

ON REGULARITY AND FLATNESS

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الملخص

يقال للحلقة R بأنها من النمط SF-اليمنى ، إذا كان كل مقاس بسيط إيمن فيها مسطحاً. من المعروف أن كل حلقة منتظمة بمفهوم فون نيومان تكون حلقة من النمط SF-اليمنى واليسرى. في هذا البحث أعطينا شروطاً أخرى لكي تكون كل حلقة من النمط SF-اليمنى حلقة منتظمة بقوة . ومن النتائج الأخرى التي حصلنا عليها هي خواص أخرى جديدة للحلقات من النمط SF-اليمنى.

ABSTRACT

A ring R is called a right SF-ring if all its simple right R -modules are flat. It is well known that Von Neumann regular rings are right and left SF-rings. In this paper we study conditions under which SF-rings are strongly regular. Finally, some new characterstic properties of right SF-rings are given.

1. INTRODUCTION

In this paper all rings are assumed to be associative with identity, and all modules are unital right R -modules.

Following [2], a ring R is called a right (left) SF-ring if all of its simple right (left) R -modules are flat . It is well known that a ring R is Von Neumann regular if and only if every right (left) R -module is flat [3]. Ramamurthi in [8] asked whether left and right SF-ring is Von Neumann regular. Many authors have given various conditions for SF-rings to be regular (see, e.g. Chen [1], Ming [4], Rege [9] and Xu-[10]). In this paper, to the list of equivalent conditions, we shall add several news. We recall that:

- 1- A ring R is called reduced if R contains no non-zero nilpotent elements.
- 2- R is said to be Von Neumann regular (or just regular) if $a \in aRa$ for every $a \in R$, and R is called strongly regular if $a \in a^2R$. Clearly, every strongly regular ring is a regular reduced ring.
- 3- R is said to be right duo-ring if every right ideal is a two-sided ideal.
- 4- $r(a)$ and $L(a)$ will denote right and left annihilator of a respectively.
- 5- Following [9], for any ideal I of R , R/I is flat if and only if for each $a \in I$, there exists $b \in I$ such that $a = ba$.
- 6- Y and J will stand for the right singular ideal and Jacobson radical of R .

2. RINGS WHOSE SIMPLE MODULES ARE FLAT

Following [7], a ring R is called ERT-ring if every essential right ideal of R is a two-sided ideal.

Ming [6] proved the following:

Proposition 2.1. If R is a right duo-ring then R/Y is a reduced ring.

We use a similar method of proof in Prop.2.1 to establish the following lemma.

Lemma 2.2: If R is an ERT-ring, then R/Y is a reduced ring.

Proof. Suppose that R/Y is not reduced, then there exists an element

$Y \neq a + Y \in R/Y$, $a \in R$, such that $(a+Y)^2 = Y$. This implies that $a \notin Y$ and $a^2 \in Y$. So $r(a^2)$ is essential right ideal of R . Since R is ERT, then $r(a^2)$ is a two-sided ideal. Let I be any subideal of $r(a^2)$

Such that

$a \notin r(a)$, then $a \in r(a) \subseteq I$, this means that $r(a)$ is essential in $r(a^2)$ and hence in R , this contradicts $a \notin Y$.

The following theorem gives the condition of being right SF-rings are strongly regular.

Theorem 2.3: Let R be a ring. Then the following are equivalent.

- (1) R is strongly regular.
- (2) R is a right SF- and ERT ring.

Proof. (1) $\Rightarrow$ (2) is obvious.

- (2) $\Rightarrow$ (1) By Lemma 2.2, R/Y is a reduced ring. We claim that $Y=0$. Suppose that $Y \neq 0$ then by [5], there exists $0 \neq y \in Y$ such that $y^2 = 0$.

