

A Note on non – Singular Rings

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الملخص

في هذا البحث درسنا المثاليات المنفردة والحلقات غير المنفردة وأعطينا خواص جديدة لها ، ثم أعطينا العلاقة بين المثاليات المنفردة وجذر جاكوبسون. و أخيرا أعطينا الشرط اللازم لكي تكون الحلقات غير المنفردة مختزلة .

ABSTRACT

In this paper several new properties of singular ideals and non-singular rings are obtained, a connection between a singular ideal and the Jacobson radical is considered , and a sufficient condition for a non-singular ring to be reduced is given.

1. Introduction:

Throughout this work all rings are assumed to be associative rings with identity. Recall that: 1- A ring R is said to be reduced if R contains no non- zero nilpotent element ; 2- A ring R is said to be a duo- ring if every right and left ideal is a two-sided ideal; 3- $J(R)$ and N will stand respectively for the Jacobson radical ideal of R and the set of all nilpotent elements; 4- The right and left annihilators of a in R will be denoted by $r(a)$ and $l(a)$, respectively ; 5- An ideal I is said to be a right (left) pure if , for every $a \in I$, there exists $b \in I$ such that $a=ab(ba)$.

2. The Singular Ideals:

In this section, new properties of singular ideals are given , and a connection between singular ideals and the Jacobson radical is obtained .

Definition 2.1:

A non-zero elements a of R is said to be right singular if $r(a)$ is an essential right ideal of R . The set of all right singular elements of R is

denoted by $Y(R)$. The set of all left singular elements will be denoted by $Z(R)$.

Clearly $Y(R)$ and $Z(R)$ are ideals of R .

We shall begin this section with the following lemma .

Lemma 2.2:

If $a \in Y(R)$, then $r(1-a) = 0$.

Proof: let a be a non- zero element in $Y(R)$, then $r(a)$ is a non-zero essential right ideal of R . Let $x \in r(a) \cap r(1-a)$, then $a.x = 0$ and $(1-a).x = 0$, this implies that $x = 0$. Therefore , $r(a) \cap r(1-a) = 0$; since $r(a)$ is a non-zero essential right ideal of R , then $r(1-a) = 0$.

Corollary 2.3: Let R be a ring and $Y(R) = 0$, the only idempotent element in $Y(R)$ is zero .

Proof: Let $0 \neq a \in Y(R)$, and $a = a^2$, this implies that $a.(1-a) = 0$.

Hence $a \in r(1-a)$. By [Lemma 2.2.] ; $a = 0$.

Next, we give the following result .

Proposition 2.4 :

Let R be a ring with every right non–zero element is invertible, then $Y(R) \subseteq J(R)$.

Proof: let $0 \neq a \in Y(R)$. Then by Lemma 2.2 $r(1-a) = 0$, and hence $1-a$ is invertible. Whence $a \in J(R)$.

Recall the following result of Ferreno and Puczyłowski in [1].

Lemma 2.5:

Let I be a sem – prime ideal of R , then :

1- $Y(I) = I \cap Y(R)$.

2- $Z(I) = I \cap Z(R)$.

In view of the above lemma , we have the following :

Proposition 2.6:

Let I and J be semi – prime ideals of R , then :

If $I \subseteq J$, then $Y(I) \subseteq Y(J)$.

If $I \subseteq J$, then $Z(I) \subseteq Z(J)$.

Proof: Let $I \subseteq J$, and let $a \in Y(I)$, then by Lemma 2.5. $Y(I) = I \cap Y(R)$, this implies that $a \in I$ and $a \in Y(R)$, and hence $a \in J \cap Y(R)$. Whence $a \in J \cap Y(R) = Y(J)$.

3. Non – Singular Rings:

This section is devoted to study non-singular rings, we give condition for non-singular rings to be reduced, and we characterize non-singular rings in terms of maximal pure ideals and essential right ideals of R .

Definition 3.1:

A ring R is said to be a right (left) non-singular if $Y(R) = 0$, ($Z(R) = 0$). A ring R is said to be non – singular if $Y(R) = Z(R) = 0$.

Example:

The ring of integers module 6, Z_6 is a non-singular ring. Recall the following result of Ming in [2].

Lemma 3.2:

If $Y(R) \neq 0$, then there exists $y \in Y(R)$ such that $y^2 = 0$.

The following result characterizes non-singular rings in terms of maximal pure ideals.

Theorem 3.3:

Let R be a ring, such that for every nilpotent element y of R , there exists a maximal pure right ideal M such that $r(y) \subseteq M$. Then R is a right non-singular ring.

