The product of para - compact Spaces

Mohammad G.Dakhel

University of Thi – Qar/ College of Education

Abstract

The product of m-paracompact and m-strongly paracompact are briefly discussed.

Introduction

In this paper we give a necessary and sufficient condition for the product of two m-paracompact (m-metacompact, m-strongly paracompact) spaces to be an m-paracompact (m--metacompact, m-strongly paracompact) space.

Also, we provide simple proofs for the results of Morita [4], katuta[3], Wenjen[10], and smith and Krajewski[9]. Theorems 2.4 and 2.7 are the main results of this paper.

Definition 1.1:

A Collection {At:t∈T} of subsets a space X is called order locally finite, if one can introduce a total order ≤on T such that for each t in T, {As:s≤t} is locally finite at each point of At.

Definition 1.2:

Let F be a subset of a space (X,T), Then:

(I) F is called relatively m-paracompact if every open cover of cardinality ≤m of F by members of T has an open locally finite refinement by members of T Covering F.

(II) F is called relatively m-metacompact if every open cover of cardinality ≤ m of F by members T has an open point-finite refinement by members of T covering F.

(III) F is called relatively m-strongly paracompact if for every open cover U of cardinality ≤ m of F by members of T there is an open star-finite cover of X such that {v∩F; v∈V} refines U.

If a set F is relatively m-paracompact (relatively m-metacompact, relatively m-strongly paracompact) for each cardinal number m, then it is called relatively paracompact(relatively metacompact, relatively strongly paracompact).

Definition 1.3:

Let m be an infinite cardinal number and let S be a subspace of a space X.Then S is said to be p^m-embedded in X if every m-separable continuous pseudo-metric on S can be

extended to an m-separable continuous pseado-metric on X.

We say S is p-embedded on X if every continuous pseudo-metric on S can be extended to X.

Lemma 1.1:

Let U be an open cover of X and let V={vi:i is in N} be a star-finite open cover of X .Let W be a star –finite open cover of X satisfying:
{vi∩w: w€W} is a refinement of U for each i in N. Then U has an open star-finite refinement.

Proof:

The proof is easy as $R = U_{i=1}^{\infty} R_i$ where Ri ={vi∩w: w€W} is a refinement of U by assumption , it is star- finite (because W is star-finite).■

Lemma1.2:

Let U be an open star-finite cover of X .Then U={U_t: t€T} where U_t={U_{ti}:i€ N} and the sets U_t^* are open and disjoint.

For the proof see (p.228 , lemmas 1 and 2 of [2]).

2. Some generalizations of perfect marppings:

Definition 2.1:

A continuous function f: X $\rightarrow$ Y is called m-paraperfect (m-metaperfect) if for every open cover U of X of cardinality $\leq$ m there is an open cover V of Y and an open refinement Wof U such that for each v in V, f^{-1}(v) is contained in the Union of a subfamily R of W, where R is locally finite (point – finite).

Definition 2.2:

A continuous function f: X $\rightarrow$ Y is called m-strongly paraperfect if for every open cover U of X of cardinality $\leq$ m there is an open cover V of Y such that for each countable subcollection V_1 of V there is an open star-finite cover W_1 of X such that $f^{-1}(V_1) \cap W_1$ is a refinement of U.

Note: We shall state and prove results for m-paracompact spaces except in cases where others need special consideration.

Theorem 2.1:

Any continous function from an m-paracompact space X onto any space Y is an m-paraparfect function.

The proof is obvious.

Theorem2.2:

Let f: X $\rightarrow$ Y be an m-paraperfect function .Then X is m-paracompact if Y is paracompact .

The proof of the above theorem in the paraperfect and metaperfect case is simple. The case for strong paraperfectness follows from lemmas 1.1 and 1.2.

Theorem2.3:

Let M be a closed m-paracompact subset of a space X and let F be a closed subset of the interior G of M. Then F is relatively m-paracompact if M is paracompact.

For the proof see [1].

