



A Generalization of the Concave Integral in Terms of Decomposition of the Integrated Function for Bipolar Scales

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Article information

Article history:

Received: July, 07, 2021

Accepted: November, 16, 2021

Available online: December, 04, 2021

Keywords:

Capacities,
Concave integral,
Bi-capacities,
Bipolar concave integral,
Multiple-criteria decision analysis

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Abstract

In the context of the multiple-criteria decision aid (MCDA), several fuzzy integrals concerning capacities (non-additive measures) have been introduced by various researchers in the last sixty years. Recently, Lehrer has proposed a new integral for capacities known as concave integral. The concave integral is based on the decomposition of random variables into simple ingredients. The concave integral concerning capacity is defined as the maximum value obtained among all its decompositions. The paper aims to model a new integration based on the decomposition of random variables into simple ingredients for multi-criteria decision making support when underlying scales are bipolar. This paper proposes a generalization of the concave integral in terms of decompositions of the integrated function to be suitable for bipolar scales. We show that the random variable is analyzed as a combination of indicators, where each allowed decomposition has a value determined by the bi-capacity. Lastly, we illustrate our framework by a practical example.

1. Introduction

The applications of fuzzy integrals are widely used in many fields, (e.g. image processing, face recognition, economic, computational intelligence, multi-criteria decision making problem, biology, education, finance, pattern recognition, data fusion, operations research, etc.). Several fuzzy integrals concerning capacities (non-additive measures) have been introduced by various researchers in the last sixty years [1-7]. The fuzzy integrals with respect to non-additive measures have been studied and applied in diverse fields [8-10].

Grabisch and Labreuche [11] proposed the concept of bi-capacity as a generalization of capacity. The bipolar fuzzy integrals for bi-capacities and their possible applications have been introduced and discussed in recent literature [12-14] as an extension of the fuzzy integral for cases in which the underlying scale is bipolar.

Recently, Lehrer [15] introduced a new integral for capacities based on a decomposition of random variables into simple ingredients. The concave integral for capacity is defined as the maximum value obtained among all its decompositions. In this paper, we propose a generalization of the concave integral in terms of decompositions of the integrated function to be suitable for bipolar scales. In this integration, we show that the random variable is analyzed as a combination of indicators, where each allowed decomposition has a value determined by the bi-capacity.

The paper is organized as follows. The following section recalls the background that is needed in this paper. Section 3 discusses the notion of bi-capacity. In section 4, we propose a generalization of the concave integral for bi-capacity. Section 5 gives a practical example of the generalization of the concave integral for bipolar scales. Section 6 presents the results and discussion of the application of the generalized concave integral. Lastly, the paper finished with some conclusions. Throughout the paper, $\mathbb{N}$ is the set of natural numbers, R is the set of real numbers, $X = \{x_1, x_2, \dots, x_n\}$ be a universal set of n elements and $P(X)$ denotes the power set of X .

2. Background and basic concepts

In this section, we present the background of the study to the construction of the generalizations of the concave integral that we shall discuss in the paper.

Let $(X, \mathbb{S})$ be a measurable space, where X is a non-empty set and $\mathbb{S}$ is a σ - algebra of subsets of X . A capacity [1] (or fuzzy measure [2]) is a generalization of classical measure using non-additivity property instead of additivity property. The definition of the capacity is as follows.

Definition 1. [2] Let $X = \{x_1, x_2, \dots, x_n\}$ be a universal set. A capacity is a function $\mu : \mathbb{S} \rightarrow [0, 1]$ satisfies:

1. $\mu(\emptyset) = 0, \mu(X) = 1,$
2. for all $A, B \in \mathbb{S}, A \subseteq B, \mu(A) \leq \mu(B).$

In decision analysis, especially in multiple-criteria decision analysis, has introduced several non-additive integrals for capacities in the last 60 years. One of them is concave integral known as Lehrer integral. The concave integral concept has been proposed in [15] as a novel approach to integrals for capacities, as follows.

