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The Core of Inner ideal of Real Four-Dimensional Lie Algebra with One Dimensional Derived

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ABSTRACT: Suppose that V is any inner ideal of L. The core of V is an inner ideal of L with special requirement. In this paper we prove. If L is a 4-dimension Lie algebra a with 1-dimensional derived, then the core of every inner ideal of L is zero. Moreover L containing a sandwich element if $L' \nsubseteq Z$ and every element in L is sandwich if $L' \subseteq Z$.

Keywords: Lie algebras, The $core_L$(V), Sandwich element, four dimensional

1. INTRODUCTION

The idea of inner ideals was initially introduced in 1976 by American scientist Georgia Benkart [1]. According to Benkart, an inner ideal or (simply $I - ideal$) is a subspace V of a Lie algebra L such that $[V,[V,L]] \subseteq V$. If $[V,V] = 0$, then an inner ideal V is said to be commutative. Inner ideals of the Lie subalgebra of simple associative rings were studied in [1] and fully classified in 2009 by Benkart and Fernandiz Lopez in [2]. In 2016, Brox, Fernandez Lopez, and Gomez Lozano examined the case of centrally closed prime rings with involution of characteristic not equal 2, 3, or 5 [3].

Premet explains in [4] and [5] that the inner ideals of Lie algebras perform a similar role to the one-sided ideal of algebras of associative relations; hence, by taking into account the inner ideals of Lie algebras, one can provide instruction for Artinian theory for Lie algebras. Lie algebra is called Artinian if and only if it possesses a descending chain of inner ideals [6]. Shlaka, and Mousa, in 2023 study inner ideals of the special linear Lie algebras of associative simple finite dimensional algebras (see [7]). It is demonstrated in [8, Proposition 2] that each one-sided ideal of a finite dimensional associative algebra A accepts a Levi decomposition and, under certain minimal circumstances, can be created by an idempotent. Baranov and Shlaka achieve the same results for inner ideals in [8], demonstrating that each inner ideal of the Lie algebra $[A,A]$ permits Levi decomposition and may be created by a pair of idempotent elements (if it satisfies some minimal conditions) [9].

Additional incentive to study inner ideals arises from [10], where Fernández López et al demonstrated that when L is any non-degenerate Lie algebra over an abelian ring F, equipped with two and three invertible elements, then every nonzero commutative inner ideal V of finite length in L is complemented by a commutative inner ideal [10].

In 2023, Shlaka and Saeed [11] studied inner ideal of the four-dimensional Lie algebras depending on Schobel classification of Lie algebras in [15]. Shlaka, and Kareem, described abelian non-Jordan-Lie inner ideals of the orthogonal finite dimensional Lie algebras in [13]. The notion of core of an inner ideal is introduced in [9] by Baranov and Shlaka. Let V be an inner ideal of a Lie algebra L. It is well-known that $[V,[V,L]]$ is also an $I - ideal$ of L. Take $V_0 = V$ and consider the following $I - ideal$ of L: $V_n = [V_{n-1},[V_{n-1},L]] \subseteq V$, for all integers n ≥ 1. Then $V = V_0 \supseteq$

$V_1 \supseteq V_2 \supseteq \cdots$. As L is finite dimension, this series terminates, so there is an integer n such that $V_n = V_{n+1}$. Such V_n is said to be the core of V , is denoted by $core_L(V)$.

In this paper, we study the $core_L(V)$ of the 4- dimensional Lie algebras with 1-dimensional derived. We also study sandwich elements of the four-dimensional Lie algebras with 1-dimensional derived. In section two, we start with some preliminary information about inner ideal (or simply I -ideal), the $cor_L(V)$ of I-ideal, sandwich elements. The third section we state basic concept about 4-dimensional Lie algebra with a 1-dimensional derived. In section four, we showed that $core_L(V) = 0$ for every I-ideal of a real 4-dimensinal Lie algebra with a 1-dimensional derived. Section five we proved that if L is a 4-dimensinal Lie algebra with a 1-diensional derived then L contain sandwich element if $L' \nsubseteq Z$. Moreover, every element in L is sandwich elements if $L' \subseteq Z$.

2. PRELIMINARIES

In this section we state some definitions about I-ideal, the cor_L(V) of inner ideal, extremal and zero divisor or (sandwich)element.

Definition 2.1 [8]: Suppose that V is a subspace of L. Then V is said to be an $I - ideal$ of L when $[V,[V,L]] \subseteq V$. We denoted by I-ideal to be an inner ideal of L . The I-ideal is said to be commutative if $[V,V] = 0$.

Let V be an inner ideal of L. Then $[V,[V,L]] \subseteq V$. It's widely recognized $[V,[V,L]] \subseteq V$ is an I-ideal of L (see [11, Lemma 1.1]). Take $V_0 = V$ and suppose that the following $I - ideals$ of : $V_n = [V_{n-1},[V_{n-1},L]] \subseteq V_{n-1}$ for all $n \geq 1$. Then $= V_0 \supseteq V_1 \supseteq V_2 \supseteq \cdots$. As L is finite dimension, this series terminates. This motivates the following definition.

Definition 2.2. [9]: let V be an I-ideal of L and L as a finite dimension Lie algebra. Then there is an integer n such that $V_n = V_{n-1}$. Thus V_n is called the core of V, denoted by $core_L(V)$.

Definition 2.3.[14]: Let $x \in L$ be non zero element, then x is called extremal if $[x,[x,L]] \subseteq Fx$, where $Fx = span\{x \mid x \in L\}$.
The element x is said to be zero-divisor (sandwich) if $\{x \in L \mid [x,[x,L]] = 0\}$.

