)) = X$ show that A is not s-closed set because $int(cl(A)) = X \not\subset A$.

And it is not g-closed set too because $A \subseteq \{a,b\}$ but $cl(A)=X \not\subset A$.

Hence we can say that the s-closed set and g-closed set are independent completely consepts.

4- S –T_0 space

Definition (4-1):

A topolgical space X is called S-T_0 space if and only if for each x and y are distinct points in X , there exist an s-open set W in X contaning one of them and not the other .

Remark (4-2):

Every T_0 space is S-T_0 ,but the coverse may not be true as the following example .

Example (4-3):

Let X={1,2,3} and let T={ $\phi, X, \{1\}$ } be a topology on X , clear (X,T) is S-T_0 space because the s-open sets in X are : $\{\phi, X, \{a\}, \{b\}, \{c\}, \{a,b\}\{a,c\}\}$, hence every two distinct point in X , there exist s-open set in X contaning one of them and not the other, and it is not T_0 space . Note that b and c are two different points in X , and we can't find an open set in X which contains one of them and not the other. Then X is S-T_0 space but it is not T_0 space.

Theorem (4-4) :

Every open subspace of S-T_0 space is S-T_0 space.

Proof

Let Y be an open subspace of S-T_0 space X , and let x , y be two distinct point of Y , then there exist an s-open set A in X containing one of them and not the other, let it be containing x but not y then $A \cap Y$ is s-open set in Y [theorem(2 – 11)] containing x but not y .

Hence Y is S-T_0 space .

Thorem (4-5) :

Let $f : X \to Y$ be injective function and S^*- open function ,
If X is S-T_0 space then Y is S-T_0 space.

Proof

Let y_1 , y_2 be two distinct points in Y , since f is injective function then there exist two distint points x_1 , x_2 in X .
such that $y_1 = f(x_1)$, $y_2 = f(x_2)$, but X is S-T_0 space and x_1 , x_2 are two distinct point in it , then there exist g-open set V in X containing one of them and not the other (i.e $x_1 \in V , x_2 \notin V$)
then $f(x_1) \in f(V)$ and $f(x_2) \notin f(V)$.
since f is S^*- open function and V is s-open set in X , then $f(V)$ is open set in Y . $\therefore$ $f(V)$ is s-open set(remark 2-5), since $y_1 = f(x_1)$, $y_2 = f(x_2)$ then $y_1 \in f(V)$, $y_2 \notin f(V)$. Hence Y is S-T_0 space .

Corollary(4-6):

S-$T_\cdot$ is a topological property where the function is injective and S^*-open function .

5- S -T_1 space

Definition (5-1) :

A topological space X is called S-T_1 space if and only if for each x and y are distinct point in X there exist two s-open sets G_1 , G_2 in X such that $x \in G_1$, $y \notin G_1$ and $x \notin G_2$, $y \in G_2$.

Remarks (5-2):

a. Every S-T_1 space is S-T_0 space .
b. Every T_1 space is S-T_1 space , but the converse my not be true as in the following example .

Example (5-3):

Let X= { 1,2,3} , T = $\{\phi,\{1\},\{2\},\{1,2\}\}$ be topology on X , then the space X is S-T_1 but not T_1 space.

Theorem (5-4):

A topological space X is S-T_1 space if and only if $\forall x \in X$, singleton {x} is s-closed set in X.

proof : $\Longrightarrow$

Let X be S-T_1 space and let $x \in X$, to prove that {x} is s- closed set we will prove X-{x} is s-open set in X , let $y \in$ X-{x} $\Longrightarrow x \neq y \in$ X, and since X is S-T_1 space then their exist two s-open sets G_1, G_2 such that $x \notin G_1$, $y \in G_2 \subseteq$ X-{x}. since $y \in G_2 \subseteq$ X-{x} then X-{x} is s-open set , Hence {x} is s-closed set.

$\Longleftarrow$ conversely:

Let $x \neq y \in$ X then {x},{y} are s-closed sets ,
i.e X-{x} is s-open set clearly $x \notin$ X-{x} and $y \in$ X-{x}.
simlarly X-{y} is s-open set , $y \notin$ X-{y} and $x \in$ X-{y}.
Hence X is S-T_1 space.

Theorem (5-5):

Let X be S-T_1 space and $f : X \to Y$ be an injective function and S^*- open function then Y is S-T_1 space.

Proof:

Let y_1, y_2 be two distinct points in Y , since f is injective function , then there exist two distinct points x_1, x_2 in X , such that $y_1=f(x_1)$, $y_2=f(x_2)$, but X is S-T_1 space and x_1, x_2 are distinct points in it ,then there exist two s-open sets V_1, V_2 in X such that $x_1 \in V_1$, $x_2 \notin V_1$ and $x_2 \in V_2$, $x_1 \notin V_2$.
(i.e $f(x_1) \in f(V_1)$, $f(x_2) \notin f(V_1)$ *and* $f(x_2) \in f(V_2)$, $f(x_1) \notin f(V_2)$).

