Journal University of Kerbala , Vol. 18 Scientific 2020

The Effect of Pauli gates on the superposition for four-qubit in Bloch sphere

تأثير بوابات باولي على حالة التراكب لأربع بتات كمومية في كرة بلوخ

Basem R. Kazem*, Mezher B. Saleh**
* **Department of Physics, College of Science, Al- Mustansiriyah University
*E-mail: basem79@uomustansiriyah.edu.iq
**E-mail: mezher69@uomnstansiriyah.edu.iq

Abstract

In this paper, we show the results that can be obtained as four-sphere form Bloch sphere by using four-electrons after the happening of the superposition between them, showing the effect of Pauli gates on the superposition states. Each electron here represents as a single qubit. The superposition here is represented by sixteen states based on the base 2^n (n the number of electrons). We can select the specific state with one electron from the four electrons, for which the gate will applied to it and show the outputs as a four sphere forms from Bloch sphere. With the possibility of using be two gates and two electrons for a specific case. We are able to use two different electrons or the same electron twice then apply the two gates on them, and we can use the same gate twice. When applying X gate on the any qubit from the superposition state, it will rotate the qubit around the x-axis. The same way, when the Y gate is applied for any qubit from the superposition state, it will rotate the specified qubit around the Y-axis but appearance the Factor (i), which has no effect here because we focus on drawing the state not the measurement. In case affecting the qubit by using the Y gate and for two consecutive times will return the qubit to its original state, but to the opposite point, this is because of the Negative signal. As for the Z gate, if affecting a particular qubit it will remain with no change for the state$|0\rangle$, but the change in phase of state from $|0\rangle$ to $-|1\rangle$ i.e. the point of state will be on the opposite point and the reason is the appearance of the negative signal in front of the state. In case the Z gate is repeated for two consecutive times and on the same qubit, with if the state$|1\rangle$, the qubit will return to the original state. All the results here will appear as drawings with mentioning the gate used, together with mentioning the former state and the final states after applying gates on them.

Keywords: Qubit states, Bloch sphere, superposition for four-qubit, Pauli gates, Quantum Circuit.

الملخص

في هذا البحث، يمكن عرض النتائج التي يمكن الحصول عليها على شكل أربع كرات من كرات بلوخ باستخدام أربعة إلكترونات بعد حدوث التراكب بينها، موضحا عليها تتأثر بوابات باولي على حالات التراكب. يمثل كل إلكترون هنا كيوبت واحد. يمثل التراكب هنا من خلال ستة عشر حالة وفق القاعدة n^2 (n عدد الإلكترونات). يمكننا تحديد حالة معينة ولألكترون واحد من الإلكترونات الأربعة، والتي سيطبق عليها البوابة وتظهر المخرجات في صورة أشكال كروية من كرات بلوخ. مع إمكانية أستخدم برنامج ثاني يحتوي على بوابتين ولإلكترونين ولحالة معينة. يمكننا استخدام إلكترونين مختلفين أو الإلكترون نفسه لمرتين ثم نطبق البوابات عليهما، مع إمكانية استخدام نفس البوابة لمرتين. عند تطبيق بوابة X على أي كيوبت من حالة التراكب، فإنه سيتم تدوير الكيوبت حول محور x. بنفس الطريقة، عندما يتم تطبيق البوابة Y لأي كيوبت من حالة من حالات التراكب، فسوف تقوم بتدوير الكيوبت المحدد حول المحور y ولكن تظهر العامل (i)، والذي لا يؤثر هنا لأننا نركز على رسم الحالة وليس القياس. في حالة

التأثير على الكوبيت باستخدام بوابة Y ولمرتين متعاقبتين، سيعيد الكبت إلى حالته الأصلية، ولكن إلى النقطة المعاكسة، يكون ذلك بسبب الإشارة السالبة. بينما بالنسبة للبوابة Z ، إذا كان تأثيرها على كيوبت معين سيبقى دون أي تغيير للحالة $|0\rangle$، ولكن التغيير في مرحلة الحالة من $|1\rangle$، إلى $-|1\rangle$ ، أي نقطة الحالة ستكون على النقطة المعاكسة والسبب هو ظهور الاشارة السلبة أمام الحالة. في حالة تكرار البوابة Z مرتين متتاليتين وعلى نفس الكوابت، وإذا كانت الحالة $|1\rangle$، فإن الكوبيت سيعود إلى الحالة الأصلية. ستظهر جميع النتائج هنا كرسومات مع ذكر البوابة المستخدمة، والحالة السابقة والحالات النهائية بعد تطبيق البوابات عليها.

1. Introduction

The superposition principle is an important feature in the work of quantum computers. Here we will look at the superposition and represent it as the Bloch sphere. The superposition principle can provide n of the states (for n qubits, we can make calculations 2^nparallel by quantum algorithms) and the impossibility of transcription for any case of states. The quantum computer can be in several states at the same time, thus speed up calculations because of the huge quantum parallelism. [1]

There are many physical systems that can provide the qubits, such as spin the electrons for up or down, state for atoms ground or excited, a polarization of photons and other systems that are subject to quantum mechanics laws.