3. BASIC CONCEPT OF REAL FOUR DIMENSIONAL LIE ALGEBRA WITH 1-DIMENSIONAL DERIVED

In this paper $p \geq 0$, L, V and Z are field of any characteristic, the characteristic of R. Lie algebra over R. Inner ideal of L, and the center of L, respectively. Recall that the center of L. Z_n n-dimensional center. We denoted by U_2 is two-dimensional Lie algebra with the following property $[u_1,u_2] = u_1$, where is a basis for U_2. H_j is j-dimensional Lie algebra with the following property $[h_2,h_3] = [h_4,h_5] = \cdots = [h_{j-1},h_j] = h_1$, and $[h_n,h_m] = 0$ otherwise, where $h_1,h_2,\dots h_j$ are a basis for H_j.
Let L be a four-dimensional. The dimension of L' may be 1,2 3 or 4 . In [15] Schöbel classified the real four-dimensional Lie algebra by relating the dimension of L'. Suppose that the dimensional of L' is 1. Then we have the following result. For the proof see [15].

Theorem 3.1. *[15]: Consider L as a real n-dimensional Lie algebra with a 1-dimensional derived. Then L is one of the following :*

1. If $L' \nsubseteq Z$, then $L = U_2 \oplus Z_{n-2}$.
2. If $L' \subseteq Z$, then $L = H_j \oplus Z_{n-j}$ ($j = 2m - 1, m \geq 2$) .

Where in the case (1) we have $L' \nsubseteq Z$, while in the case (2) we have $L' \subseteq Z$,

Proposition 3.4. *[11]: If $L = U_2 \oplus Z_2$ and $L' \nsubseteq Z$, then*

1. L contains a 1-dimensional I-ideal.
2. L contains a 2-dimensional non-commutative I-ideal.

3. L contains a 3-dimensional non commutative I-ideal.

Remark 3.5. According to [11], if L is a real 4-dimensinal with 1-dimensional derived and $L' \nsubseteq Z$, then L has basis $\{u_1, u_2, z_1, z_2\}$ and the Lie multiplication of this basis is $[u_1, u_2] = u_1$ and otherwise is zero. Thus

1. $span\{u_1\}$, $span\{z_1\}$ and $span\{z_2\}$ are all 1-dimensional I-ideal. However, $span\{u_2\}$ is un I-ideal as proved in [11, Remark 3.3].
2. The only non-commutative I-ideal of a 2-dimensional I-ideal of L is $span\{u_1, u_2\}$.
3. The only non-commutative I-ideal of L are span $\{u_1, u_2, z_1\}$ and $span\{u_1, u_2, z_2\}$.

Proposition 3.6. *[11]: Let $L = H_3 \oplus Z_1$ and $L' \subseteq Z$. Then every 1-dimensional subspace of L is an I-ideal.*

Proposition 3.7. *[11] Let $L = H_3 \oplus Z_1$ and $L' \subseteq Z$. Then every 2-dimensional subspace of L is an I-ideal.*

Proposition 3.8. *[11] Let $L = H_3 \oplus Z_1$ and $L' \subseteq Z$. Then every 3-dimensional subspace of L is an I-ideal.*

4. THE CORE OF THE REAL FOUR-DIMENSIONAL LIE ALGEBRA WITH ONE-DIMINSIONAL DERIVED

In this section we show that, if L is a real 4-dimensitional with 1-dimensional derived. Then L has a $core_L(V) = 0$ for every I-ideal.

Proposition 4.1. *Let $L = U_2 \oplus Z_2$ and $L' \nsubseteq Z$, then*

1. $cor_L(V) = 0$ for every 1-dimensional I-ideal.
2. $cor_L(V) = 0$ for every 2-dimensional I-ideal.
3. $cor_L(V) = 0$ for every 3-dimensional I-ideal.

Proof. By Theorem 3.1 L has basis $\{u_1, u_2, z_1, z_2\}$ such that $[u_1, u_2] = u_1$ and otherwise is zero, and by Proposition 3.2, if $L = U_2 \oplus Z_2$ and $L' \nsubseteq Z$, then L contain a 1, 2 and 3-dimensional I-ideal. Suppose that $y \in L$. Then $y = \lambda u_1 + \mu u_2 + \alpha z_1 + \beta z_2$ for some $\lambda, \mu, \alpha, \beta \in R$.

1. Let V be a 1-dimensional I-ideal of L. Then by Remark 3.5 $V = span\{u_1\}$ or $V = span\{z_1\}$ or $V = span\{z_2\}$. Since $z_1, z_2 \in Z(L)$, it is clear that $core_L(V) = 0$ if $V = span\{z_1\}$ or $span\{z_2\}$. It remains to $V = span\{u_1\}$. We need to show $cor_L(V) = 0$. Let $x \in cor_L(V)$, $a, b \in V$. Then $a = \lambda_1 u_1$ and $b = \mu_1 u_1$ for some $\lambda_1, \mu_1 \in R$.

   Since $x = [a, [b, y]] = [\lambda_1 u_1, [\mu_1 u_1, \lambda u_1 + \mu u_2 + \alpha z_1 + \beta z_2]]$

   $$= [\lambda_1 u_1, \mu_1 \lambda [u_1, u_1] + \mu_1 \mu [u_1, u_2] + \mu_1 \alpha [u_1, z_1] + \mu_1 \beta [u_1, z_2]]$$

   $$= [\lambda_1 u_1, \mu_1 \mu u_1] = \lambda_1 \mu_1 \mu [u_1, u_1] = 0,$$

so $cor_L(V) = 0$.

