sp-Weakly Regular Rings

Raida D. Mahmood
College of Computer sciences and Mathematics
University of Mosul

Abdullah M. Abdul-Jabbar
College of Sciences
University of Salahaddin

Received on:17/5/2006 **Accepted on:16/8/2006**

الملخص

الغرض من هذا البحث هو دراسة صنف جديد من الحلقات التي تكون لكل $a \in R$ ، $a^n \in a^n R a^{2n} R$ لبعض قيم n الصحيحة الموجبة. ويطلق على هكذا حلقات اسم حلقات منتظمة ضعيفة من النمط -sπ وكذلك نعطي بعض الخواص الأساسية لهذه الحلقات ثم نجد العلاقة بين الحلقات المنتظمة الضعيفة من النمط -sπ و الحلقات المنتظمة بقوة من النمط π و مع حلقات القسمة.

ABSTRACT

The purpose of this paper is to study a new class of rings R in which, for each $a \in R$, $a^n \in a^n R a^{2n} R$, for some positive integer n. Such rings are called sπ-weakly regular rings and give some of their basic properties as well as the relation between sπ-weakly regular rings, strongly π-regular rings and division rings.

1. Introduction

Throughout this paper, R is an associative ring with identity. A ring R is said to be right (left) s-weakly regular if for each $a \in R$, $a \in aRa^2R$ ($a \in Ra^2Ra$). This concept was introduced by V. Gupta [5] and W. B. Vasantha Kandasamy [9]. Recall that:

(1) An ideal I of a ring R is a right pure, if for every $a \in I$, there exists $b \in I$ such that $a = ab$. (2) R is called reduced if R has no nonzero nilpotent element. (3) For any element a in R, the right annihilator of a is $r(a) = \{x \in R: ax = 0\}$ and likewise for the left annihilator $\ell(a)$. (4) According to Cohn [3], a ring R is called reversible if $ab = 0$ implies $ba = 0$ for $a, b \in R$.. It is easy to see that R is reversible if and only if right (left) annihilator of a in R is a two-sided ideal [3]. (5) Following [4], a ring R is a right (left) weakly π-regular if $a^n \in a^n R a^n R$ ($a^n \in R a^n R a^n$), for every $a \in R$ and a positive integer n.

2. sp-Weakly Regular Rings

In this section we introduce a new generalization of s-weakly regular rings which is called sπ-weakly regular, and is denoted by sπWR-rings. We give some of its basic properties, as well as a connection between s-weakly regular rings and sπWR-rings.

Definition 2.1:

An element b of a ring R is said to be sπ-weakly regular if there exists a positive integer n and c, d $\in$ R such that $b^n = b^n c\, b^{2n} d$.

A ring R is said to be right(left) sπ-weakly regular, if for each a $\in$ R, there exists a positive integer n, n = n(a), depending on such that

$a^n \in a^n R a^{2n} R$ ($a^n \in R a^{2n} R a^n$).

A ring R is called sπ-weakly regular if it is both right and left sπ-weakly regular.

Remark:

From now on, sπWR-rings mean right sπ-weakly regular rings unless other stated.

Example (1):

$$\text{Let } R = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : a, b \in \mathbf{R} \text{ and } a, b \neq 0 \right\}$$, where **R** is the

set of all real numbers. Then, R is sπWR-rings, since for any positive integer n

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}^n = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}^n \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a^2 & 0 \\ 0 & b^2 \end{pmatrix}^n \begin{pmatrix} \frac{1}{a^{2n}} & 0 \\ 0 & \frac{1}{b^{2n}} \end{pmatrix}$$

Obviously every s-weakly regular ring is sπWR-rings, however the converse is not true in general as the following example shows.

Example (2):

Let $\mathbf{Z}_4$ be the ring of integers modulo 4. Then, $\mathbf{Z}_4$ is sπWR-rings, but it is not s-weakly regular.

We now consider a necessary and sufficient condition for sπWR-rings to be s-weakly regular.

Theorem 2.2:

Let R be a ring. If $r(a^n) \subseteq r(a)$ and $a^nR = aR$, for every $a \in R$ and a positive integer n. Then, every sπWR-rings is s-weakly regular.

