

On Third-Order Differential Subordinations and Superordinations Results for Univalent Analytic Functions Defined by a New Operator

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Abstract: In the current paper, we obtain sandwich theorems for univalent functions by using some of the finding of Third-order differential subordination and superordination for univalent functions defined by a new operator $R^n f(z)$.

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1-Introduction and Definitions

Let $\hat{W} = \hat{W}(U)$ be the class functions which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. For $n \in \mathbb{N}$ and $a \in \mathbb{C}$. Let $\hat{W}[a, n]$ be the subclass of $\hat{W}$ and

$$\hat{W}[a, n] = \{f \in \hat{W} : f(z) = a + a_k z^k + a_{k+1} z^{k+1} + \dots\}, (a \in \mathbb{C})$$

Let A denote the subclass of $\hat{W}$ of functions f of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, (a_k \geq 0, a \in \mathbb{C}), \quad (1.1)$$

Suppose f and g are analytic functions in $\hat{W}$. We say that f is subordinate to g , or g is superordinate to f in U , and write $f < g$, or $f(z) < g(z)$, if there exists a Schwarz function w in U , which with $w(0) = 0$, and $|w(z)| < 1$, ($z \in U$), where $f(z) = g(w(z))$. Furthermore, if g is univalent in U , we have (see [8]), $f(z) < g(z)$ if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$ $z \in U$.

Definition (1.1)[1]: Let $\mathcal{O}: \mathbb{C}^4 \times U \rightarrow \mathbb{C}$ and suppose that the function $K(z)$ be univalent in U . If $p(z)$ is analytic in U and satisfies the following:

$$\mathcal{O}(p(z), z p'(z), z^2 p''(z), z^3 p'''(z); z) < K(z), \quad (1.2)$$

then $p(z)$ is called a solution of the differential subordination (1.2). Furthermore, given univalent function $p(z)$ is called a dominant of the solution of the differential subordination (1.2) if, $p(z) < q(z)$ for all $p(z)$ satisfying (1.2). A dominant $q(z)$ that satisfies $q(z) < q(z)$ for all dominant $q(z)$ of (1.2) is said to be the bsst dominant.

Definition (1.2)[3,24]: For $f \in A$, we define An operator $R^n: A \rightarrow A$

$$D^n f(z) = z + \sum_{k=2}^{\infty} [1 + (k-1)\delta]^n a_k z^k, (n \in \mathbb{N}). \quad (1.3)$$

When $\delta = 1$, we get Salagean's differential operator (see[3]), and

$$J_{s,a} f(z) = z + \sum_{k=2}^{\infty} \left(\frac{k+a}{p+a} \right)^s a_k z^k, \quad (1.4)$$

($z \in U, a \in \mathbb{C} \setminus \mathbb{Z}_0, \mathbb{Z}_0 = \{0, -1, -2, \dots\}, s \in \mathbb{C}$ (see[24])).

We define the new operator

$$R^n = D^n * J_{s,a} \\ R^n f(z) = z + \sum_{k=2}^{\infty} \left(\frac{k+a}{1+a} \right)^s [1 + (k-1)\delta]^n a_k z^k \quad (1.5)$$

$$z(R^n f(z))' = \frac{1}{\delta} R^{n+1} f(z) - \frac{1-\delta}{\delta} R^n f(z). \quad (1.6)$$

Definition(1.3)[1]: Let Q the set of all functions q that are analytic and univalent on $\underline{U}|E(q)$, where $\underline{U} = U \cup \{z \in \partial U\}$, and

$$E(q) = \{z \in \partial U : q(z) = \infty\} \quad (1.7)$$

and $\min |q'(z)| = p > 0$ for $z \in \partial U \setminus E(q)$. Further, let the subclass of Q for which $q(0) = a$ be denoted by $Q(a)$, and $Q(0) = Q_0, Q(1) = Q_1$.

The subordination methodology is applied to appropriate classes of admissible function.

The following class of admissible functions is given by Antoniuo and Miller (see[1]).

Definition (1-4) [1]: Let Ω be a set in $\mathbb{C}$. Also $q \in Q$ and $n \in \mathbb{N} \setminus \{1\}$, N being the set of positive integers. The class $H_n[\Omega, q]$ of admissible functions consists of those functions $K: \mathbb{C}^4 \times U \rightarrow \mathbb{C}$, which satisfy following admissibility conditions :

$$K(e, s, r, t) \notin \Omega,$$

$$\text{whenever } e = q(\xi), \quad s = \xi q'(\xi), \quad R\left(\frac{r}{s} + 1\right) \geq \xi R\left(\frac{\xi q''(\xi)}{q'(\xi)} + 1\right)$$

$$\text{and} \quad R\left(\frac{t}{s}\right) \geq \xi^2 R\left(\frac{\xi^2 q'''(\xi)}{q'(\xi)}\right) \quad (\text{where } z \in U, \xi \in \partial U|E(q), k \geq 2).$$

Lemma (1.5), below is the foundation result in the theory of third order differential subordination.