Let $\mathbb{F}$ be a class of all finite nonnegative real-valued measurable functions on measurable space $(X, \mathbb{S})$. For any $f \in \mathbb{F}$, the sum $\sum_{i=1}^n \alpha_i I_{A_i}$ is a decomposition of f , if $\alpha_i \geq 0$ for every $n \in \mathbb{N}$ and

$\sum_{i=1}^n \alpha_i I_{A_i} = f$, where I_{A_i} denoted to the indicator of A_i , which is the random variable that takes 1 over A_i , and value 0 otherwise. The concave integral of f with respect to μ is an optimal decomposition $\sum_{i=1}^n \alpha_i \mu(A_i)$ among all decompositions of f , as given in the following definition.

Definition 2. [15] Let $(X, \mathbb{S})$ be a measurable space and $f \in \mathbb{F}$. The concave integral of f respect to capacity μ is defined by

$$(L) \int f d\mu = \bigvee \left\{ \sum_{i=1}^n \alpha_i \mu(A_i) : \sum_{i=1}^n \alpha_i I_{A_i} \leq f, \alpha_i \geq 0, \{A_i\}_{i=1}^n \subset \mathbb{S}, n \in \mathbb{N} \right\}. \quad (1)$$

3. Definition of bi-capacities

Even though capacities can catch a wide group of decision activities, they may be inefficient in some circumstances, specifically when the scales are bipolar. Hence, in many practical cases, it is natural to use a scale going from negative (bad) to positive (good) values, including a central neutral value, to encode the bipolarity of the effect. Such a scale is called a bipolar scale, typical examples are $[-1, 1]$ (bounded cardinal), R (unbounded cardinal) or $\{\text{very bad, bad, medium, good, excellent}\}$ (ordinal). “Let us take for simplicity the $[-1, 1]$ scale, with neutral value 0”.

Then it is natural to use a scale going from negative (bad) to positive (good) values, including a central neutral value, to encode the bipolarity of the effect. Such a scale is called a bipolar scale, typical examples are $[-1, 1]$ (bounded cardinal), R (unbounded cardinal) or $\{\text{very bad, bad, medium, good, excellent}\}$ (ordinal).

In a more general model, by considering that independence between positive and negative parts does not hold, so that we have to consider ternary alternatives $(1_A, -1_B, 0_{(A \cup B)^c})$, and assign to each of them a number in $[-1, 1]$. We denote this number as $\mu(A, B)$, i.e., a two-argument function, whose first argument is the set of totally satisfied criteria, and the second one the set of totally unsatisfied criteria, the remaining criteria being at the neutral level.

In the field of multi-criteria decision making, Grabisch and Labreuche[11] have presented the definition of bi-capacity as follows.

Let $Q(X) = 3^X = \{(A_1, A_2) \in P(X) \times P(X) \mid A_1 \cap A_2 = \emptyset\}$ be the set all disjoint pairs of sets and equip it with a binary relation $\sqsubseteq$ for arbitrary $(A_1, A_2), (B_1, B_2) \in Q(X)$ such that:

$$(A_1, A_2) \sqsubseteq (B_1, B_2) \text{ iff } A_1 \subseteq B_1 \text{ and } A_2 \supseteq B_2 \quad (2)$$

In this order on the structure $(Q(X), \sqsubseteq)$ becomes a lattice. *Sup* and *Inf* are denoted by $\sqcup, \sqcap$, respectively.

The Supremum is given by

$$(A_1, A_2) \sqcup (B_1, B_2) = (A_1 \cup B_1, A_2 \cap B_2) \quad (3)$$

Infimum is given by

$$(A_1, A_2) \sqcap (B_1, B_2) = (A_1 \cap B_1, A_2 \cup B_2) \quad (4)$$

The top and bottom are, $(X, \emptyset)$ & $(\emptyset, X)$ respectively, (for more details [11]).