Proof:

Let R be sπWR-ring. Then, for every $a \in R$, there exists a positive integer n such that $a^n = a^n b a^{2n} c$, for some $b, c \in R$. But, $a^{2n} c = a^n (a^n c) \in a^nR = aR$ and $a^n c \in a^nR = aR$. Therefore, $a^{2n} c = a^2 d$, for some $d \in R$. Now, we obtain $a^n = a^n b a^2 d$. This implies that $a^n(1\text{-}b a^2 d) = 0$ and hence $1\text{-}b a^2 d \in r(a^n) \subseteq r(a)$. Therefore, $1\text{-}b a^2 d \in r(a)$. Whence it follows that $a = ab a^2 d$ and hence R is s-weakly regular. ♦

Theorem 2.3:

Let R be a right duo, sπWR-ring, then for all $a \in R$, there exists a positive integer n such that the principal ideal a^nR is idempotent.

Proof:

Assume that R is sπWR-ring. Let I be a right ideal of R such that $I = a^nR$ with $a \in R$, and a positive integer n, clearly $I^2 \subseteq I$. On the other hand, since $I = a^n R$, $1 \in R$ and $a^n \in I$. But R is sπWR-ring, then $a^n = a^n b a^{2n} c$, for some $b, c \in R$ and R is a right duo ring, then $b a^{2n} = a^{2n} x$, for some $x \in R$, so $a^n = a^n a^{2n} xc$. If we set $y = xc$, then $a^n = a^n a^{2n} y$. Now, $a^n \in I$ and $a^n = a^n a^{2n} y = a^n a^n z \in I^2$ $(a^{2n} y = a^n a^n y = a^n z)$. Therefore, $I \subseteq I^2$. Hence $I^2 = I$. ♦

Proposition 2.4:

If R is a ring in which $a^n = a^{3n}$, for every $a \in R$, then R is sπWR-ring.

Proof:

It is obvious; since R is a ring with identity as for every $a \in R$ and a positive integer n, we have $a^n \in a^n R a^{2n} R$ $(a^n = a^n 1 a^{2n} 1)$. ♦

Theorem 2.5:

Let R be a ring without divisors of zero. The ring R is sπWR-ring if and only if $a^{2n} = 1$ or $b a^{2n} c = 1$ or $a^{2n} c = 1$, for every $a \in R$ and a positive integer n.

Proof:

Given R is a ring with identity; which has no proper divisors of zero. Now, let us assume that R is sπWR-ring; to prove $a^{2n} = 1$ or $b a^{2n} c = 1$, for every $a \in R$ and a positive integer n. Given R is sπWR-ring, hence $a^n \in a^n R a^{2n} R$ for every $a \in R$ and a positive integer n. Thus, $a^n = a^n b a^{2n} c$, for every $a \in R$; if $b = c = 1$, then we have $a^n = a^{3n}$. This implies that $a^n(1- a^{2n}) = 0$, but R has no zero divisors; hence $a^{2n} = 1$. If $b \neq 1$, $c \neq 1$; then $a^n = a^n b\, a^{2n} c$ implies that $a^n(1- b\, a^{2n} c) = 0$, since R has no zero divisors, then $b\, a^{2n} c = 1$. If $b = 1$ and $c \neq 1$, then $a^n = a^n a^{2n} c$ implies that $a^n(1- a^{2n} c) = 0$, then $a^{2n} c = 1$.

Conversely, if $a^{2n} = 1$, for every $a \in R$, then $1- a^{2n} = 0$ implies that $a^n = a^{3n}$ and $a^n = a^n 1\, a^{2n} 1$. Therefore, R is sπWR-ring. Now, if $1- b\, a^{2n} c = 0$ or $a^{2n} c = 1$, we get immediately R to be sπWR-ring using the fact that R has no zero divisors. ♦

We recall the following result of [7].

Lemma 2.6:

Let R be a reduced ring. Then, for every $a\in R$ and a positive integer n,

(1) $r(a^n) = \ell(a^n)$

(2) $r(a^n) \subseteq r(a)$

(3) $\ell(a^n) \subseteq \ell(a)$

Theorem 2.7:

Let R be a reduced ring and let $I = R\, a^{2n} R$. Then, R is sπWR-ring if and only if $r(a^n)$ is a direct summand for every $a\in R$ and a positive integer n.

Proof:

Assume that R is sπWR-ring, then for every $a \in R$, there exists a positive integer n such that $a^n = a^n t_1\, a^{2n} t_2$, for some $t_1, t_2 \in R$. So, $(1- t_1\, a^{2n} t_2) \in r(a^n)$. Therefore, $1 = t_1\, a^{2n} t_2 + (1- t_1\, a^{2n} t_2)$. Hence, $R = R\, a^{2n} R + r(a^n)$. Now, let $b \in R\, a^{2n} R \cap r(a^n)$ implies $a^n b = 0$ and $a^n b\, t = 0$, for all $t \in R$, so $bt \in r(a^n) = \ell(a^n) = \ell(a^{2n})$. Then, $bt\, a^{2n} = 0$ and $bt\, a^{2n} c = 0$ implies $b\,(t\, a^{2n} c) = b^2 = 0$. Since R is reduced, then $b = 0$. Therefore, $R\, a^{2n} R \cap r(a^n) = 0$. Thus, $r(a^n)$ is a direct summand.