Lemma(1.5)[1]: Let $\hat{W}[a, n]$ with $n \geq 2$ and $q \in Q_a$ satisfy the following conditions :

$$R\left(\frac{\xi q''(\xi)}{q'(\xi)}\right) \geq 0, \quad \left| \frac{z p''(z)}{q'(z)} \right| \leq k$$

(where $z \in U, \xi \in \partial U|E(q), k \geq 2$). If Ω is a set in $\mathbb{C}$, $\emptyset \in H_n[\Omega, q]$

$$\text{and} \quad \{\emptyset(p(z), zp'(z), z^2 p''(z), z^3 p^3(z); z)\} \subset \Omega.$$

Then $p(z) \prec q(z)$.

The notion of the third-order differential subordination can be found in the work of Ponnurasy and Juneja [10]. The recent work by several authors (see example [4,5,6,7,8,15,22,23,26,27]) on the differential subordination attracted many researchers in this field. For example (see [2,3, 9,10,11,12,14,16,17,18,20,21,25,26,27]).

In the present paper, we investigate suitable classes of admissible functions associated with new an operator $R^n f(z)$

2-Results Related to the Third-Order Subordination:

In this section, we start with a given set Ω , and given function q , and we determine a set of admissible new operator, so that (1.2) holds true. For this purpose, we introduce the following new class of admissible functions which will be required to prove the main third-order differential subordination theorems for new the operator $R^n f(z)$ defined by (1.5).

Definition (2.1): Let Ω be a set in $\mathbb{C}$, $q \in Q_0 \cap \hat{W}_0$. The class $H_i[\Omega, q]$ of admissible functions consist of those functions:

$$\phi : \mathbb{C}^4 \times U \rightarrow \mathbb{C}.$$

That satisfy the following admissibility condition: $\phi(a, b, c, d; z) \notin \Omega$,

whenever $a = q(\epsilon)$, $b = \delta \epsilon q'(\epsilon)$,

$$Re \left(\frac{c - (1 - \delta)[2b - (1 - \delta)a]}{\delta^2 b - \delta^2(1 - \delta)a} \right) \geq k Re \left(\frac{\epsilon q''(\epsilon)}{q'(\epsilon)} + 1 \right),$$

And

$$Re \left(\frac{d - 3c - (1 - \delta)[(1 - \delta)b + (1 + \delta)a] + 2(2 - \delta)b}{\delta^2 b - \delta^2(1 - \delta)a} \right) \geq k^2 Re \left(\frac{\epsilon^2 q'''(\epsilon)}{q'(\epsilon)} \right),$$

(where $z \in U$, $\epsilon \in \partial U|E(q)$, $k \geq 2$).

Theorem (2.1): Let $\phi \in H_i[\Omega, q]$. If the function $f \in A$, and $q \in Q_0$, satisfy the following conditions :

$$R \left(\frac{\epsilon q''(\epsilon)}{q'(\epsilon)} \right) \geq 0, \quad \left| \frac{R^n f(z)}{q'(z)} \right| \leq k, \quad (2.1)$$

And

$$\{\phi(R^n f(z), R^{n+1} f(z), R^{n+2} f(z), R^{n+3} f(z); z) : z \in U\} \subset \Omega, \quad (2.2)$$

Then

$$R^n f(z) < q(z) \quad (z \in U).$$

Proof: Define the analytic function $p(z)$ in U by

$$p(z) = R^n f(z). \quad (2.3)$$

From equation (1.6) and (2.3), we have

$$R^{n+1} f(z) = \delta z p'(z) + (1 - \delta) p(z). \quad (2.4)$$

By a similar argument, we get

$$R^{n+2} f(z) = \delta^2 z^2 p''(z) + \delta(2 - \delta) z p'(z) + (1 - \delta)^2 p(z), \quad (2.5)$$

and

$$\begin{aligned} \mathbb{R}^{n+3}f(z) &= \delta^3 z^3 \mathbb{P}'''(z) + 3\delta^2 z^2 \mathbb{P}''(z) + \delta[(2-\delta) + (1-\delta)^2]z\mathbb{P}'(z) \\ &\quad + (1-\delta)^3 \mathbb{P}(z). \end{aligned} \quad (2.6)$$