Theorem2.4:

Let f: X $\rightarrow$ Y be a continuous and closed mapping of X onto Y. Then f is an m-paraperfect function if $f^{-1}(y)$ is relatively m-paracompact for each y in Y.

Proof: Let U be an open cover of X of cardinality ≤m. since $f^{-1}(y)$ is relatively m-paracompact, U has an open refinement Ry in X which is locally finite and covers $f^{-1}(y)$. Define Oy=y-f(X-R^*y) foe each y in Y. Since f is closed , Oy is open and f^{-1}(Oy) is contained in R^*y. Now R=$\cup$ {Ry:y in Y} is an open refinement of U which covers X and {Oy:y in Y} is an open cover of Y such that for each Oy, f^{-1}(Oy) is contained in R^*y where Ry is locally finite.■

Corollary 2.5:

Let f:X $\longrightarrow$Y be a closed continuous function.Then X is m-para-compact if Y is paracompact and $f^{-1}(y)$ is relatively m-para-compact for each y in Y.

Remark:

In view of theorem 2.7 in [7], if a closed m-paracompact subset S is P^m-embedded in a normal space X, Then S is relatively m-paracompact.

Corollary 2.6:

Let f: X$\rightarrow$Y be a continuous and closed function. Then:

(I) X is paracompact if Y is paracompact and $f^{-1}(y)$ is compact for all y in Y. (Hanai [6])
(II) A normal space X is paracompact if Y is paracompact and $f^{-1}(y)$ is paracompact and p-embedded in X for all y in Y. (shapiro [8])

The next theorem is proved only for paracompact spaces.

Theorem 2.7:

Let f: X $\rightarrow$ Y be a continuous function, where X is a regular space. Let {v_s :s Є S} and { h_s :s Є S} be two coverings of Y satisfying:

a) {v_s :s Є S } is closed covering of y and { h_s :s Є S}is an open ordered locally finite covering of y such that v_s is contained hs for each s in S.

b) $f^{-1}(v_s)$ is relativety paracompact for each s in S. Then X is paracompact.

Proof:

Let U be an open cover of X since $f^{-1}(v_s)$ is relativety paracompact for each s in S.
i.e.: $\forall\, s\in S$, there is an open locally finite refinement say W(v_s) in X of U which covers $f^{-1}(v_s)$.

Now, we have $f^{-1}(v_s) \subseteq W(v_s)$ and $f^{-1}(Vs) \subset f^{-1}(hs)$ $\forall\, s\in S$.Define A={^{1}(hs)$\cap$w: w is in W(v_s) and s is in S}. Evidently, A is an ordered locally finite refinement of U which covers X. Hence X is paracompact by [3].■

3. Application of the results of seIn this section all spaces are assumed T1.

Theorem 3.1:

Let X and Y be two spaces and let P_x:XxY $\rightarrow$ X be the projection map. Then P_x is an m-paraperfect function if for each x in X, there is a neighborhood O_x such that $P^{-1}_x(O_x)$ is relativity m-paracompact.

The proof follows easily.

The following theorem gives us as corollaries the results of Morita [4], Wenjen[10], and smith and krajewski[9].

Journal of University of Thi-Qar No.4 Vol.2

Theorem 3.2:

Let X and Y be a paracompact spaces.Then X×Y is m-paracompact if and only if for each x in X, there is a neighberhood U_x of x such that U_x×Y is relativety m-paracompact.

The proof follows from theorem 2.2 and the observation that if X xY is m-paracompact then it is relativly m-paracompact.

Theorem 3.3:

Let {v_s :s is in S} be a closed covering of X and let {h_s: s is in S} be an open ordered locally finite covering X such that $v_s \subset h_s$ for each s in S and $p_x^{-1}(v_s)$ is relatively paracompact in XxY for each s in S. Then XxY is paracompact if and only if Y is paracompact.

The proof follows from theorem 2.1.

Finally, we state an analogue of ponomarev's.

Theorem 3.4:

Let X be a space, then X is m-paracompact if and only if there exists an m-paraperfect function from X onto and a paracompact space Y.

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