The quantum bit representation provides the possibility of storing and downloading information in large quantities, taking the advantage from the two important characterization of superposition and the entanglement. Some companies tried to build a quantum computer consisting of a small number of bits, which led to the acceleration in the design of algorithms and control programs and quantum circuits dealing with these bits. In this research, the researcher try to show the effect of Pauli gates on the superposition for four-qubit in Bloch sphere which help to build a control system that serves as a miniature model that paves the way for larger models that can built in the near future to use in quantum computer device.

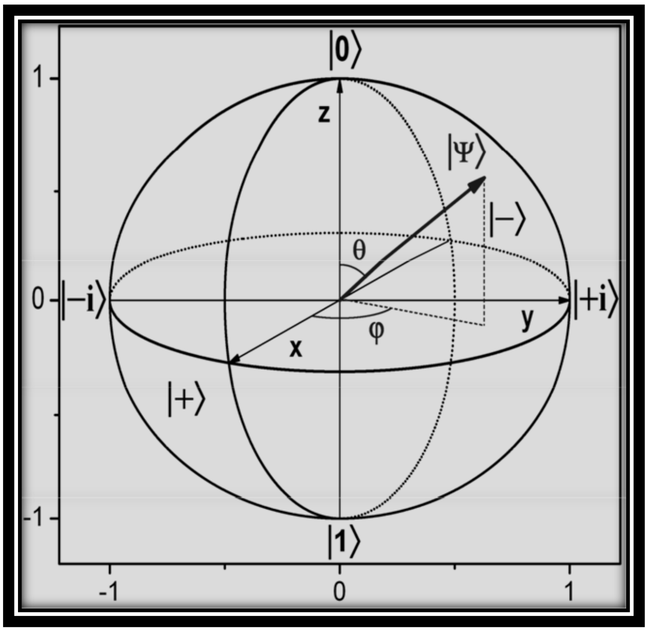

Figure **1** [Bloch sphere for single qubit].

2. Theoretical Part

According to the Schrodinger equation ($\mathcal{H}|\psi\rangle = i\hbar|\psi\rangle$), the eigenstates of the system Hamiltonian are state $|0\rangle$ and state $|1\rangle$. [2]

We can represent the pure qubit according to the superposition as:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \quad \textbf{(1)}$$

Where α and β are complex coefficients of eigenstates. The probability of eigenstates given as:

$$\|\alpha\|^2 + \|\beta\|^2 = 1 \quad \textbf{(2)}$$

, and we can represent the qubit as a two point in Bloch sphere as follows:

$$|\psi\rangle = cos\frac{\theta}{2}\,|0\rangle + e^{\phi}\,sin\frac{\theta}{2}\,|1\rangle \quad \textbf{(3)}$$

, where $0 \leq \theta \leq \pi$, and $0 \leq \emptyset \leq 2\pi$. As shown in Figure 1. [3]

2.1 The superposition for n qubit

We can represent the quantum superposition by the tensor product of its counterpart qubits, and consequently, for n qubit, the superposition according to the base rule (2^n):

$$|\psi_n\rangle = \sum_{i=0}^{2^n-1} a_i\,|i\rangle \quad \textbf{(4)}$$

$$\sum_{i=0}^{2^n-1}|a_i|^2 = 1 \quad \textbf{(5)}$$

In matrix, the state $|0\rangle$ and the state $|1\rangle$ show as:

$$|0\rangle \equiv \begin{pmatrix}1\\0\end{pmatrix} \; and \; |1\rangle \equiv \begin{pmatrix}0\\1\end{pmatrix}$$

So that we get sixteen states for four-qubit, each state consists of the tensor product for four-qubit. We can describe the state $|1100\rangle$ as: [4]

$$|1100\rangle = |1\rangle\otimes|1\rangle\otimes|0\rangle\otimes|0\rangle \quad \textbf{(6)}$$

2.2 The three Pauli gates

The Pauli gates is the simplest type of quantum gate fulfilling a unitary transformation on single-qubit states, further important for quantum computation. When applying it on a single-qubit, it rotates by π radians about the x, y, and z-axes respectively on the Bloch sphere.

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The Pauli-X known as a bit-flip, and Pauli-Z is sometimes called phase-flip. [5]