Since f is S^*-open function and V_1,V_2 are two s-open set in X then $f(V_1)$, $f(V_2)$ are two open sets in Y (Def 2-13-b).
Hence $f(V_1)$ and $f(V_2)$ are s-open sets in Y , but $y_1 = f(x_1)$ and

$y_2 = f(x_2)$ then $y_1 \in f(V_1)$, $y_2 \notin f(V_1)$ and $y_2 \in f(V_2)$, $y_1 \notin f(V_2)$.

Hence Y is S-T_1 space.

Corollary(5-6):

S-T_1 is a topological property where the function is injective and S^*-open function .

Theorem(4-7):

Every open subspace of a S-T_1 space is S-T_1 space.

Proof :

Let X be S-T_1 space and let G be an open subspace of X ,

let $x \in$ G , since X is S-T_1 space , X-{x} is s-open set in X.

G $\cap$ (X-{x}) = G-{x} and it is s-open set in G [Theorem 2-11]

Then {x} is s-closed in G.

Hence G is S-T_1 space [Theorem 5-4].

6- S – T_2 space ((S – Hausdorff space))

Defintion (6-1):

A topological space X is called S-T_2 space (S-Hausdorff) ,

if $\forall x_1 \neq x_2 \in X$, $\exists$ two s-open sets H_1 , H_2 in X such

that $x_1 \in H_1$ and $x_2 \in H_2$ and $H_1 \cap H_2 = \phi$.

Remarks (6-2) :

1- Every S-T_2 space is S-T_1space .

2- Every T_2 space is S-T_2 space , but the converse may not be true as the following example.

Example (6-3) :

Let X={a,b,c} , T_i is indiscrete topology on X , then (X, T_i) is S-T_2 space but it is not T_2-space .

Theorem (6-4):

Every open subspace of S-T_2 space is S-T_2 space.

Proof :

By the same method in proveing theorem (5-7)

Remark (6-5):

Every singleton subset of S-T_2 space is S-closed set.

Theorem (6-6)

Let X be S-T_2 space and $f : X \rightarrow Y$, be injactive , S^*-open

function , where X and Y are topological spaces then Y is
S-T_2 space .

Proof:

Let y_1,y_2 be two distinct points in Y , since *f* is injective function , then there exist two distinct points x_1,x_2 in X ,such that $y_1=f(x_1)$,
$y_2=f(x_2)$, , but X is S-T_2 space and x_1 , x_2 are distint point in it,
then there exist two s-open sets H_1,H_2 in X such that $x_1 \in H_1$,
$x_2 \in H_2$ amd $H_1 \cap H_2 = \phi$ since f is S^*-open function then $f(H_1)$ and $f(H_2)$ are S-open sets in Y ,but $y_1 = f(x_1)$ then $y_1 \in f(H_1)$,
since $x_2 \in H_2$ then $f(x_2) \in f(H_2)$, but $y_2 = f(x_2)$ then $y_2 \in f(H_2)$,
to prove that $f(H_1) \cap f(H_2) = \phi$, since $H_1 \cap H_2 = \phi$ then $f(H_1 \cap H_2) = \phi$ then $f(H_1) \cap f(H_2) = \phi$.

Hence Y is S-T_1 space.

Corollary (6-7):

S-T_2 is topological property where the function is injective and S^*-open function .

Proof:

Clear frome theorem (6-6)

The following diagram shows the relation between usual separation axiom and S-separation axiom :

$$\begin{tikzcd} T_0 \arrow[d] & T_1 \arrow[l] \arrow[d] & T_2 \arrow[l] \arrow[d] \\ S\text{-}T_0 & S\text{-}T_1 \arrow[l] & S\text{-}T_2 \arrow[l] \end{tikzcd}$$

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نوع آخر من بديهيات الفصل

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المستلخص

إن الهدف الرئيس لهذا العمل هو دراسة خواص أنماط أخرى من المجموعات المغلقة (المفتوحة) وهي المجموعات المغلقة-s (المفتوحة-s) أي شبه المغلقة(شبه المفتوحة) والمجموعات المغلقة-g (المفتوحة-g)وتقديم عدد من المبرهنات والنتائج والملاحظات والأمثلة الخاصة بذلك ودراسة العلاقة بين هذين النمطين وقد بينا بأنهما مستقلان تماماً.

تم أيضا تقديم نوع آخر جديد من بديهيات الفصل يسمى بديهيات الفصل-s ودراسة خواص هذا المفهوم وتقديم بعض الفضاءات الخاصة به ومنها الفضاءات $S\text{-}T_0$, $S\text{-}T_1$, $S\text{-}T_2$ ودراسة العلاقة فيم بينهما وعلاقتها مع بديهيات الفصل الاعتيادية ، وتم أيضا دراسة تأثير بعض الدوال الخاصة بالمحافظة على هذه الخاصية ومنها الدالة المفتوحة-s^* ، وقد بينا أيضا بان خاصية $S\text{-}T_2$, $S\text{-}T_1$, $S\text{-}T_0$ خاصية تبولوجية إذا كانت الدالة متباينة ومفتوحة-s^* .

كلمات مفتاحيه:

مجموعات شبه مغلقه (مفتوحة) ، مجموعات مغلقه-s (مفتوحة-s)، بديهيات فصل-S .