2. Let V be a 2-dimensional I-ideal of L. Then by Remark 3.5 $V = span\{u_1, u_2\}$ is only non-commutative I-ideal of L . We claim that $cor_L(V) = span\{u_1\}$ and $cor_L(V) \subseteq V_1$, we need to show $V_1 \subseteq cor_L(V)$.
   Since
   $$V_1 = [V, [V, L]]$$
   let $x \in V_1$, then there exists $a, b \in V$ and $y \in L$. Then $a = \alpha_1 u_1 + \beta_1 u_2$, $b = \alpha_2 u_1 + \beta_2 u_2$ and $y = \lambda u_1 + \mu u_2 + \alpha z_1 + \beta z_2$ for some $\alpha_1, \alpha_2, \beta_1, \beta_2, \lambda, \mu, \alpha, \beta \in R$.

Thus

$x = [a, [b, y]]$

$= [\, a, [\, \alpha_2\, u_1 \,+\, \beta_2\, u_2 \,, \lambda\, u_1 \,+\, \mu\, u_2 \,+\, \alpha\, z_1 \,+\, \beta\, z_2 \,]\,]$
$= [\, \alpha_1\, u_1 \,+\, \beta_1\, u_2 \,, \alpha_2\, \mu\, u_1 \,-\, \beta_2\, \lambda\, u_1 \,]$
$= \alpha_1\, \alpha_2\, [\, u_1 \,, u_1 \,] \,-\, \alpha_1\, \beta_2 [\, u_1 \,, u_1 \,] \,+\, \beta_1\, \alpha_2\, [\, u_2 \,, u_1 \,] \,-\, \beta_1\, \beta_2\, \lambda\, [\, u_2 \,, u_1 \,]$
$= -\, \beta_1\, \alpha_2\, u_1 \,+\, \beta_1\, \beta_2\, \lambda\, u_1 \,=\, (\, \beta_1\, \beta_2\, \lambda \,-\, \beta_1\, \alpha_2 \,) u_1 \,\in\, span\{\, u_1 \,\}$.
and

$$V_2 \,=\, [\, V_1 \,, [\, V_1 \,, L \,]\,] \,.$$

Let $x_1 \,\in\, V_2$, then there exists $a_1 \,, b_1 \,\in V_1$ and $y \,\in\, L$. Then $a_1 \,=\, \alpha_1\, u_1 ,\, b_1 \,=\, \alpha_2\, u_1$ and $y = \,\lambda\, u_1 \,+\, \mu\, u_2 \,+\, \alpha\, z_1 \,+\, \beta\, z_2$ for some $\alpha_1 \,, \beta_1 \,, \lambda \,, \mu \,, \alpha \,, \beta \,\in\, R$.
Thus
$x_1 = [\, a_1 \,, [\, b_1 \,, y \,]\,]$
$= [\, \alpha_1\, u_1 \,, [\, \beta_1\, u_1 \,, \lambda\, u_1 \,+\, \mu\, u_2 \,+\, \alpha\, z_1 \,+\, \beta\, z_2 \,]\,]$
$= [\, \alpha_1\, u_1 \,, \beta_1\, \mu\, u_1 \,] \,=\, \alpha_1\, \beta_1\, [\, u_1 \,, u_1 \,] \,= 0,$
so $co\, r_L\, (\, V \,) \,= 0$.

3. Let V be a 3-dimensional I-ideal of L. Then by Remark 3.5 $V = span\{\, u_1 \,, u_2 \,, z_1 \,\}$ and $V = span\{\, u_1 \,, u_2 \,, z_2 \,\}$ are only non-commutative I-ideal of L . We claim that $co\, r_L\, (\, V \,) \,= span\{\, u_1 \,\}$ and $co\, r_L\, (\, V \,) \,\subseteq\, V_1$, we need to show $V_1 \,\subseteq\, co\, r_L\, (\, V \,)$. Since

$$V_1 \,= [\, V, [\, V, L \,]\,]$$

let $x \,\in\, V_1$, then there exists $a, b \,\in\, V$ and $y \,\in\, L$. Then $a = \,\alpha_1\, u_1 \,+\, \beta_1\, u_2 \,+\, \gamma_1\, z_1 \,, b = \,\alpha_2\, u_1 \,+\, \beta_2\, u_2 \,+\, \gamma_2\, z_1$ and $y = \lambda\, u_1 \,+\, \mu\, u_2 \,+\, \alpha\, z_1 \,+\, \beta\, z_2$ for some $\alpha_1 \,, \alpha_2 \,, \beta_1 \,, \beta_2 \,, \gamma_1 \,, \gamma_2 \,, \lambda \,, \mu \,, \alpha \,, \beta \,\in\, R$.
Thus
$x = [\, a, [\, b, y \,]\,]$
$= [\, a, [\, \alpha_2\, u_1 \,+\, \beta_2\, u_2 \,+\, \gamma_2\, z_1 \,, \lambda\, u_1 \,+\, \mu\, u_2 \,+\, \alpha\, z_1 \,+\, \beta\, z_2 \,]\,]$
$= [a, \alpha_2\, \lambda\, [\, u_1 \,, u_1 \,] \,+\, \alpha_2\, \mu\, [\, u_1 \,, u_2 \,] \,+\, \alpha_2\, \alpha\, [\, u_1 \,, u_1 \,] \,+\, \alpha_2\, \beta\, [\, u_1 \,, z_2 \,] \,+\, \beta_2\, \lambda\, [\, u_2 \,, v_1 \,]$
$+\, \beta_2\, \mu\, [\, u_2 \,, v_2 \,] \,+\, \beta_2\, \alpha\, [\, u_2 \,, z_1 \,] \,+\, \beta_2\, \beta\, [\, u_2 \,, z_2 \,] \,+\, \gamma_2\, \lambda\, [\, u \,, v_1 \,] \,+\, \gamma_2\, \mu\, [\, u_1 \,, u_2 \,]$
$+\, \gamma_2\, \alpha\, [\, z_1 \,, z_1 \,] \,+\, \gamma_2\, \beta\, [\, z_1 \,, z_2 \,]$
$= [\, \alpha_1\, u_1 \,+\, \beta_1\, u_2 \,+\, \gamma_1\, z_1 \,, \alpha_2\, \mu\, u_1 \,-\, \beta_2\, \lambda\, u_1 \,]$
$= -\, \beta_1\, \alpha_2\, \mu\, u_1 \,+\, \beta_1\, \beta_2\, \lambda\, u_1 \,=\, (\, \beta_1\, \beta_2\, \lambda \,-\, \beta_1\, \alpha_2\, \mu) u_1 \,\in\, span\{\, u_1 \,\}$.
and