There is another order relation on the structure $Q(X)$ has introduced by Bilbao et al. [12]. This order is simply the product order on $P(X) \times P(X)$:

$$(A_1, A_2) \sqsubseteq (B_1, B_2) \text{ iff } A_1 \subseteq B_1 \text{ and } A_2 \subseteq B_2 \quad (5)$$

Infimum is given by

$$(A_1, A_2) \sqcap (B_1, B_2) = (A_1 \cap B_1, A_2 \cap B_2) \quad (6)$$

always exists in $Q(X)$, while the Supremum does not exist in general. Hence, this ordered set is an inf-semilattice with a bottom element $(\emptyset, \emptyset)$. For more details [16 & 17].

Definition 5. [4] A bi-capacity on X is a set function $\nu_b: Q(X) \rightarrow [-1, 1]$ if

- 1- $\nu_b(\emptyset, \emptyset) = 0$, $\nu_b(N, \emptyset) = 1$ and $\nu_b(\emptyset, N) = -1$
- 2- $\forall (A_1, A_2), (B_1, B_2) \in Q(X)$, $(A_1, A_2) \sqsubseteq (B_1, B_2) \Rightarrow \nu_b(A_1, A_2) \leq \nu_b(B_1, B_2)$.

4. A generalization of concave integral

In this section, we propose an expression of the bipolar concave integral in terms of decompositions of the integrated function as a generalization of the concave integral.

Let us denote by $I_{(A,B)}$ for the indicator of (A, B) , which is the random variable that takes 1 over A , -1 over B , and value 0 otherwise. That is,

$$I_{(A,B)} = \begin{cases} 1 & \text{iff } i \in A \\ -1 & \text{iff } i \in B \\ 0 & \text{otherwise} \end{cases}$$

Where, A represents the positive part and B represent the negative part. Therefore, $\alpha_i I_{(A_i, B_i)}$ is a decomposition of f . This means that a particular decomposition of f is used to calculate the bipolar concave integral for the bi-capacity.

Thus, we can write the bipolar concave integral in terms of decompositions of the integrated function f for ν_b as the following definition.

Definition 2. Let $\nu_b: Q(X) \rightarrow [-1, 1]$ be a bi-capacity and $f \in \mathbb{F}$. the bipolar concave integral of f respect to bi-capacity μ_b is defined by

$$(BL) \int f d\nu_b = \bigvee \left\{ \sum_{i=1}^n \alpha_i \nu_b(A_i, B_i) : \sum_{i=1}^n \alpha_i I_{(A_i, B_i)} \leq f, \alpha_i \geq 0, n \in \mathbb{N} \right\}. \quad (7)$$

5. Practical Example

Let us try to illustrate the generalized concave integral (bipolar concave integral) by a case of evaluation of students from an example very well known in the specialized literature [13].

In the admission process, the Director of the College seeks to make a sound decision regarding the order of the students in the College who apply for Postgraduate Studies in Economics where certain prerequisites are required. There are 3 criteria (Mathematics (M), Statistics (S), and Languages (L)) serve as a basis for the preselection of the candidates. We suppose that the students are evaluated on the same bipolar scale from -10 to 10, with a neutral value of 0.

Let us consider five students (A, B, C, D, and E) having the evaluations presented in Table 1, and assume that the bi-capacity values as shown in Table 2. In the next section, we show that the generalized concave integral can model and solve this problem.

Table 1: Degrees of students

Subjects Students	Mathematic (M)	Statistic (S)	Language (L)
Student A	4	6	-3
Student B	3	4	-2
Student C	-1	6	-3
Student D	-1	5	-2
Student E	5	3	-1

Table 2: Bi-capacity values

$v_b(A, B)$	$\emptyset$	{M}	{S}	{L}	{M, S}	{M, L}	{S, L}	{M,S,L}
$\emptyset$	0	-2	-2	-2	-6	-4	-6	-10
{M}	2	-	-2	-2	-	-	-6	-
{S}	2	-3	-	-3	-	-6	-	-