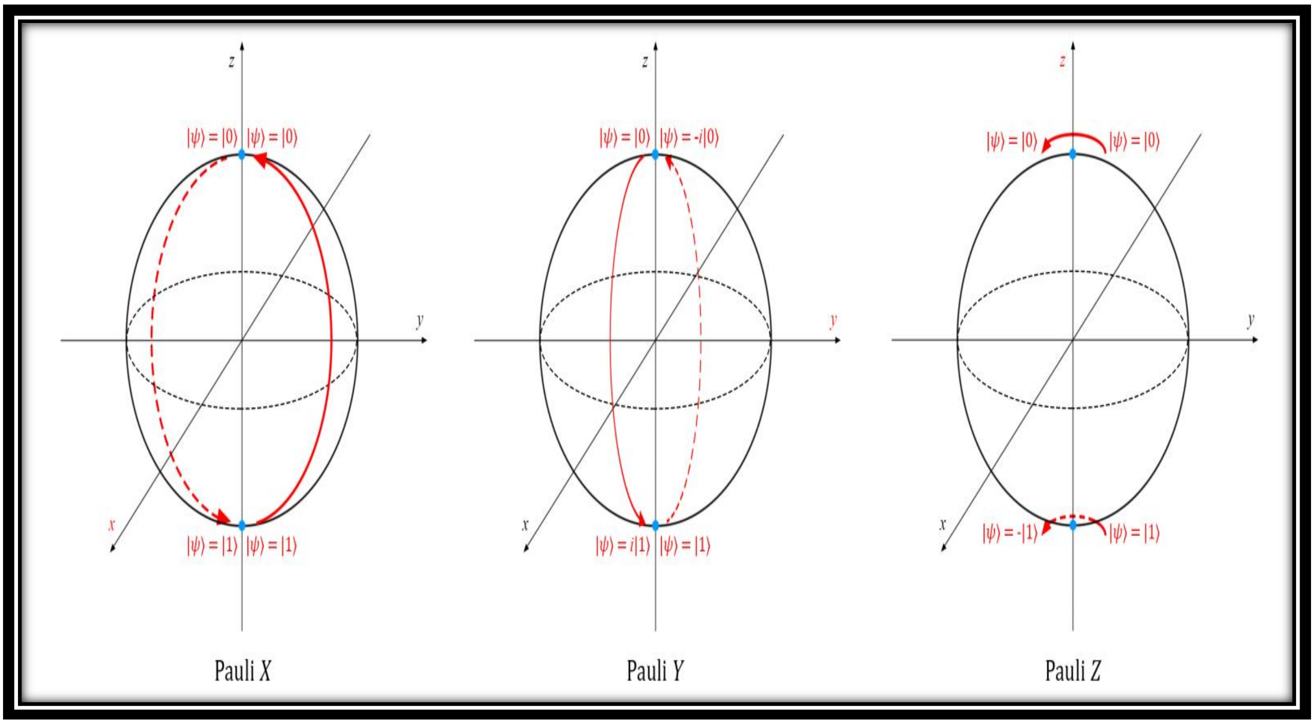

Figure **2** [Representation the effect Pauli gates on the three axes in Bloch sphere]. [6]

2.3 Quantum Circuits

Quantum Circuits are a combination of quantum gates that analyze and process quantum information in quantum computers, where these gates manipulate with qubits. In order to build the quantum circuits, two important limitations must be imposed: First, feedback of a part of the circuit into another not allowed, i.e. the circuit is cyclic. Secondly, it is not possible to copy the qubits. These circuits can contain one or more gates and there are kinds of quantitative gates, which operate on one or two qubits such as the gates of Pauli and Hadamard gates. In addition, we can operate the gates on three qubits such as Toffoli and Fredkin gates. In this research, we will address the gates of Pauli only. This gate is working to turn the state of the qubit into a new state. We can describe Quantum circuits and gates by linear transformations. These linear transformations are represented by unitary matrixes only. Denoted for the matrix that describe the state transformation a transfer matrix. [7]

3. The practical part divided into three cases

3.1 Case (1)

For four electrons, each electron considered a single qubit we can be representing as:

$$|\psi_1\rangle = \alpha_1|0\rangle + \beta_1|1\rangle \quad \textbf{(7)}$$

$$|\psi_2\rangle = \alpha_2|0\rangle + \beta_2|1\rangle \quad \textbf{(8)}$$

$$|\psi_3\rangle = \alpha_3|0\rangle + \beta_3|1\rangle \quad \textbf{(9)}$$

$$|\psi_4\rangle = \alpha_4|0\rangle + \beta_4|1\rangle \quad \textbf{(10)}$$

To find the superposition for four-qubit we use the tensor product: [8]

$$|\Psi\rangle = |\psi_1\rangle\otimes|\psi_2\rangle\otimes|\psi_3\rangle\otimes|\psi_4\rangle \quad \textbf{(11)}$$

$$|\Psi\rangle = (\alpha_1|0\rangle + \beta_1|1\rangle)\otimes(\alpha_2|0\rangle + \beta_2|1\rangle)\otimes(\alpha_3|0\rangle + \beta_3|1\rangle)\otimes(\alpha_4|0\rangle + \beta_4|1\rangle)$$

$$where\ \alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2, \beta_3, \beta_4 \in \mathbb{C}$$

$$and\ (|\alpha_1|^2 + |\beta_1|^2)(|\alpha_2|^2 + |\beta_2|^2)(|\alpha_3|^2 + |\beta_3|^2)(|\alpha_4|^2 + |\beta_4|^2) = 1$$

The following matrix describes all states that we get from the equation (12).

$$\Psi = \begin{bmatrix} 1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1 \end{bmatrix} \qquad (12)$$

Each column of this matrix represents a state and is described as follows:

$$|0101\rangle = |0\rangle \otimes |1\rangle \otimes |0\rangle \otimes |1\rangle = \begin{bmatrix}1\\0\end{bmatrix} \otimes \begin{bmatrix}0\\1\end{bmatrix} \otimes \begin{bmatrix}1\\0\end{bmatrix} \otimes \begin{bmatrix}0\\1\end{bmatrix} \qquad \textbf{(13)}$$

$$|0101\rangle = [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]^T \qquad \textbf{(14)}$$

This is represents the state$|0101\rangle$.