$$V_2 \,=\, [\, V_1 \,, [\, V_1 \,, L \,]\,] \,.$$

Let $x_1 \,\in\, V_2$, then there exists $a_1 \,, b_1 \,\in\, V$ and $y \,\in\, L$. Then $a_1 \,=\, \alpha_1\, u_1 \,, b_1 \,=\, \alpha_2\, u_1$ and $y = \,\lambda\, v_1 \,+\, \mu\, v_2 \,+\, \alpha\, z_1 \,+\, \beta\, z_2$ for some $\alpha_1 \,, \beta_1 \,, \lambda \,, \mu \,, \alpha \,, \beta \,\in\, R$.
Thus
$x_1 \,= [\, a_1 \,, [\, b_1 \,, y \,]\,]$
$= [\, \alpha_1\, u_1 \,, [\, \beta_1\, u_1 \,, \lambda\, u_1 \,+\, \mu\, u_2 \,+\, \alpha\, z_1 \,+\, \beta\, z_2 \,]\,]$
$= [\, \alpha_1\, u_1 \,, \beta_1\, \mu\, u_1 \,] \,=\, \alpha_1\, \beta_1\, [\, u_1 \,, u_1 \,] \,= 0,$
so $co\, r_L\, (\, V \,) \,= 0$.

Proposition 4.2. *Let* $L = \, H_3 \,\oplus\, Z_1$ *and* $L' \,\subseteq\, Z$. *Then every 1-dimensional subspace of* L *has* $cor\, e_L\, (\, V \,) = 0$ *for every I-ideal.*

Proof. Let V be a 1-dimensional subspace of L and let $v \,\in\, V$ be a nonzero. Then $\{v\}$ from a basis of V. We extend $\{v\}$ to form a basis $\{\, h_1 \,, h_2 \,, h_3 \,, v\}$, where $h_1 \,, h_2 \,, h_3 \in H_3$ such that the Lie multiplication of this basis satisfy the condition of Theorem 3.1 and by Proposition 3.4 let $L = \, H_3 \,\oplus\, Z_1$ and $L' \,\subseteq\, Z$. Then every 1-dimensional subspace of L is an I-ideal. we need to show $cor\, e_L\, (\, V \,) \,= 0$. Let $x \,\in\, core_L\, (\, v \,)$, $a, b \,\in\, V$ and $y \,\in\, L$. Then $a = \,\lambda\, v, b = \,\mu\, v$ and $y = \,\alpha\, h_1 \,+\, \beta\, h_2 \,+\, \gamma\, h_3 \,+\, \delta\, v$ fore some $\lambda \,, \mu \,, \alpha \,, \beta \,, \gamma \,, \delta \,\in\, R$. Then

$$x = [a, [\, b, y \,]\; (1)$$

By Theorem 3.1 we need to consider four cases dependent on the multiplication of $h_1 \,, h_2 \,, h_3$ and v whether $[\, h_1 \,, h_2 \,] \,=\, h_1 \,, v \,\in\, Z_1$, or $[\, h_2 \,, h_3 \,] \,= v$; $h_1 \,\in\, Z_1$ or $[\, v, h_3 \,] \,=\, h_1 \,, h_2 \,\in\, Z_1$ or $[\, h_2 \,, v \,] \,=\, h_1 \,, h_3 \,\in\, Z_1$.

Case 1. Suppose initially that $V \,\in\, Z_1$ and $[\, h_2 \,, h_3 \,] \,= h_1$, otherwise is zero. By Equation 1
$x = [\, a, [\, b, y \,]\,] \,= [\, a, [\mu\, v, \alpha\, h_1 \,+\, \beta\, h_2 \,+\, \gamma\, h_3 \,+\, \delta\, v \,]\,]$
$= [\, a, \mu\, \alpha\, [\, v, h_1 \,] \,+\, \mu\, \beta\, [\, v, h_2 \,] \,+\, \mu\, \gamma\, [\, v, h_3 \,] \,+\, \mu\, \delta\, [\, v, v \,]\,]$
$= [\, a, 0 \,] \,= 0, so\; co\; r\; L\, (\, V \,) \,= 0.$

Suppose now that $h_1 \in Z_1$ and $[\, h_2 \,, h_3 \,] \,= v$, otherwise is zero. By Equation 1
$x = [\, a, [\, b, y \,]\,] \,= [\, a, 0 \,] \,= 0,$
$so\; cor\; e\; L\, (\, V \,) \,= 0.$
Suppose Next that $h_2 \,\in Z_3$ and $[\, v, h_3 \,] \,=\, h_1$, otherwise is zero. By Equation1

$$x = [\,a, [\,b, y\,]\,] = [\,a, [\,\mu\, v, \alpha\, h_1 + \beta\, h_2 + \gamma\, h_3 + \delta\, v\,]\,]$$
$$= [\,\lambda\, v, \mu\, \gamma\,[\,v, h\,3\,]\,] = [\,\lambda\, v, \mu\, \gamma\, h\,1\,] = \lambda\, \mu\, \gamma\,[\,v, h\,1\,] = 0$$
$$so\ cor\ e\ L\,(\,V\,) = 0\,.$$

Finally let $h_3 \in Z_1$ and $[\,h_2\,, v\,] = h_1$, otherwise is zero. By Equation1
$$x = [\,a, [\,b, y\,]\,] = [\,\lambda\, v, [\,\mu\, v, \alpha\, h_1 + \beta\, h_2 + \gamma\, h_3 + \delta\, v\,]\,]$$
$$= [\,\lambda\, v, -\,\mu\, \beta\, h_1\,] = -\,\lambda\, \mu\, \beta\,[\,v, h\,1\,] = 0,$$
so $cor\ e_L\,(\,V\,) = 0$.