{L}	2	-2	-2	-	-4	-	-	-
{M, S}	6	-	-	3	-	-	-	-
{M, L}	4	-	2	-	-	-	-	-
{S, L}	6	3	-	-	-	-	-	-
{M,S,L}	10	-	-	-	-	-	-	-

6. Results and discussion

The decompositions of the considered functions (students: A, B, C, D, and E) are shown in the following tables (Table 3, 4, 5, 6, 7):

Table 3: Decomposition of student A

1.	$4v_b(\{M\}, \emptyset) + 6v_b(\{S\}, \emptyset) + 3v_b(\emptyset, \{L\})$	$4(2) + 6(2) + 3(-2) = 14$
2.	$4v_b(\{M, S\}, \emptyset) + 2v_b(\{S\}, \emptyset) + 3v_b(\emptyset, \{L\})$	$4(6) + 2(2) + 3(-2) = 22$
3.	$3v_b(\{M, S\}, \{L\}) + v_b(\{M, S\}, \emptyset) + 2v_b(\{S\}, \emptyset)$	$3(3) + (6) + 2(2) = 19$
4.	$3v_b(\{M, S\}, \{L\}) + v_b(\{M\}, \emptyset) + 3v_b(\{S\}, \emptyset)$	$3(3) + (2) + 3(2) = 17$
5.	$4v_b(\{M, S\}, \emptyset) + 2v_b(\{S\}, \{L\}) + v_b(\emptyset, \{L\})$	$4(6) + 2(-3) + (-2) = 16$
6.	$3v_b(\{M, S\}, \emptyset) + 3v_b(\{S\}, \{L\}) + v_b(\{M\}, \emptyset)$	$3(6) + 3(-3) + (2) = 11$
7.	$3v_b(\{S\}, \{L\}) + 4v_b(\{M\}, \emptyset) + 3v_b(\{S\}, \emptyset)$	$3(-3) + 4(2) + 3(2) = 5$
8.	$3v_b(\{M\}, \{L\}) + v_b(\{M, S\}, \emptyset) + 5v_b(\{S\}, \emptyset)$	$3(-2) + (6) + 5(2) = 10$
9.	$3v_b(\{M\}, \{L\}) + v_b(\{M\}, \emptyset) + 6v_b(\{S\}, \emptyset)$	$3(-2) + (2) + 6(2) = 8$

Table 4: Decomposition of student B

1.	$3v_b(\{M\}, \emptyset) + 4v_b(\{S\}, \emptyset) + 2v_b(\emptyset, \{L\})$	$3(2) + 4(2) + 2(-2) = 10$
2.	$3v_b(\{M, S\}, \emptyset) + v_b(\{S\}, \emptyset) + 2v_b(\emptyset, \{L\})$	$3(6) + (2) + 2(-2) = 16$
3.	$2v_b(\{M, S\}, \{L\}) + v_b(\{M, S\}, \emptyset) + v_b(\{S\}, \emptyset)$	$2(3) + (6) + (2) = 14$
4.	$2v_b(\{M, S\}, \{L\}) + v_b(\{M\}, \emptyset) + 2v_b(\{S\}, \emptyset)$	$2(3) + (2) + 2(2) = 12$
5.	$3v_b(\{M, S\}, \emptyset) + v_b(\{S\}, \{L\}) + v_b(\emptyset, \{L\})$	$3(6) + (-3) + (-2) = 13$

6.	$2v_b(\{M, S\}, \emptyset) + 2v_b(\{S\}, \{L\}) + v_b(\{M\}, \emptyset)$	$2(6) + 2(-3) + (2) = 8$
7.	$2v_b(\{S\}, \{L\}) + 3v_b(\{M\}, \emptyset) + 2v_b(\{S\}, \emptyset)$	$2(-3) + 3(2) + 2(2) = 4$
8.	$2v_b(\{M\}, \{L\}) + v_b(\{M, S\}, \emptyset) + 3v_b(\{S\}, \emptyset)$	$2(-2) + (6) + 3(2) = 8$
9.	$2v_b(\{M\}, \{L\}) + v_b(\{M\}, \emptyset) + 4v_b(\{S\}, \emptyset)$	$2(-2) + (2) + 4(2) = 6$