Define the transformation from $\mathbb{C}^4$ to $\mathbb{C}$ by

$$a(e, s, r, t) = e, \quad b(e, s, r, t) = \delta s + (1-\delta)e$$

$$c(e, s, r, t) = \delta^2 r + \delta(2-\delta)s + (1-\delta)^2 e, \quad (2.7)$$

$$\begin{aligned} d(e, s, r, t) &= \delta^3 t + 3\delta^2 r + \delta[(2-\delta) + (1-\delta)^2]s \\ &\quad + (1-\delta)^3 e. \end{aligned} \quad (2.8)$$

Let $\mathbb{K}(e, s, r, t) = \mathcal{O}(a, b, c, d)$

$$\begin{aligned} &= \mathcal{O}(e, \delta s + (1-\delta)e, \delta^2 r + \delta(2-\delta)s + (1-\delta)^2 e, \delta^3 t + 3\delta^2 r + \delta[(2-\delta) \\ &\quad + (1-\delta)^2]s + (1-\delta)^3 e). \end{aligned} \quad (2.9)$$

The proof will make use of lemma (1.5) using the equations (2.3), and (2.6), and from the equation (2.9), we have

$$\begin{aligned} &\mathbb{K}(\mathbb{P}(z), z\mathbb{P}'(z), z^2\mathbb{P}''(z), z^3\mathbb{P}'''(z); z) \\ &= \mathcal{O}\left(\mathbb{R}^n f(z), \mathbb{R}^{n+1} f(z), \mathbb{R}^{n+2} f(z), \mathbb{R}^{n+3} f(z)\right). \end{aligned} \quad (2.10)$$

Hence, clearly (2.2), becomes

$$\mathbb{K}(\mathbb{P}(z), z\mathbb{P}'(z), z^2\mathbb{P}''(z), z^3\mathbb{P}'''(z); z) \in \Omega,$$

we note that

$$\frac{r}{s} + 1 = \frac{\varsigma - (1-\delta)[2b - (1-\delta)a]}{\delta[b - (1-\delta)a]},$$

and

$$\frac{t}{s} = \frac{d - 3\zeta - (1 - \delta)[(1 - \delta)b + (1 + \delta)a] + 2(2 - \delta)b}{\delta^2[b - (1 - \delta)a]}.$$

Thus clearly, the admissibility condition for $\phi \in H_1[\Omega, q]$, in definition (2-1) is equivalent to admissibility condition $\zeta \in H_2[\Omega, q]$ as given in definition (1.4), with $n = 2$.

Therefore, by using (2.1), and lemma (1-5), we have

$$R^n f(z) < q(z).$$

This proof is complete of theorem (2.1).

Our next result is consequence of theorem (2.1), when the behavior of $q(z)$ on ∂U is not known.

Corollary (2.1): Let $\Omega \in \mathbb{C}$, and let the function q be univalent in U with $q(0) = 1$. Let $\phi \in H_1[\Omega, q_p]$, for some $p \in (0, 1)$, where $q_p(z) = q(pz)$. If the function $f \in A_p$, and q_p , satisfies the following conditions :

$$\operatorname{Re} \left(\frac{\varepsilon q_p''(\varepsilon)}{q_p'(\varepsilon)} \right) \geq 0, \quad \left| \frac{R^{n+1} f(z)}{q_p'(\varepsilon)} \right| \leq k, \quad (z \in U, \varepsilon \in \partial U \setminus E(q_p), k \geq 2),$$

$$\text{and } \phi(R^n f(z), R^{n+1} f(z), R^{n+2} f(z), R^{n+3} f(z)) \in \Omega.$$

Then

$$R^n f(z) < q(z), \quad (z \in U).$$

Proof: By applying theorem (2.1), we get

$$R^n f(z) < q(z) \quad (z \in U)$$

The result asserted by Corollary (2-1), is now deduced from following subordination property

$$q_p(z) < q(z), \quad (z \in U).$$

This **proof** is complete of corollary (2.1).

If $\Omega \neq \mathbb{C}$, is simply connected domain, the $\Omega = M(U)$, for some conformal mapping $M(z)$ of U on to Ω . In this case the class $H_i[M(U), q]$ is written as $H_i[M, q]$.

This leads to the following immediate consequence of theorem (2.1).

Theorem(2.2): Let $\phi \in H_i[M, q]$. If the function $f \in A$, and $q \in Q_0$, satisfy the following conditions :

$$\operatorname{Re} \left(\frac{\varepsilon q_p''(\varepsilon)}{q_p'(\varepsilon)} \right) \geq 0, \quad \left| \frac{R^{n+1}f(z)}{q_p'(z)} \right| \leq k, \quad (2.11)$$

and

$$\phi(R^n f(z), R^{n+1}f(z), R^{n+2}f(z), R^{n+3}f(z); z) < M(z), \quad (2.12)$$

then

$$R^n f(z) < q(z), \quad (z \in U).$$

The next result is an immediate consequence of corollary (2.1).