Conversely, assume that $r(a^n)$ is a direct summand for every $a \in R$ and positive integer n. Then, $R\, a^{2n} R + r(a^n) = R$ and $1 = t_1\, a^{2n} t_2 + d$, with $t_1, t_2 \in R$ and $d \in r(a^n)$. Multiplying by a^n we obtain, $a^n = a^n t_1\, a^{2n} t_2 + a^n d$, so $a^n = a^n t_1\, a^{2n} t_2$. Whence R is sπWR-ring. ♦

Lemma 2.8:

If R is a semi-prime reversible ring, then R is reduced.

Proof:

See [6, Lemma 2.7]. ♦

Proposition 2.9:

If R is a semi-prime reversible ring and every maximal right ideal of R is a right annihilator, then R is sπWR-ring.

Proof:

Let $a \in R$, we shall prove that $R a^{2n} R + r(a^n) = R$, for some positive integer n. If not, there exists a maximal right ideal M of R containing $R a^{2n} R + r(a^n)$. If $M = r(b)$, for some $0 \neq b \in R$, we have $b \in \ell(R a^{2n} R + r(a^n)) \subseteq \ell(a^n)$ [8]. Since R is semi-prime and reversible ring, then by Lemma 2.8, R is reduced. Therefore, by Lemma 2.6(1), $b \in r(a^n)$, which implies that $b \in M = r(b)$, then $b^2 = 0$ and hence $b = 0$, a contradiction. Therefore, $R a^{2n} R + r(a^n) = R$. In particular, $c a^{2n} d + x = 1$, for some c, d $\in$ R and $x \in r(a^n)$, then $a^n = a^n c a^{2n} d$. Whence R is sπWR-ring. ♦

3. The Relation Between sp**WR-rings and Other Rings**

In this section, we consider the connection between sπWR-rings, strongly π-regular rings and division rings.

We start this section with the following definition.

Definition 3.1:

A ring R is called strongly π-regular [1] if for every $a \in R$, there exists a positive integer n, depending on a and an element $b \in R$ such that $a^n = a^{n+1}b$. Or equivalently R is strongly π-regular if and only if $a^n R = a^{2n} R$ [8]. It is easy to see that R is strongly π-regular if and only if $R a^n = R a^{2n}$.

Recall that R is weakly right duo (briefly, WRD) [2] if for any $a \in$ R, there exists a positive integer n such that $a^n R = R a^n R$.

Theorem 3.2:

Let R be WRD. Then, R is strongly π-regular if and only if R is sπWR-ring.

Proof:

Assume that R is strongly π-regular, then for every $a \in R$, there exists a positive integer n such that $a^n R = a^{2n} R$. Now,

$a^n R = a^{2n} R$ (R is strongly π-regular)

$= a^n a^n R$

$= a^n (R a^n R)$ (R is WRD; $a^n R = R a^n R$)

$= a^n R a^n R$

$= a^n R(a^{2n} R)$ (R is strongly π-regular; $a^n R = a^{2n} R$)

$= a^n R a^{2n} R$

Therefore, R is sπWR-ring.

Conversely, assume that R is sπWR-ring. Then, $a^{m}R = a^{m}Ra^{2m}R$, for some positive integer m. Sine R is WRD, then $a^{n}R = Ra^{n}R$, for some positive integer n. Now,

$$\begin{aligned} a^{2n}R &= a^{n}a^{n}R \\ &= a^{n}Ra^{n}R \\ &= (Ra^{n}R)a^{n}R \\ &= Ra^{n}a^{n}R \\ &= Ra^{2n}R \end{aligned}$$

So, $a^{kn}R = Ra^{kn}R$, for some positive integer k ...(1)

$$\begin{aligned} a^{2m}R &= a^{m}a^{m}R \\ &= a^{m}(a^{m}Ra^{2m}R) \quad (\text{R is s}\pi\text{WR-ring}) \\ &= a^{2m}Ra^{2m}R \end{aligned}$$

In particular, $a^{km}R = a^{km}Ra^{km}R$ ...(2)

Now,

$$\begin{aligned} a^{mn}Ra^{mn}R &= a^{mn}(Ra^{mn}R) \\ &= a^{mn}(a^{mn}R) \\ &= a^{2mn}R \end{aligned}$$

$$\begin{aligned} a^{mn}Ra^{mn}R &= a^{mn}R(a^{mn}Ra^{mn}R) \\ &= a^{mn}Ra^{mn}(a^{mn}R) \\ &= a^{mn}Ra^{2mn}R \\ &= a^{mn}R \quad (\text{R is s}\pi\text{WR-ring}) \end{aligned}$$

Therefore, $a^{2mn}R = a^{mn}R$, set $mn = l$. Thus, $a^{2l}R = a^{l}R$. So, R is strongly π-regular. ♦

Corollary 3.3:

Let R be a WRD and sπWR-ring. Then, R is π-regular.