Then we draw all states by using a program in Matlab, the figures 3 and 4 show some result for this program. [9]

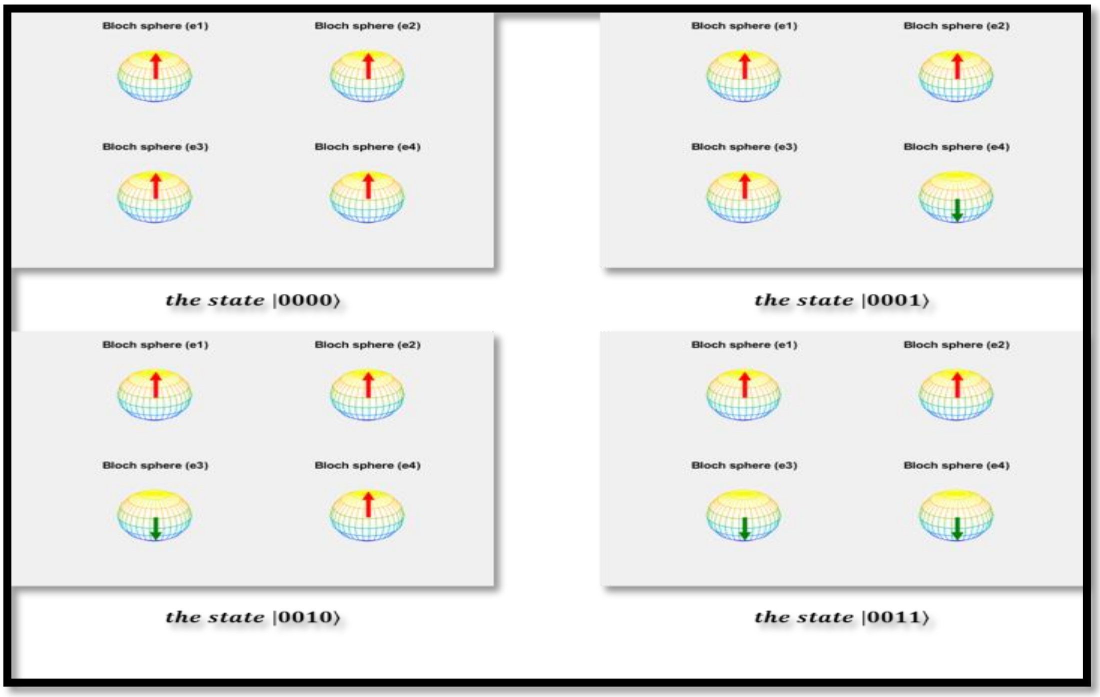

Figure **3** [The states $|0000\rangle$, $|0001\rangle$, $|0010\rangle$ and $|0011\rangle$].

