On Generalized PF – Rings

Nazar H. Shuker Husam Q. Mohammad
Dep. of Mathematics
College of Computer Sciences and Mathematics
University of Mosul

Received on: 11/6/2002 **Accepted on: 20/7/2002**

الملخص

الهدف من هذا البحث هو توسيع بعض النتائج المعروفة في الحلقات من النمط **GPF**. فضلاً عن ذلك درسنا العلاقات بين الحلقات من النمط **GPF** والحلقات المنتظمة من النمط **π**، والحلقات من النمط **PF** وكذلك المثاليات من النمط **GP**. ومن النتائج الأخرى التي حصلنا عليها هي " لتكن **R** حلقة موحدة ، فان **R**حلقة من النمط **GPF** إذا وإذا فقط فإنَ كل عنصر في **R** إما إلا يكون من قواسم الصفر وإما أن يكون عنصراً معدوم القوى " .

ABSTRACT

The aim of this paper is to extend several known results on GPF –rings. π-regular rings, PF-rings and GP-ideals are also considered. Among other results we prove that: If R is a uniform ring, then R is a GPF-ring if and only if every element of R is either non-zero divisor or nilpotent.

1. INTRODUCTION

Throughout this paper R denotes a commutative ring with identity and module means unitary R-module. Recall that (1) an ideal I of the ring R is said to be pure if for every $a \in I$, there exists $b \in I$ such that a = ab [2];(2) R is called π-regular if for every $a \in R$, there exists a positive integer n such that $a^n \in a^n R a^n$; (3) An R-model M is called general right principally injective (briefly GP-injective)if,for any $0 \neq a \in R$, there exists a positive

integer n such that $a^n \neq 0$ and any R-homomorphism of a^nR into M extends to one of R into M [4]; (4) For any $a \in R$, ann(a) will denote the annihilator of a; (5) For any ideal I of R, R/I is a flat R-module if and only if for every $a \in I$, there exists $b \in I$ such that a=ab [6] ;(6) R is called quniform ring if every non-zero ideal of R is an essential ideal;(7) Z(R) and J(R) will stand respectively for the singular ideal of R, and the Jacobson radical of R.

2.GPF-RIGS (BASIC PROPERTIES)

Recall that a ring R is said to be a PF-ring if for every $a \in R$, the principal ideal aR is flat R-module .

A ring R is called generalized PF-ring (GPF-ring) if for any $a \in R$, there exists a positive integer n such that a^nR is a flat R-module. Clearly every PF-ring is a GPF-ring[2] .

Recall the following result of Naoum [5].

Lemma 2.1: A ring R is a PF-ring if and only if for every $a \in R$ ann(a) is pure ideal of R.

In [2] AL – Ezeh proved that:

Lemma 2.2: A ring R is a GPF-ring if and only if for every a $\in$ R, there exists a positive integer n such that ann(a^n) is a pure ideal.

In [1]Aritico and Marconi proved that:
Theorem2.3.A ring R is a PF-ring if and only if
ann(a) +ann(b)=R, whenever ab = 0
The next results generalize Theorem 2.3

Theorem 2.4: A ring R is a GPF-ring if and only if there exists a positive integer n, whenever $a^nb = 0$, ann(a^n)+ann(b)=R.

Proof. Let R be a GPF - ring, and Let $0 \neq a \in R$, then there exists a positive integer n such that ann(a^n) is pure . For any $b \in R$, if

$a^n b=0$, we claim that $ann(a^n) + ann(b) =R$. Suppose that $ann(a^n) + ann (b) \neq R$, then there exists a maximal ideal M containing $ann(a^n) +ann(b)$, since $a^n b=0$, then $b\in ann(a^n)$ by purity of $ann(a^n)$, there exists $c\in ann(a^n)$ such that $b=bc$. This implies that $1-c\in ann(b)\subseteq M$, but $c\in ann(a^n) \subseteq M$, whence $1\in M$ which contradicts $M\neq R$. Consequently $ann(a^n)+ann(b)=R$. Conversely, assume that $ann(a^n)+ann(b)=R$ where $a^n b=0$.