Proposition 4.3. *Let* $L = H_3 \oplus Z_1$ *and* $L' \subseteq Z$. *Then every 2-dimensional subspace of L has* $cor\ e_L\,(\,V\,) = 0$ *for every I-ideal.*

Proof. Let V be a two-dimension subspace of L , and let $v_1\,, v_2 \in V$ be a non-zero. Then $\{\,v_1\,, v_2\,\}$ from a basis of V. We extend $\{\,v_1, v_2\,\}$ to form a basis $\{\,h_1\,, h_2\,, v_1, v_2\,\}$ such that the Lie multiplication of this basis satisfies the condition of Theorem 3.1 and by Proposition 3.5 let $L = H_3 \oplus Z_1$ and $L' \subseteq Z$. Then every 2-dimensional subspace of L is an I-ideal. We need to show that $cor\ e_L\,(\,v\,) = 0$.
Let $x \in cor\ e_L\,(\,V\,)$, $a, b \in V$ and $y \in L$. Then $a = \lambda_1\, v_1 + \mu_1\, v_2$, $b = \lambda_2\, v_1 + \mu_2\, v_2$ and $y = \alpha\, h_1 + \beta\, h_2 + \gamma\, v_1 + \delta\, v_2$ for $some\ \lambda_1\,, \lambda_2\,, \mu_1\,, \mu_2\,, \alpha\,, \beta\,, \gamma\,, \delta \in R$.
Then
$$x = [a, [\,b, y\,]\ (2)$$
By Theorem 3.1 we need to consider six cases depending on the multiplication of $h_1\,, h_2\,, v_1$ and v_2 whether $[\,v_2\,, h_1\,] = v_1$; $h_2 \in Z_1$ or $[\,h_1, v_2\,] = v_1$; $h_2 \in Z_1$ or $[\,h_1\,, h_2\,] = v_1$; $v_2 \in Z_1$ or $[\,v_1\,, v_2\,] = h_1$; $h_2 \in Z_1$ or $[\,v_1\,, h_2\,] = h_1$; $v_2 \in Z_1$ or $[\,h_2\,, v_1\,] = h_1$; $v_2 \in Z_1$

Case 1. Suppose first that $h_2 \in Z_1$ and $[\,v_2\,, h_1\,] = v_1$, otherwise is zero. By Equation 2
$$x = [a, [\,b, y\,]$$
$$= [\,a, [\,\lambda_2\, v_1 + \mu_2\, v_2\,, \alpha\, h_1 + \beta\, h_2 + \gamma\, v_1 + \delta\, v_2\,]\,]$$
$$= \lambda\,1\,\mu\,1\,[\,v_1\,, v_1\,] + \lambda_1\, \mu_2\, \alpha\,[\,v_2\,, v_1\,] = 0,$$
$$so\ co\ r_L\,(\,V\,) = 0.$$

Case 2. Suppose now that $h_2 \in Z_1$ and $[\,h_1, v_2\,] = v_1$, otherwise is zero. By Equation 2
$$x = [a, [\,b, y\,] = [\,a, [\,\lambda_2\, v_1 + \mu_2\, v_2\,, \alpha\, h_1 + \beta\, h_2 + \gamma\, v_1 + \delta\, v_2\,]\,]$$
$$= [\,a, -\,\mu_2\, \alpha\, v_1\,] = [\,\lambda_1\, v_1 + \mu_1\, v_2\,, -\,\mu_2\, \alpha\, v_1\,]$$
$$= -\,\lambda_1\, \mu_2\, \alpha\,[\,v_1\,, v_1\,] - \mu_1\, \mu_2\, \alpha\,[\,v_2\,, v_1\,] = 0,$$
$$so\ cor\ e_L\,(\,V\,) = 0.$$

Case 3. Suppose that $v_2 \in Z_1$ and $[\,h_1\,, h_2\,] = v_1$, otherwise is zero. By Equation 2
$$x = [a, [\,b, y\,] = [\,a, 0\,] = 0,$$
$$so\ cor\ e\ L\,(\,V\,) = 0.$$

Case 4. Suppose that $h_2 \in Z_1$ and $[\,v_1\,, v_2\,] = h_1$, otherwise is zero. By Equation 2
$$x = [a, [\,b, y\,] = [\,a, [\,\lambda_2\, v_1 + \mu_2\, v_2\,, \alpha\, h_1 + \beta\, h_2 + \gamma\, v_1 + \delta\, v_2\,]\,]$$
$$= \lambda_1\, \mu_2\,[\,v_1\,, h_1\,] - \lambda_1\, \mu_2\,[\,v_1\,, h_1\,] - \mu_1\, \mu_2\, \delta\,[\,v_2\,, h_1\,] - \mu_1\, \mu_2\, \gamma\,[\,v_2\,, h_1\,] = 0,$$
$$so\ cor\ e_L\,(\,V\,) = 0.$$