Theorem 3.4:

Let R be a WRD ring. If R is sπWR-ring with $r(a^{n}) = 0$, for every $a \in R$ and a positive integer n. Then, R is a division ring.

Proof:

Let R be sπWR-ring. Then, by Theorem 3.2, R is strongly π-regular. Therefore, $a^{n}R = a^{2n}R$, for every $a \in R$ and a positive integer n. Then, $a^{n} = a^{2n}b$, for some $b \in R$ and hence $1 - a^{n}b \in r(a^{n}) = 0$ implies that $1 = a^{n}b \in a^{n}R$. Thus, $a^{n}R = R$ (right invertible). Now, since $a^{n}x = 1$, we have $a^{n}xa^{n} = a^{n}$, which implies that $(1 - xa^{n}) \in r(a^{n}) = 0$. Therefore, $1 - xa^{n} = 0$, whence $xa^{n} = 1$, so $Ra^{n} = R$. Whence R is a division ring. ♦

Proposition 3.5:

Let R be a commutative ring. If x is not nilpotent and right sπ-weakly regular element, then x^{2n} is invertible in R.

Proof:

Assume that x is a right sπ-weakly regular element, there exists b, c $\in$ R and a positive integer n such that $x^n = x^n \, b \, x^{2n} c$. Then, $x^n (1- b \, x^{2n} c) = 0$. Since x is not nilpotent element, then $x^n \neq 0$. Therefore, $1- b \, x^{2n} c = 0$ ($x^n \neq 0$). So, $1 = b \, x^{2n} c$ implies that x^{2n} is invertible. ♦

Theorem 3.6:

Let R be a reduced ring with every essential right ideal is pure. Then, R is sπWR-ring.

Proof:

Let $a \in R$ and $I = R \, a^{2n} R + r(a^n)$. We claim that I is an essential right ideal of R. Suppose this is not true, there exists a nonzero ideal K of R such that $I \cap K = (0)$. Then, $(R \, a^{2n} R) K \subseteq IK \subseteq I \cap K = (0)$. Since $a^{2n} R \subseteq R \, a^{2n} R$, then $a^{2n} R \cap K = (0)$. But, $(a^{2n} R) K \subseteq a^{2n} R \cap K = (0)$ implies $K = (0)$.This contradiction proves that I is an essential right ideal, that is I is pure. Since $a \in I$, there exists $b \in I$ such that a = ab. In particular, $b = c \, a^{2n} d + h$, for some $c, d \in R$ and $h \in r(a^n)$. Therefore, $a^n = a^n b = a^n c \, a^{2n} d + a^n h = a^n c \, a^{2n} d$. Whence R is sπWR-ring. ♦

REFERENCES

[1] Azumaya, G. (1954) "Strongly π-regular rings", *J. Fac. Sci. Hokkaida Univ*. vol. 13, 34-39.

[2] Chen, J. L. and Ding N. Q. (1999) "On general principally injective rings", *Comm. Algebra*, 27(5), 2097-2116.

[3] Cohn, P. M. (1999) "Reversible rings", *Bull. London Math.* Soc., 31, 641-648.

[4] Gupta, V. (1977) "Weakly π-regular rings and group rings", *Math. J. Okayama Univ*. 19, 123-127.

[5] Gupta, V. (1984) "A generalization of strongly regular rings", *Acta. Math. Hung.*, vol. 43 (1-2), 57-61.

[6] Kim, N. K. and Lee Y. (2003) "Extensions of reversible rings", *Journal of Pure and Applied Algebra* 185, 207-223.

[7] Mahmood, A. S. (1990) "On von Neumann regular rings", *M. Sc. Thesis, Mosu University*.

[8] Mohammed, R. M. (1996) "On π-regular rings", *M. Sc. Thesis, Mosul University.*

[9] Vasantha Kandasamy, W. B. (1993) "s-weakly regular group rings", *Arch. Math. (BRNO)*, 39-41.