In particular $c_1+c_2=1$ for some $c_1\in ann(a^n)$ and $c_2\in ann (b)$. Multiplying by b, we get $bc_1=b$. Hence $ann(a^n)$ is pure ideal. Therefore R is GPF-ring.

Following [3], an ideal I of the ring R is said to be generalized pure ideal (GP-ideal) if for every $a\in I$ there exists $b \in I$ and a positive integer n such that $a^n=a^n b$.

The following theorem characterizes GPF-ring in terms of GP-ideal.

Theorem 2.5: Let R be a GPF-ring then for every $a \in R$, ann(a) is a GP-ideal.

Proof. Let $0 \neq a\in R$, and let $b\in ann(a)$.Since R is a GPF-ring, then there exists a positive integer n such that $ann(b^n)$ is pure(Lemma2.2). Applying Theorm2.5, we get $ann(b^n)+ann(a) =R$. In particular there exists $c_1\in ann(b^n)$ and $c_2\in ann(a)$ such that $c_1+c_2=1$ multiplying by b^n we have $b^n c_2=b^n$. Therefore ann(a) is a GP-ideal of R.

Next we consider the singular ideal of GPF-ring.

Proposition 2.6 Let R be a GPF-ring. Then Z(R) is a nilideal.

Proof. Let a be a non –zero element of Z(R) then $ann(a^n)$ is pure, for some positive integer n. We claim that $ann(a^n) \cap a^n R=(0)$.Let $x \in ann (a^n)\cap a^n R$. Then $x a^n = 0$ and $x = a^n r$ for some $r\in R$. On the other hand, since $ann(a^n)$ is pure then there exists $y\in ann(a^n)$ such that $x=xy$. Whence $x=a^n ry=ra^n y=0$, yielding $ann (a^n) \cap a^n R=0$.

Since $a^n \in Z(R)$,then $ann(a^n)$ is an essential ideal of R. Whence it follows that $a^n R= 0$, thus $a^n = 0$. Z(R), therefore, is a nilideal.

3. THE CONNECTION BETWEEN GPF – RINGS AND OTHER RINGS

In this section we give further properties GPF-ring and the link between GPF-rings and other rings.

We shall begin this section with the following result, which gives the connection between GPF-ring and PF-rings.

Theorem 3.1: Let R be a GPF-ring, and let, $J^2=0$. Then R/J is PF-ring.

Proof. Let $a+J \in R/J$, $a \in R$. Since R is a GPF-ring, then there exists a positive integer n such that $ann(a^n)$ is pure. In order to prove R/J is a PF-ring we need to prove that ann(a+J) is pure. Let $x+J \in$ ann(a+J), then $ax \in J$. Since $J^2=0$, we have $(ax)^2=a^2 x^2=0$. Whence $x^2 \in ann(a^2)$. But $ann(a^2) \subseteq ann(a^n)$ for $n \geq 1$. On the other hand, since $ann(a^n)$ is pure, there exists $y \in ann(a^n)$ such that $x^2 y=x^2$, and this implies that $x^2(y-1)=0$. But $[x(y-1)]^2= x^2(y-1)^2=0$. Thus $x(y-1) \in N \subseteq J$. Whence it follows that $(x+J)(y+J)= x+J$ Since R/J is reduced,then $ann(a^n+J)=ann(a+J)$, whence it follows that $y+J \in ann(a^n+J)=ann(a+J)$ hence ann(a+J) is pure. R/J is, therefore, a PF-ring.

We now consider a necessary and sufficient condition for GPF-ring to be π-regular.

Before stating this result, the following lemma is needed.

Lemma 3.2: For any $a \in R$, if $ann(a^n)$ pure,
then $ann(a^n) = ann(a^{n+1})$.

Proof. Obviously $ann(a^n) \subseteq ann(a^{n+1})$.

Let $x \in ann(a^{n+1})$, then $xa^{n+1}=0$, which implies that $xa \in ann(a^n)$. Since $ann(a^n)$ is pure, there exists $y \in ann(a^n)$ such that $xa=yxa$, and this implies that $xa^n=yxa^n=0$. Hence $x \in ann(a^n)$. Therefore $ann(a^n)=ann(a^{n+1})$.