case 5. Suppose that $v_2 \in Z_1$ and $[\,v_1\,, h_2\,] = h_1$, otherwise is zero. By Equation2
$$x = [a, [\,b, y\,] = [\,a, [\,\lambda_2\, v_1 + \mu_2\, v_2\,, \alpha\, h_1 + \beta\, h_2 + \gamma\, v_1 + \delta\, v_2\,]\,]$$
$$= [a, \lambda_2\, \alpha\,[\,v_1\,, h_1\,] + \lambda_2\, \beta\,[\,v_1\,, h_2\,] + \lambda_2\, \gamma\,[\,v_1\,, v_1\,] + \mu_2\, \delta\,[\,v_1\,, v_2\,] + \mu_2\, \alpha\,[\,v_2\,, h_1\,] + \mu_2\, \beta\,[\,v_2\,, h_2\,] + \mu_2\, \gamma\,[\,v_2\,, v_1\,] + \mu_2\, \delta\,[\,v_2\,, v_2\,]$$
$$= [\,\lambda_1\, v_1 + \mu_1\, v_2\,, \lambda_2\, \beta\, h_1\,]$$
$$= \lambda_1\, \lambda_2\, \beta\,[\,v_1\,, h_1\,] + \mu_1\, \lambda_2\, \beta\,[\,v_2, h_1\,] = 0,$$
$$so\ cor\ e\ L\,(\,V\,) = 0.$$

Case 6. Suppose that $v_2 \in Z_1$ and $[\,h_2\,, v_1\,] = h_1$, otherwise is zero. By Equation 2
$$x = [a, [\,b, y\,] = [\,a, [\,\lambda\,2\,v\,1 + \mu\,2\,v\,2\,, \alpha\, h\,1 + \beta\, h\,2 + \gamma\, v\,1 + \delta\, v\,2\,]\,]$$
$$= [a, \lambda_2\, \alpha\,[\,v_1\,, h_1\,] + \lambda_2\, \beta\,[\,v_1\,, h_2\,] + \lambda_2\, \gamma\,[\,v_1\,, v_1\,] + \mu_2\, \delta\,[\,v_1\,, v_2\,] + \mu_2\, \alpha\,[\,v_2\,, h_1\,] + \mu_2\ \beta\,[\,v_2\,, h_2\,] + \mu\,2\,\gamma\,[\,v\,2\,, v\,1\,] + \mu\,2\,\delta\,[\,v\,2\,, v\,2\,]$$
$$= [\,\lambda_1\, v_1 + \mu_1\, v_2\,, -\,\lambda_2\, \beta\, h_1\,]$$
$$= -\,\lambda_1\, \lambda_2\, \beta\,[\,v_1\,, h_1\,] - \mu_1\, \lambda_2\, \beta\,[\,v_2\,, h_1\,] = 0,$$

$so\ cor\ e\ L\ (\ V\ )\ =0.$

Proposition 4.4. *Let* $L=\ H_3\ \oplus\ Z_1$ *and* $L'\ \subseteq\ Z$*. Then every 3-dimensional subspace of* L *has* $cor\ e_L\ (\ V\ )\ =\ 0$ *for every I-ideal.*

Proof. Let V be a 3-dimensinal subspace of L and $\{\ v_1\ ,v_2\ ,v_3\ \}$ be a basis of V . we extend $\{\ v_1\ ,v_2\ ,v_3\ \}$ to form a basis $\{\ h_1\ ,v_1\ ,v_2\ ,v_3\ \}$ such that the Lie multiplication of this basis satisfies the condition Theorem 3.1 and by Proposition prop 3.6 let $L=\ H_3\ \oplus\ Z_1$ and $L'\subseteq Z$. Then every 3-dimensional subspace of L is an I-ideal.
We need to show $cor\ e_L\ (\ V\ )=0$. Let $x\ \in\ cor\ e_L\ (\ v\ )$,$a,b\ \in\ V$ and $y\ \in\ L$. Then $a=\ \alpha_1\ v_1\ +\ \alpha_2\ v_2\ +\ \alpha_3\ v_3$, $b=\ \beta_1\ v_1\ +\ \beta_2\ v_2\ +\ \beta_3\ v_3$ and $y=\ \lambda_1\ h_1\ +\ \lambda_2\ v_1\ +\ \lambda_3\ v_2\ +\ \lambda_4\ v_3$ for some $\alpha_1\ ,\alpha_2\ ,\beta_1\ ,\beta_2\ ,\lambda_1\ ,\lambda_2\ ,\lambda_3\ ,\lambda_4\ \in R$. Then

$$x=[a,[\ b,y\ ]\ (3)$$

By Theorem 3.1 we need to consider four cases depending on the multiplication of $h_1\ ,v_1\ ,v_2$ and v_3 whether $[\ v_2\ ,v_3\ ]\ =v_1\ ;\ h_1\ \in\ Z_1$ or $[\ v_2\ ,h_1\ ]\ =\ v_1\ ;v_3\ \in\ Z_1$ or $[\ h_1\ ,v_2\ ]\ =\ v_1\ ;v_3\ \in\ Z_1$ or $[\ v_1\ ,v_2\ ]\ =\ h_1\ ;v_3\ \in\ Z_1$.