Table 5: Decomposition of student C

1.	$v_b(\emptyset, \{M\}) + 6v_b(\{S\}, \emptyset) + 3v_b(\emptyset, \{L\})$	$(-2) + 6(2) + 3(-2) = 4$
2.	$v_b(\{S\}, \{M, L\}) + 2v_b(\{S\}, \{L\}) + 3v_b(\{S\}, \emptyset)$	$(-6) + 2(-1) + 3(2) = -2$
3.	$v_b(\{S\}, \{M, L\}) + 5v_b(\{S\}, \emptyset) + 2v_b(\emptyset, \{L\})$	$(-6) + 5(2) + 2(-2) = 0$
4.	$3v_b(\{S\}, \{L\}) + 3v_b(\{S\}, \emptyset) + v_b(\emptyset, \{M\})$	$3(-3) + 3(2) + (-2) = -5$
5.	$v_b(\emptyset, \{M, L\}) + 2v_b(\{S\}, \{L\}) + 4v_b(\{S\}, \emptyset)$	$(-4) + 2(-3) + 4(2) = -2$
6.	$v_b(\emptyset, \{M, L\}) + 6v_b(\{S\}, \emptyset) + 2v_b(\emptyset, \{L\})$	$(-4) + 6(2) + 2(-2) = 4$
7.	$v_b(\{S\}, \{M\}) + 3v_b(\{S\}, \{L\}) + 2v_b(\{S\}, \emptyset)$	$(-3) + 3(-3) + 2(2) = -8$
8.	$v_b(\{S\}, \{M\}) + 5v_b(\{S\}, \emptyset) + 3v_b(\emptyset, \{L\})$	$(-3) + 5(2) + 3(-2) = 1$

Table 6: Decomposition of student D

1.	$v_b(\emptyset, \{M\}) + 5v_b(\{S\}, \emptyset) + 2v_b(\emptyset, \{L\})$	$(-2) + 5(2) + 2(-2) = 4$
2.	$v_b(\{S\}, \{M, L\}) + v_b(\{S\}, \{L\}) + 3v_b(\{S\}, \emptyset)$	$(-6) + (-3) + 3(2) = -3$
3.	$v_b(\{S\}, \{M, L\}) + 4v_b(\{S\}, \emptyset) + v_b(\emptyset, \{L\})$	$(-6) + 4(2) + (-2) = 0$
4.	$2v_b(\{S\}, \{L\}) + 3v_b(\{S\}, \emptyset) + v_b(\emptyset, \{M\})$	$2(-3) + 3(2) + (-2) = -2$
5.	$v_b(\emptyset, \{M, L\}) + v_b(\{S\}, \{L\}) + 4v_b(\{S\}, \emptyset)$	$(-4) + (-3) + 4(2) = 1$
6.	$v_b(\emptyset, \{M, L\}) + 5v_b(\{S\}, \emptyset) + v_b(\emptyset, \{L\})$	$(-4) + 5(2) + (-2) = 4$
7.	$v_b(\{S\}, \{M\}) + 2v_b(\{S\}, \{L\}) + 2v_b(\{S\}, \emptyset)$	$(-3) + 2(-3) + 2(2) = -5$
8.	$v_b(\{S\}, \{M\}) + 4v_b(\{S\}, \emptyset) + 2v_b(\emptyset, \{L\})$	$(-3) + 4(2) + 2(-2) = 1$