Proof: Let $Y(R) \neq 0$, then by Lemma 3.2., there exists a non-zero element y in $Y(R)$ such that $y^2 = 0$, then by the hypothesis there exists a maximal pure right ideal M of R such that $r(y) \subseteq M$. Since $y^2 = 0$, then $y \in r(y) \subseteq M$, and since M is a right pure, there exists $m \in M$ such that $y = ym$. So, $y(1-m) = 0$. This implies that $1-m \in r(y) \subseteq M$, which means that $1 \in M$, a contradiction. Therefore, $Y(R) = 0$.

Next, we give another condition for R to be a non-singular ring.

Proposition 3.4:

Let $Y(R)$ be a left pure ideal , then R is a right non – singular ring.

Proof: Let $Y(R)$ be a non- zero left pure ideal , and let a be a non – zero element in $Y(R)$, then there exists $b \in Y(R)$ such that $a = ba$. Since $b \in Y(R)$, then $r(b)$ is essential right ideal of R .We claim that $r(b) \cap aR = 0$.Let $x \in r(b) \cap aR$, then $b.x = 0$ and $x = a.r$ for some $r \in R$, hence $ba.r = 0$.But $ba = a$, then we have $a.r = x = 0$. Now, since $r(b)$ is an essential right ideal of R , then $aR=0$; and hence $a = 0$. Therefore $Y(R) = 0$.

It is well-known that if R is a reduced ring ,then R is a non-singular ring. However, the converse is not true, as the following example shows:

Example :

Let R be the ring of 2×2 upper triangular matrices with entries in Z_2 , where Z_2 is the ring of integers modulo 2 , that is :

$$R = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \text{ where } : a, b, c \in Z_2 \right\}.$$

Then by direct calculation ,we observe that R is a non-singular ring but it is not reduced.

The following result gives a sufficient condition for non-singular rings to be reduced

Theorem 3.5:

Let R be a right non-singular ring with $l(a) \subseteq r(a)$ for every $a \in R$. Then R is a reduced ring .

Proof: Let $a \in R$, and let $a^2 = 0$. We shall prove first that $r(a)$ is an essential right ideal of R ; if not , then there exists a right ideal I of R such that $r(a) \cap I = 0$. Since $a \in l(a)$, then $i.a \in l(a)$ for all $i \in I$, which implies that $I.a \subseteq l(a) \subseteq r(a)$. So $I.a \subseteq I \cap r(a) = 0$; this gives $I.a = 0$, therefore $I \subseteq l(a) \subseteq r(a)$, and hence $I=0$. Therefore $r(a)$ is an essential right ideal of R . This implies that $a \in Y(R) = 0$. Thus R is reduced.

Recall that an element a is said to be regular (in the sense of Von Neumann) if $a \in aRa$.

Before closing this section, we present two additional results.

Theorem 3.6:

Let R be duo right singular ring, then any nilpotent element of R is regular.

Proof: Let $a \in R$, such that $a^n = 0$ for some positive integer n , and let $s = a^{n-1} \neq 0$.

If aR is not an essential right ideal of R , then there exists a non-trivial right ideal K of R such that $aR \oplus K$ is an essential right ideal of R . Suppose that a is not a regular element in R , then $aR \oplus K \neq R$. Observe that $sK \subseteq K$ and $aR = 0$.

This implies that $K \subseteq r(s)$.

Now, since $a^n = 0$, then $s.a = 0$, hence $a \in r(s)$, this gives $aR \subseteq r(s)$. This means that $r(s)$ is an essential right ideal of R . Whence it follows that $s \in Y(R) = 0$. This is a contradiction. Therefore, a is a regular element of R .

Theorem 3.7:

A ring R is right non-singular, if and only if $L(I) = 0$ for every essential right ideal I of R .

Proof: Suppose that R is a right non-singular ring, then $Y(R) = 0$. Let I be an essential right ideal of R , such that $L(I) \neq 0$, then there exists a non-zero element a in $L(I)$. This implies that $a.I = 0$, and hence $I \subseteq r(a)$. Since $Y(R) = 0$, then $r(a)$ is not essential right ideal of R , and hence there exists a non-trivial right ideal K of R , such that $r(a) \cap K = 0$. Since $I \subseteq r(a)$ and $K.I \subseteq K \cap I$ then $K.I \subseteq K \cap I \subseteq K \cap r(a) = 0$, which means that $K \cap I = 0$, a contradiction.

Conversely, assume that I is an essential right ideal of R , and let $L(I) = 0$. Suppose that $Y(R) \neq 0$, and let a be a non-zero element in $Y(R)$, then $r(a)$ is an essential right ideal of R . Since $L(I) = 0$, then $L(r(a)) = 0$, then for every $y \in r(a)$, $a.y = 0$, hence $a \in L(y) = 0$, a contradiction. Therefore $Y(R) = 0$.

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