Suppose first that $v_1,v_2,v_3\notin Z$, and $[\ v_2\ ,v_3\ ]\ =\ v_1$ otherwise is zero, and $x_1\ \in\ Z_1$. By Equation 3

$$x=[\ a,[\ b,y\ ]\ ]$$
$$=[a,[\ \beta_1\ v_1\ +\ \beta_2\ v_2\ +\ \beta_3\ v_3\ ,\lambda_1\ h_1\ +\ \lambda_2\ v_1\ +\ \lambda_3\ v_2\ +\ \lambda_4\ v_3\ ]$$
$$=[a,\beta_1\ \lambda_1\ [\ v_1\ ,h_1\ ]\ +\ \beta_1\ \lambda_2\ [\ v_1\ ,v_1\ ]\ +\ \beta_1\ \lambda_3\ [\ v_1\ ,v_2\ ]\ +\ \beta_1\ \lambda_4\ [\ v_1\ ,v_3\ ]\ +\ \beta_2\ \lambda_1\ [\ v_2\ ,h_1\ ]\ +\ \beta_2\ \lambda_2\ [\ v_2\ ,v_1\ ]$$
$$+\ \beta_2\ \lambda_3\ [\ v_2\ ,v_2\ ]\ +\ \beta_2\ \lambda_4\ [\ v_2\ ,v_3\ ]\ +\ \beta_3\ \lambda_1\ [\ v_3\ ,h_1\ ]\ +\ \beta_3\ \lambda_2\ [\ v_3\ ,v_1\ ]\ +\ \beta_3\ \lambda_3\ [\ v_3\ ,v_2\ ]$$
$$+\ \beta_3\ \lambda\ 4\ [\ v_3\ ,v_3\ ]$$
$$=[\ \alpha_1\ v_1\ +\ \alpha_2\ v_2\ +\ \alpha_3\ v_3\ ,\beta_2\ \lambda_4\ v_1\ -\ \beta_3\ \lambda_3\ v_1\ ]$$
$$=\ \alpha_1\ \beta_2\ \lambda_4\ [\ v_1\ ,v_1\ ]-\ \alpha_1\ \beta_3\ \lambda_3\ [\ v_1\ ,v_1\ ]+\ \alpha_2\ \beta_2\ \lambda_4\ [\ v_2\ ,v_1\ ]\ -\ \alpha_2\ \beta_3\ \lambda_3\ [\ v_1\ ,v_1\ ]\ +\ \alpha_3\ \beta_3\ \lambda_3\ [\ v_3\ ,v_1\ ]\ =0,$$
$$so\ co\ r_L\ \ (\ V\ )\ =0.$$

Suppose now that $v_1,v_2\notin Z$, and $[\ v_2\ ,h_1\ ]=\ v_1$ otherwise is zero, and $v_3\in Z_1$. Then

$$x=[\ a,[\ b,y\ ]\ ]\ =[\ \alpha_1\ v_1\ +\ \alpha_2\ v_2\ +\ \alpha_3\ v_3\ ,\beta_2\ \lambda_1\ v_1\ ]$$
$$=\ \alpha_1\ \beta_2\ \lambda_1\ [\ v_1\ ,v_1\ ]\ +\ \alpha_2\ \beta_2\ \lambda_1\ [\ v_2\ ,v_1\ ]\ +\ \alpha_3\ \beta_2\ \lambda_1\ [\ v_3\ ,v_1\ ]\ =0,$$
$$so\ co\ r_L\ (\ V\ )\ =0.$$

Suppose next that $v_1,v_2\notin Z$, and $[\ h_1\ ,v_2\ ]\ =\ v_1$ otherwise is zero, and $v_3\ \in\ Z_1$. By Equation 3

$$x=[\ a,[\ b,y\ ]\ ]$$
$$=[\ \alpha_1\ v_1\ +\ \alpha_2\ v_2\ +\ \alpha_3\ v_3\ ,-\ \beta_2\ \lambda_1\ v_1\ ]$$
$$=[-\ \alpha_1\ \beta_2\ \lambda_1\ [\ v_1\ ,v_1\ ]\ -\ \alpha_1\ \beta_2\ \lambda_1\ [\ v_2\ ,v_1\ ]\ -\ \alpha_3\ \beta_2\ \lambda_1\ [\ v_3\ ,v_1\ ]\ =0,$$
$$so\ cor\ e_L\ (\ V\ )\ =0.$$

Finally let $v_1,v_2\notin Z$, and $[\ v_1\ ,v_2\ ]\ =\ h_1$ otherwise is zero, and $[\ v_1\ ,v_2\ ]\ =\ h_1$. By Equation 3

$$x=[\ a,[\ b,y\ ]\ ]$$
$$=[\ \alpha_1\ v_1\ +\ \alpha_2\ v_2\ +\ \alpha_3\ v_3\ ,\beta_1\ \lambda_3\ h_1\ -\ \beta_2\ \lambda_2\ h_1\ ]$$
$$=\ \alpha_1\ \beta_1\ \lambda_3\ [\ v_1\ ,h_1\ ]-\ \alpha_1\ \beta_2\ \lambda_2\ [\ v_1\ ,h_1\ ]+\ \alpha_2\ \beta_1\ \lambda_3\ [\ v_2\ ,h_1\ ]-\ \alpha_2\ \beta_2\ \lambda_2\ [\ v_2\ ,h_1\ ]+\ \alpha_3\ \beta_1\ \lambda_3\ [\ v_1\ ,h_1\ ]$$
$$-\ \alpha_1\ \beta_2\ \lambda_2\ [\ v_2\ ,h_1\ ]\ =0,$$
$$so\ cor\ e_L\ (\ V\ )\ =0.$$

Theorem 4.5. Suppose that L is a real 4-dimansional Lie algebra with a 1-dimensional derived, then $cor_L(V)=0$ for every I-ideal.

Proof. Since L is 4-dimension with a 1-dimensional derived. Then by Theorem 3.1 either $L=U_2\oplus Z_2$ or $L=H_3\oplus Z_1$.
Suppose first that $L=U_2\oplus Z_2$. Then by proposition 4.1. $cor_L(V)=0$ for every I-ideal.
Suppose now that $L=H_3\oplus Z_1$.
Thus

1- By Proposition 4.2 every 1-dimensional subspace has $core_L(V)=0$ for every I-ideal.
2- By Proposition 4.3 every 2-dimensional subspace has $core_L(V)=0$ for every I-ideal.
3- By Proposition 4.4 every 3-dimensional subspace has $core_L(V)=0$ for every I-ideal.