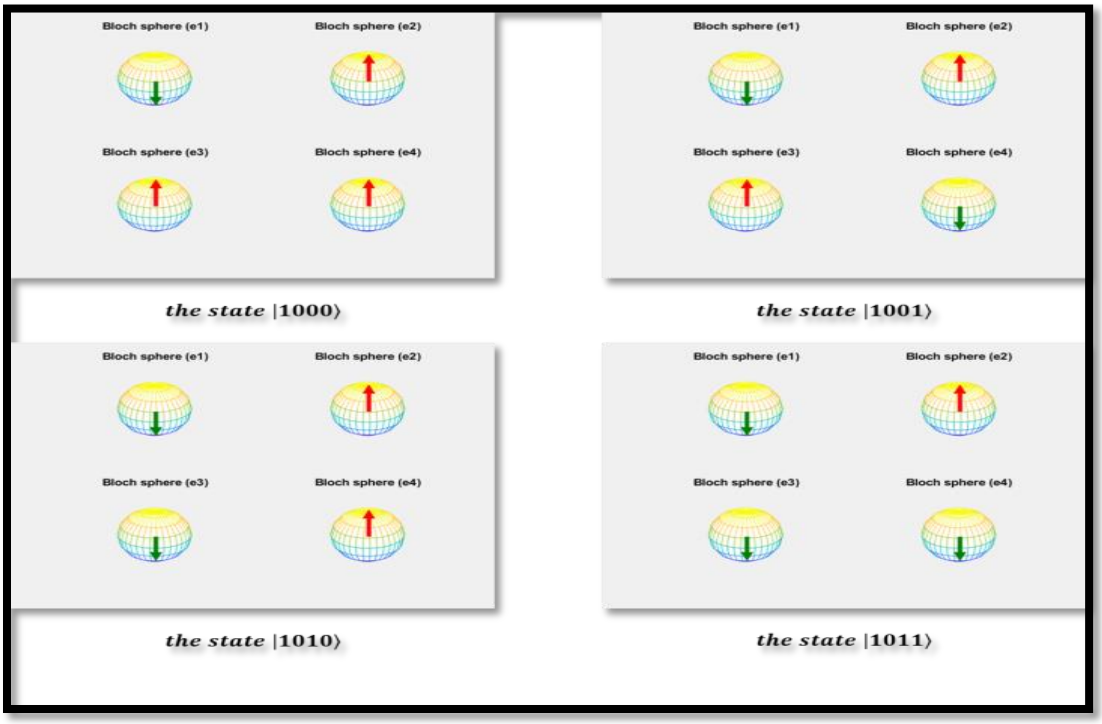

Figure **4** [The states $|1000\rangle$, $|1001\rangle, |1010\rangle$ and $|1011\rangle$].

3.2 Case (2)

The effect of Pauli gate on different order of qubit from the four-qubits can be described by these equations
(15, 16,17and 18), and the figure (6) appear effect X gate on the third qubit.

$$X|1\rangle\otimes|0\rangle\otimes|1\rangle\otimes|0\rangle = |0\rangle\otimes|0\rangle\otimes|1\rangle\otimes|0\rangle = |0010\rangle \qquad \textbf{(15)}$$

$$|1\rangle\otimes Y|0\rangle\otimes|0\rangle\otimes|1\rangle = |1\rangle\otimes i|1\rangle\otimes|0\rangle\otimes|1\rangle = |1101\rangle \qquad \textbf{(16)}$$

$$|0\rangle\otimes|1\rangle\otimes Z|0\rangle\otimes|0\rangle = |0\rangle\otimes|1\rangle\otimes|0\rangle\otimes|0\rangle = |0100\rangle \qquad \textbf{(17)}$$

$$|0\rangle\otimes|1\rangle\otimes|1\rangle\otimes Z|1\rangle = |0\rangle\otimes|1\rangle\otimes|1\rangle\otimes -|1\rangle = |0111\rangle \qquad \textbf{(18)}$$

Table **1** [shows the effect of Pauli gate on different order of qubit from the four-qubits].

Input	Rank the electron	Type of the gate	Output
$\|0101\rangle$	On the third qubit	X-gate	$\|0111\rangle$
$\|1001\rangle$	On the first qubit	Y-gate	$\|0001\rangle$
$\|1100\rangle$	On the fourth qubit	Z-gate	$\|1100\rangle$

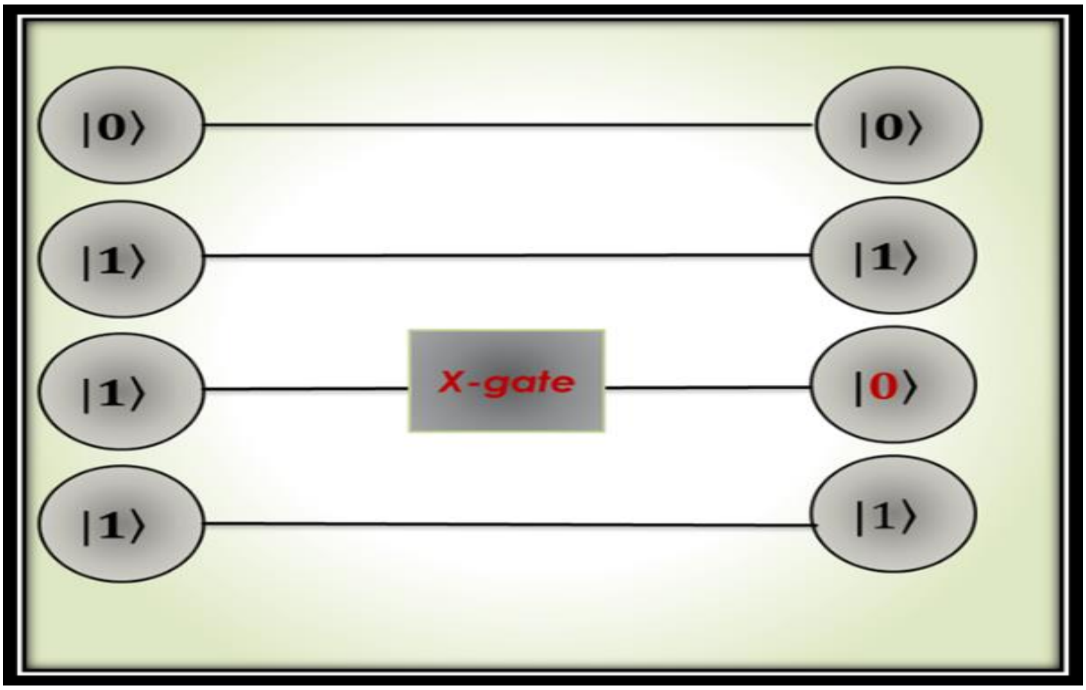

Figure **5** [Chart showing the X gate acting on the third qubit and change the state from$|0111\rangle$ to state$|0101\rangle$].

Table **2** [shows the effect of different gates on one qubit from the four qubits].