Let M be a maximal right ideal containing $r(y)$. Since $r(y)$ is an essential two-sided ideal of R , then M must be an essential two-sided ideal of R . On the other hand, since R/M is flat module, and since $y \in M$, there exists $c \in M$ such that $y=yc$, whence $1-c \in r(y) \subseteq M$, yielding $1 \in M$ which contradicts $M \neq R$. This proves that R is a reduced ring. In order to show that R is regular we need to prove that $aR+r(a) = R$ for any $a \in R$. Suppose that $aR + r(a) \neq R$, then there exists a maximal right ideal L containing $aR + r(a)$. But $a \in L$ and R/M is flat, there exists $b \in L$ such that $a=ba$, whence $1-b \in L(a) = r(a) \subseteq M$. Yielding $1 \in M$ which contradicts $L \neq R$. In particular $ar+d = 1$, for some $r \in R$ and $d \in r(a)$, whence $a^2r=a$. This proves that R is a strongly regular ring.

We now consider an other condition for right SF-ring to be strongly regular.

Theorem 2.4: Let R be a right SF-ring with every nilpotent element of R is central. Then R is strongly regular.

Proof. Let a be a non-zero element in R with $a^2=0$, and let M be a maximal right ideal containing $r(a)$. Since $a \in r(a) \subseteq M$, and

since R/M is flat, there exists $b \in M$ such that $a = ba$. This implies that $1-b \in L(a)$. But every nilpotent is central gives $r(a)=L(a)$. Whence $1-b \in r(a) \subseteq M$, yielding $1 \in M$, and this contradicts $M \neq R$. Therefore, R is a reduced ring. By a similar method of proof used in Theorem 2.3, R is strongly regular.

3. BASIC PROPERTIES

Recall that a ring R is a right uniform if every right ideal of R is essential.

We are now in a position to give new characteristic properties of a right SF-ring.

Theorem 3.1: If R is a right SF- ring, then

- 1- If $L(a) = 0$, then a is a right invertible.
- 2- Every reduced ideal of R is strongly regular.
- 3- If J is reduced, then $J = 0$.
- 4- If R is a right uniform ring, then R is a division ring.

Proof.

(1) Let $a \in R$ with $L(a)=0$. If $a \notin R^\times$, there exists a maximal right ideal M containing aR . Since $a \in M$ and R/M is flat, there exists $b \in M$, such that $a=ba$. Whence $1-b \in L(a) = 0$, yielding $1 \in M$, which contradicts $M \neq R$. Therefore $aR=R$.

(2) Follows from Theorem 2.3.

(3) Let $a \in J$, then by (2) J is strongly regular, and hence there exists $b \in J$ such that $a=a^2b$. But $a \in J$ gives $(1-ab)u = 1$ for some $u \in R$, this implies that $(a-a^2b)u=a$. Thus $a=0$, consequently, $J=0$.

(4) Suppose that $Y \neq 0$, then there exists a maximal right ideal M containing Y . For any $0 \neq y \in Y$, gives $y \in M$, but R/M is flat, then there exists $x \in M$ such that $y=xy$, whence $y \in r(1-x)$. On the other hand, since R is a right uniform, then $r(1-x)$ is an essential right ideal of R . Thus $1-x \in Y \subseteq M$, this implies that $1 \in M$, contradicting $M \neq R$. Therefore, $Y=0$. On the other hand, since R

is uniform, then for every $a \in R$, $r(a) = 0$, then by (1), R is a division ring.

Before closing this section, we present the following result.

Proposition 3.2: Let R be a reduced right SF- ring, for any $a, b \in R$ with $a.b=0$, then $r(a) + r(b) = R$.

Proof. Suppose that $a.b=0$ and $r(a) + r(b) \neq R$. Then there exists a maximal right ideal M containing $r(a) + r(b)$. Since $a \in r(b) \subseteq M$, and since R/M is flat, there exists $c \in M$ such that $a = ca$, whence $1-c \in L(a) = r(a) \subseteq M$, yielding $1 \in M$, which contradicts $M \neq R$.

Therefore $r(a) + r(b) = R$.

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