Table 7: Decomposition of student E

1.	$5v_b(\{M\}, \emptyset) + 3v_b(\{S\}, \emptyset) + v_b(\emptyset, \{L\})$	$5(2) + 3(2) + 1(-2) = 14$
2.	$v_b(\{M, S\}, \{L\}) + 2v_b(\{MS\}, \emptyset) + 2v_b(\{M\}, \emptyset)$	$(3) + 2(6) + 2(2) = 19$
3.	$v_b(\{M, S\}, \{L\}) + 2v_b(\{S\}, \emptyset) + 4v_b(\{M\}, \emptyset)$	$(3) + 2(2) + 4(2) = 15$
4.	$v_b(\{M\}, \{L\}) + 3v_b(\{M, S\}, \emptyset) + v_b(\{M\}, \emptyset)$	$1(-2) + 3(6) + 1(2) = 18$
5.	$v_b(\{M\}, \{L\}) + 4v_b(\{M\}, \emptyset) + 3v_b(\{S\}, \emptyset)$	$1(-2) + 4(2) + 3(2) = 12$
6.	$3v_b(\{M, S\}, \emptyset) + 2v_b(\{M\}, \emptyset) + v_b(\emptyset, \{L\})$	$3(6) + 2(2) + 1(-2) = 20$
7.	$v_b + 5v_b(\{M\}, \emptyset) + 2v_b(\{S\}, \emptyset)$	$1(-3) + 5(2) + 2(2) = 11$
8.	$v_b(\{S\}, \{L\}) + 2v_b(\{M, S\}, \emptyset) + 3v_b(\{M\}, \emptyset)$	$1(-3) + 2(6) + 3(2) = 15$

When we need the best candidates without restriction, the bipolar concave integral is suitable for finding the solution based on an optimal decomposition of the functions (students: A, B, C, D, and E). Therefore, we will use it to find the solution.

$$(BL) \int f dv_b = \bigvee \left\{ \sum_{i=1}^n \alpha_i v_b(A_i, B_i) : \sum_{i=1}^n \alpha_i I_{(A_i, B_i)} \leq f, \alpha_i \geq 0, n \in \mathbb{N} \right\}.$$

$$(BL) \int Adv_b = 4v_b(\{M, S\}, \emptyset) + 2v_b(\{S\}, \emptyset) + 3v_b(\emptyset, \{L\}) = 22$$

$$(BL) \int Bdv_b = 3v_b(\{M, S\}, \emptyset) + v_b(\{S\}, \emptyset) + 2v_b(\emptyset, \{L\}) = 16$$

$$(BL) \int Cdv_b = v_b(\emptyset, \{M\}) + 6v_b(\{S\}, \emptyset) + 3v_b(\emptyset, \{L\}) = 4$$

$$(BL) \int Ddv_b = v_b(\emptyset, \{M\}) + 5v_b(\{S\}, \emptyset) + 2v_b(\emptyset, \{L\}) = 4$$

$$(BL) \int Edv_b = 3v_b(\{M, S\}, \emptyset) + 2v_b(\{M\}, \emptyset) + v_b(\emptyset, \{L\}) = 20$$

Table 8: Results

Subjects Students	Mathematic (M)	Statistic (S)	Language (L)	Evaluation by bipolar concave Integral
Student A	4	6	-3	22
Student B	3	4	-2	16
Student C	-1	6	-3	4
Student B	-1	5	-2	4
Student E	5	3	-1	20

From Table 8, the highest value of bipolar concave integral values is 22. Therefore, we conclude that student A is the best of the students and should be admitted first for graduate studies in economics.

7. Conclusions

This study introduced bipolar concave integral in terms of the decomposition of the integral function and illustrated our framework by a practical example. In this integration, the random variable is analyzed as a combination of indicators, where each allowed decomposition has a value determined by the bi-capacity.

We believe that concave's integral concept will allow for new theoretical developments, wherein bipolarity is included in choices, both in multi-criteria decision-making and decisions under risk and uncertainty.

The proposed results are consistent as generalizations of the concave integral for capacities. Therefore, we trust that other integrals [12] [18-20] may be generalized in a similar manner. Also, one of the topics of our subsequent researches will be a discussion and examination of the properties and relationship of the bipolar concave integral with other integrals. Because of the high potential computational complexity, efficient algorithms are needed to compute bipolar concave integral. Therefore, some algorithms used in applications can be developed [21-23]).

Acknowledgements

The authors wish to thank the anonymous reviewers for their careful reading and valuable suggestions, which helped to improve the original version of this contribution.

Conflict of Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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