Input	Type of gate	The order of the electron	the changing in state	Output
$\|1010\rangle$	X-gate	On the first qubit	$\|1\rangle \to \|0\rangle$	$\|0010\rangle$
$\|1001\rangle$	Y-gate	On the second qubit	$\|0\rangle \to i\|1\rangle$	$\|1101\rangle$
$\|0100\rangle$	Z-gate	On the third qubit	$\|0\rangle \to \|0\rangle$	$\|0100\rangle$
$\|0111\rangle$	Z-gate	On the fourth qubit	$\|1\rangle \to -\|1\rangle$	$\|0111\rangle$

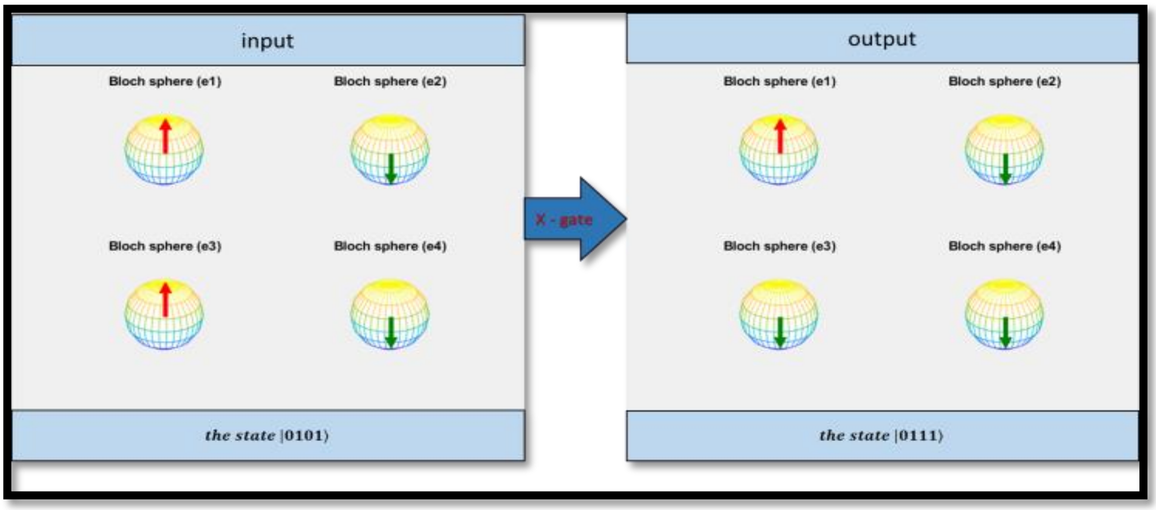

Figure **6** [The X gate acting on the third qubit and change the state from$|0101\rangle$ to state$|0111\rangle$].

3.3 Case (3)

We used a program in the Matlab, a control room that operates by two gates from the gates of Pauli. These two gates can be different or similar and has an effect on two different electrons in order or the same electron and alternately.