5. SANDWICH ELEMENTS OF A REAL FOUR DIMENSIONAL LIE ALGEBRA WITH 1-DIMENSIONAL DERIVED

In this section we proved that if L is a 4-dimensinal Lie algebra with a 1-diensional derived then L contain sandwich element if $L' \nsubseteq Z$. Moreover every elements in L is sandwich elements if $L' \subseteq Z$.

Proposition 5.1. *Let* $L = U_2 \oplus Z_2$ *and* $L' \nsubseteq Z$, *then* L *contain a sandwich element.*

Proof. By Theorem 3.1 L has basis $\{u_1, u_2, z_1, z_2\}$ such that $[u_1, u_2] = u_1$ and otherwise is zero .
Suppose that $x \in L$, and $y \in L$.
Then $x = u_1$ and $y = \lambda u_1 + \mu u_2 + \alpha z_1 + \beta z_2$ for some $\lambda, \mu, \alpha, \beta \in R$.
Since

$$
\begin{aligned}
[u_1,[u_1,y]] &= [u_1,[u_1,\lambda u_1 + \mu u_2 + \alpha z_1 + \beta z_2]] \\
&= [u_1,\lambda[u_1,u_1] + \mu[u_1,u_2] + \alpha[u_1 z_1] + \beta[u_1,z_2]] \\
&= [u_1,\mu u_1] \\
&= \mu[u_1,u_1] = 0,
\end{aligned}
$$

so u_1 is a sandwich elements.

Remark 5.2. Proposition 5.1 is not true if we state *let* $L = U_2 \oplus Z_2$ *and* $L' \nsubseteq Z$, *then every elements in* L *is a sandwich element.* As one can see in the following example.

Example 5.3. Consider $x \in L$ and $y \in L$. Then $x = u_2$ and $y = \lambda u_1 + \mu u_2 + \alpha z_1 + \beta z_2$.
Since

$$
\begin{aligned}
[u_2,[u_2,y]] &= [u_2,[u_2,\lambda u_1 + \mu u_2 + \alpha z_1 + \beta z_2]] \\
&= [u_2,\lambda[u_2,v_1] + \mu[u_2,v_2] + \alpha[u_2,z_1] + \beta[u_2,z_2]] \\
&= [\mathrm{u}_2, -\lambda \mathrm{u}_1] = -\lambda[u_2,u_1] \\
&= \lambda u_1.
\end{aligned}
$$

Thus, u_2 is non-sandwich elements.

Proposition 5.4. *Let* $L = H_3 \oplus Z_1$ *and* $L' \subseteq Z$. *Then every basis of* L *is a sandwich elements .*

Proof. By Theorem 3.1 L has basis $\{h_1, h_2, h_3, z\}$ such that $[h_2, h_3] = h_1$ and otherwise is zero. Let $y \in L$. Then $y = \alpha h_1 + \beta h_2 + \gamma h_3 + \delta z$, we need to show that each basis element is sandwich. Since $z \in Z(L)$ it is clear that z is sandwich element. It remain to show that h_1, h_2 and h_3 are all sandwich elements.

Suppose first that $h_1 \in L$ and $y \in L$. Then $y = \alpha h_1 + \beta h_2 + \gamma h_3 + \delta z$ fore some $\lambda, \mu, \alpha, \beta, \gamma, \delta \in R$.
Since

$$
\begin{aligned}
[h_1,[h_1,y]] &= [h_1,[h_1,\alpha h_1 + \beta h_2 + \gamma h_3 + \delta z] \\
&= [h_1, 0] = 0,
\end{aligned}
$$

so h_1 is sandwich element.

Suppose now that $h_2 \in L$ and $y \in L$. Then $y = \alpha h 1 + \beta h 2 + \gamma h 3 + \delta z$ fore some $\lambda, \mu, \alpha, \beta, \gamma, \delta \in R$.
Since

$$
\begin{aligned}
[h_2,[h_2,y]] &= [h_2,[h_2,\alpha h_1 + \beta h_2 + \gamma h_3 + \delta z]] \\
&= [h_2,\alpha[h_2,h_1] + \beta[h_2,h_2] + \gamma[h_2,h_3] + \delta[h_2,z]] \\
&= [h_2,\gamma h_1] = \gamma[h_2,h_1] = 0,
\end{aligned}
$$

so h_2 is sandwich elements and let

Finally let $h_3 \in V$ and $y \in L$. Then $y = \alpha h_1 + \beta h_2 + \gamma h_3 + \delta z$ fore some $\lambda, \mu, \alpha, \beta, \gamma, \delta \in R$.
Since

$$
\begin{aligned}
&[h_3,[h_3,y]] = [h_3,[h_3,\alpha h_1 + \beta h_2 + \gamma h_3 + \delta z] \\
&= [h_3,\alpha[h_3,h_1] + \beta[h_3,h_2] + \gamma[h_3,h_3] + \delta[h_3,z]] \\
&= [h_3,-\beta h_1] = -\beta[h_3,h_1] = 0,
\end{aligned}
$$

so h_3 is sandwich elements.

Theorem 5.5. Suppose that L is a real 4-dimansional Lie algebra with a1-dimensional derived, then L contain a sandwich element if $L' \nsubseteq Z$ and every element in L is a sandwich if $L' \subseteq Z$.

Proof. Since L is 4-dimensional Lie algebra with a 1-dimensional derived. Then by Theorem 3.1 either $L = U_2 \oplus Z_2$ or $L = H_3 \oplus Z_1$.

Suppose first that $L = U_2 \oplus Z_2$. Then by proposition 5.1. L contain a sandwich element.

Suppose now that $L = H_3 \oplus Z_1$. Then by Proposition 5.4. L has a sandwich element for every basis of L.

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