Theorem 3.3: A ring R is π -regular if and only if R is a GPF-ring with every maximal ideal is GP-ideal.

Proof. Let R be π -regular ring. Then for any $a\in R$, there exists a position integer n and $b\in R$ such that $a^n=a^{2n}b$ so $1-a^nb\in ann(a^n)$. Now for any $x \in ann(a^n)$, $x(1-a^n b)=x$, thus $ann(a^n)$ is pure. Therefore R is a GPF-ring.

Conversely, assume that R is a GPF-ring , and let $0 \neq a\in R$. then there exists a positive integer n such that $ann(a^n)$ is pure. We claim that $a^n R+ann(a^n)=R$. In fact, if not, there exists a maximal ideal M containing $a^nR+ann(a^n)$. On the other hand, since M is GP-ideal , and $a^n\in M$, there exists $b\in M$, and a positive integer m such that $(a^n)=(a^n)^m b$. Whence it follows that $a^{nm}(1-b)=0$, and then we have $1-b\in ann(a^{nm})$. Applying Lemma 3.2, we get $ann(a^n)=ann(a^{nm})$. Whence $1-b\in ann(a^n)$, yielding $1\in M$ which contradicts $M\neq R$. Hence $a^n R + ann(a^n)=R$, and R is, therefore, π -regular.

Our next result characterizes a uniform GPF-ring in terms of nilpotent and non-zero divisor elements.

Theorem 3.4: Let R be a uniform ring. Then R is a GPF-ring if and only if every element of R is either non –zero divisor or nilpotent.

Proof. Let R be a uniform GPF-ring, and let $0\neq a\in R$, then there exists a positive integer n such that $ann(a^n)$ is pure. Now, for any $b\in ann(a^n)$, there exists $c \in ann(a^n)$ such that b=bc. First we claim that $bR\cap ann(c)=0$. In fact, if not, let $0\neq x\in bR\cap ann(c)$ then x=br for some $r\in R$ and cx=0. But cx=cbr=br=x=0, a contradiction. On the other hand, since R is a uniform then either ann(c) =0 or bR=0. If ann(c) =0, then $a^n c=0$ gives $a^n = 0$. Thus a is a nilpotent element. Now, if bR= 0 then b=0, and hence $ann(a^n)=0$. Therefor, a is a non – zero divisor.

Conversely, for any $a \in R$, if a is a non-zero divisor, then $ann(a^n)=0$, which is pure. On the other hand, if a is nipoltent then $a^m=0$, for some positive integer m. So $ann(a^m)=R$ which is also pure. Therefore, R is a GPF-ring

Finally we introduce the following result.

Proposition 3.5: Let R be a ring with every simple singular R-module is GP-Injective. Then R is a GPF-ring.

Proof. Let $0 \neq a \in R$, and let $a^n b=0$ for some positive integer n and $b \in R$. Suppose that $ann(a^n)+ann(b) \neq R$, then there exists a maximal ideal M containing $ann(a^n)+ann(b)$. First suppose that M is not essential ideal of R, then M must be a direct summand, and hence there exists an idempotent element $0 \neq e$ in R such that $M=ann(e)$. But $a^n b=0$ gives $a^n \in ann(b) \subseteq ann(e)$. Whence it follows that $a^n e = 0$, yielding $e \in ann(a^n) \subseteq ann(e)$ which implies that $e=e^2=0$. Therefore M must be essential. Define $f: a^n R \rightarrow R/M$ by $f(a^n r)=r+M$, for all $r \in R$. Now since R/M is GP-Injective, then there exists $c \in R$ such that $f(a^n r)=c\ f\ (a^n r)$, which implies that $1+M=f(a^n)=ca^n+M$. Whence $1-ca^n \in M$ and hence $1 \in M$ which contradicts $M \neq R$. Therefore $ann(a^n)+ann(b)=R$, where $a^n b=0$. Therefore R is a GPF-ring.

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