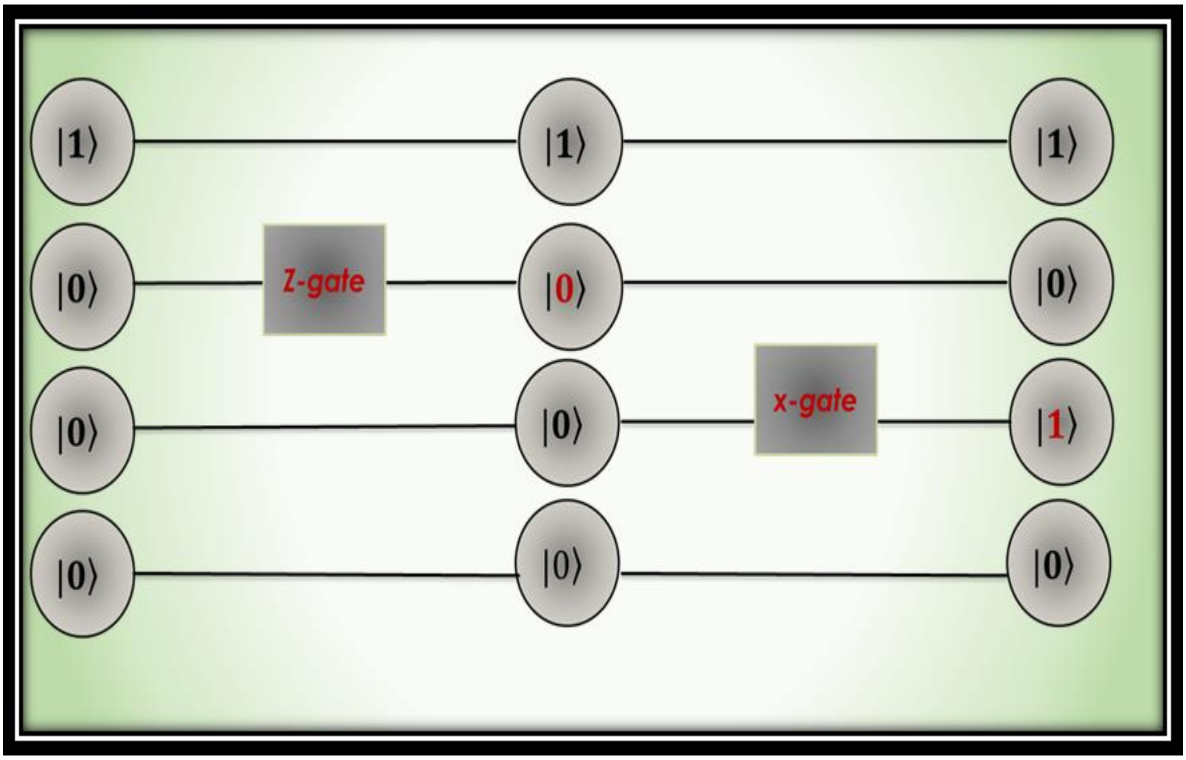

Figure **7** [The state$|1000\rangle$ not change by applied the first gate (Z-gate) on the second electron but it changed to the state$|1010\rangle$ after applied the second gate (X-gate) on the third electron].

$$X|0\rangle\otimes|0\rangle\otimes|0\rangle\otimes|0\rangle = |1\rangle\otimes|0\rangle\otimes|0\rangle\otimes|0\rangle = |1\rangle\otimes|0\rangle\otimes|0\rangle\otimes Y|0\rangle = |1001\rangle \quad \textbf{(19)}$$

$$|1\rangle\otimes|1\rangle\otimes|1\rangle\otimes X|0\rangle = |1\rangle\otimes|1\rangle\otimes|1\rangle\otimes|1\rangle = |1\rangle\otimes Y|1\rangle\otimes|1\rangle\otimes|1\rangle = |1011\rangle \quad \textbf{(20)}$$

$$|0\rangle\otimes Y|0\rangle\otimes|0\rangle\otimes|0\rangle = |0\rangle\otimes i|1\rangle\otimes|0\rangle\otimes|0\rangle = |0\rangle\otimes i|1\rangle\otimes Z|0\rangle\otimes|0\rangle = |0100\rangle \quad \textbf{(21)}$$

$$|0\rangle\otimes|0\rangle\otimes Z|1\rangle\otimes|1\rangle = |0\rangle\otimes|0\rangle\otimes - X|1\rangle\otimes|1\rangle = |0\rangle\otimes|0\rangle\otimes|0\rangle\otimes|1\rangle = |0001\rangle \quad \textbf{(22)}$$

$$|0\rangle\otimes|0\rangle\otimes Z|1\rangle\otimes|1\rangle = |0\rangle\otimes|0\rangle\otimes - Z|1\rangle\otimes|1\rangle = |0\rangle\otimes|0\rangle\otimes|1\rangle\otimes|1\rangle = |0011\rangle \quad \textbf{(23)}$$

$$|0\rangle\otimes X|0\rangle\otimes|0\rangle\otimes|1\rangle = |0\rangle\otimes X|1\rangle\otimes|0\rangle\otimes|1\rangle = |0\rangle\otimes|0\rangle\otimes|0\rangle\otimes|1\rangle = |0001\rangle \quad \textbf{(24)}$$

$$|0\rangle\otimes|0\rangle\otimes|0\rangle\otimes Y|1\rangle = |0\rangle\otimes|0\rangle\otimes|0\rangle\otimes Yi|0\rangle = |0\rangle\otimes|0\rangle\otimes|0\rangle\otimes -|1\rangle = |0001\rangle \quad \textbf{(25)}$$

Equation (19) shows the qubit one acted by the X gate firstly, which change the state$|0000\rangle$ to$|1000\rangle$, and then applied the Y gate on the fourth qubit, which change the state $|1000\rangle$ to$|1001\rangle$.

The vector (i), which is observe by applying the Y gate, has no effect on the drawing of the state. However, it indicates that the state is under the influence of rotation around the Y-axis, which represents the complex part from the dimensions of the Bloch sphere as shown from equation (21). When applying the Y gate it will return the state to where it was but with a change in the phase as shown from equation (25). The negative sign that emerged because of the application of the Z gate on the state $|1\rangle$ only, it indicates that the state $|1\rangle$ is experiencing a change in phase as shown from equation (22). When applying the Z gate for the second time on the state it will return the state to its initial state as shown from equation (23).

We can use the same gate on the same qubit sequentially twice or with two different qubits. Also we can use two different gates on the same qubit respectively or with two different qubits.

Table **3** [shows the effect of two gates on two qubit from the four qubits].

Input	**The first gate**	**The order of the electron**	**Output**	**The second gate**	**The order of the electron**	**final**
$\|1010\rangle$	**X-gate**	On the second qubit	$\|1110\rangle$	**Y-gate**	On the second qubit	$\|1010\rangle$
$\|1011\rangle$	**X-gate**	On the first qubit	$\|0011\rangle$	**X-gate**	On the fourth qubit	$\|0010\rangle$
$\|1111\rangle$	**Z-gate**	On the third qubit	$\|1111\rangle$	**Z-gate**	On the third qubit	$\|1111\rangle$
$\|1010\rangle$	**Y-gate**	On the fourth qubit	$\|1011\rangle$	**X-gate**	On the third qubit	$\|1001\rangle$

Figure **8** [the first gate (X-gate) applied on the second electron for the state$\mathbf{1010}\rangle$, after that applied (Y-gate) on the second electron for the result state$\mathbf{1110}\rangle$ that change it to same state$\mathbf{1010}\rangle$].

Figure **9** [the first gate (X-gate) applied on the first electron for the state$|\mathbf{1011}\rangle$, after that applied (X-gate) on the fourth electron for the result state$|\mathbf{0011}\rangle$ that change it to state$|\mathbf{0010}\rangle$].

4. Snapshot of Matlab codes

Figures 10 and 11 shows the snapshots of Matlab codes used in the present study.

```
set(subplotk,'DataAspectRatio',[1 1
1],'PlotBoxAspectRatio',...
    [434 342.3 342.3]);
syms e  G
j=input('Please input j: ');
e=input('Please input e: ');
Figure1('arrow',[0.3 0.3],...
    [0.74 0.84],'Color',[1 0 0],'LineWidth',4,...
    'HeadStyle','cback1');
disp(' X gate');
numer(e)=~numer(e);
if numer (1)==0
figure2,'arrow',[0.3 0.3],...
    [0.74 0.84],'Color',[1 0 0],'LineWidth',4,...
    'HeadStyle','cback1';
elseif numer(1)==1
figure2,'arrow',[0.3 0.3],...
    [0.74 0.65],'Color',[0 0.5 0],'LineWidth',4,...
    'HeadStyle','cback1';
else
    disp('no annotation');
end
```

Figure **10**

```
surf(x,y,z,'Parent',subplotk,'FaceLighting','go
uraud',...
     'EdgeLighting','flat',...
     'LineWidth',0.1,...
     'FaceColor',[1 1 1],...
     'EdgeColor','flat');
grid(subplotk,'on');
set(subplotk,'DataAspectRatio',[1 1
1],'PlotBoxAspectRatio',...
     [434 342.3 342.3]);
Figure1('arrow',[0.74 0.74],...
     [0.27 0.37],'Color',[1 0 0],...
     'LineWidth',4,...
     'HeadStyle','cback1')
title('Bloch sphere (e1)');
subplotk = subplot(2,2,k,'Parent',figure1);
axis off
hold(subplotk,'on');
surf(x,y,z,'Parent',subplotk,'FaceLighting','go
uraud',...
     'EdgeLighting','flat',...
     'LineWidth',0.1,...
     'FaceColor',[1 1 1],...
     'EdgeColor','flat');
disp('no annotation');
end
```

Figure **11**

5. Conclusion

In this research, we were able to use a control system or create a control room that can be used to process quantum information by using the concept of Bloch sphere where some states have symmetry on Bloch sphere and some have asymmetry, In other words, there are states that did not suffer any change and other states that will suffer change. Therefore, the possibility of showing the change in the state is identified by using the rotation on it. The control system or the control room consists of Pauli gates, which are applied on the result states from the superposition for four electrons. When these gates are applied to the state, it makes change in the state to another state. This means a change in quantum information stored in this state. The changing will be between the ground state$|0\rangle$ and the excited state$|1\rangle$. All states that did not have any change when applying the gates will remain at their original state, whether they are in the ground state$|0\rangle$ or excited state$|1\rangle$. This changing in state depends on the type of gate used. This work is important for quantum processing in quantum computing to solve the difficulties that cannot be solved by using the classical computer.

References

1. Kibler, Maurice Robert. "Quantum information and quantum computing: an overview and some mathematical aspects." arXiv preprint arXiv*:* 1811.08499 (2018).
2. Lou, Yuesheng, and Shuang Cong. "State transfer control of quantum systems on the Bloch sphere." Journal of Systems Science and Complexity 24.3 (2011): 506.P
3. Petrila, I. and V. Manta, Representations of the qubit states. Bulletin of the Polytechnic Institute of Iasi-Automatic Control and Computer Science Section, 2013. 59: p. 57-68.
4. Williams, Colin P. Explorations in quantum computing. Springer Science & Business Media, 2010. p. 17.
5. Strubell, E., An introduction to quantum algorithms. COS498 Chawathe Spring, 2011. 13:
6. Anita Ramanan.” Quantum Gates and Circuits: The Crash Course.” Faculty connection, 2018 Accessible via https://blogs.msdn.microsoft.com/uk_faculty_connection/2018/02/26/quantum-gates-and-circuits-the-crash-course.
7. Marinescu, D.C. and G.M. Marinescu, Lectures on quantum computing. Computer Science Department, University of Central Florida, Florida, 2003.p .74.
8. Mäkelä, H., and A. Messina. "N-qubit states as points on the Bloch sphere." Physica Scripta 2010.T140 (2010): 014054.
9. Strubell, Emma. "An introduction to quantum algorithms." COS498 Chawathe Spring 13 (2011): 19.