Corollary (2.2): Let $\Omega \in \mathbb{C}$, and let the function q be univalent in U with $q(0) = 1$. Let $\phi \in H_1[\Omega, q_p]$, for some $p \in (0, 1)$, where $q_p(z) = q(pz)$. If the function $f \in A$, and q_p , satisfies the following conditions :

$$\operatorname{Re} \left(\frac{\varepsilon q_p''(\varepsilon)}{q_p'(\varepsilon)} \right) \geq 0, \quad \left| \frac{R^{n+1}f(z)}{q_p'(z)} \right| \leq k \quad (z \in U, \varepsilon \in \partial U | E(q_p), k \geq 2),$$

$$\text{and } \phi(R^n f(z), R^{n+1}f(z), R^{n+2}f(z), R^{n+3}f(z); z) < M(z),$$

then

$$R^n f(z) < q(z) \quad (z \in U).$$

The following result yield the best dominant of differential subordination (2.12) .

Theorem(2.3): Let the function M be univalent in U .Also $\phi: \mathbb{C}^4 \times U \rightarrow \mathbb{C}$,and K given by (2.9) .Suppose that following differential equation

$$K(p(z), zp'(z), z^2 p''(z), z^3 p'''(z); z) = M(z), \quad (2.13)$$

has a solution $q(z)$ with $q(0)=1$, which satisfies the condition (2.1) .If $f \in A$, satisfies the condition (2.12) ,and if

$$\phi(R^n f(z), R^{n+1} f(z), R^{n+2} f(z), R^{n+3} f(z); z) \text{ is analytic in } U,$$

then

$$R^n f(z) < q(z), \quad (z \in U)$$

and $q(z)$ is the best dominant .

Proof: From theorem (2.1) , we see that q is a dominant of (2.12) .Since q satisfies a solution of (2.13).

Therefore , q will be dominant by all dominants .Hence q ,is the best dominant .This completes the proof of the theorem (2.3) .

In view of definition (2.1) ,and in special case when $q(z) = Gz$ ($G > 0$) , the class $H_t[\Omega, q]$,of admissible functions ,denoted by $H_t[\Omega, G]$ is expressed follows .

Definition(2.2): Let Ω be set in $\mathbb{C}$,and $G > 0$.The class $H_t[\Omega, G]$,of admissible functions consists of those functions $\phi: \mathbb{C}^4 \times U \rightarrow \mathbb{C}$ such that

$$\begin{aligned} \phi(Ge^{i\theta}, [\delta D + (1 - \delta)]Ge^{i\theta}, \delta^2 E + [\delta(2 - \delta)D + (1 - \delta)^2]Ge^{i\theta}, \delta^3 L + 3\delta^2 E \\ + (\delta[(2 - \delta) + (1 - \delta)^2]D + (1 - \delta)^3)Ge^{i\theta}) \\ \notin \Omega, \end{aligned} \quad (2.14)$$

where $(z \in U)$,

$$\operatorname{Re}(ze^{-i\theta}) \geq (k-1)kG$$

and $\operatorname{Re}(ze^{-i\theta}) \geq 0, \quad \forall \theta \in R, k \geq 2.$

Corollary(2.3); Let $\phi \in H_i[\Omega, G]$. If the function $f \in A$, satisfies the following conditions

$$|R^n f(z)| \leq kG, \quad (z \in U; k \geq 2; G > 0),$$

and

$$(R^n f(z), R^{n+1} f(z), R^{n+2} f(z), R^{n+3} f(z); z) \in \Omega,$$

then $|R^n f(z)| < G,$

In special case, when $\Omega = q(U) = \{\omega: |\omega| < G\}$, the class $H_i[\Omega, G]$ is simple denoted by $H_i[G]$. Corollary (2.3) can be rewritten in the following form.

Corollary(2.4); Let $\phi \in H_i[G]$. If the function $f \in A$, satisfies the following conditions:

$$|R^n f(z)| \leq kG, \quad (z \in U; k \geq 2; G > 0),$$

and

$$|R^n f(z), R^{n+1} f(z), R^{n+2} f(z), R^{n+3} f(z); z| < G,$$

then $|R^n f(z)| < G.$

Corollary (2.5); Let $k \geq 2, 0 \neq q \in C$, and $G > 0$. If the function $f \in A$, satisfies the following conditions:

$$|R^{n+1} f(z)| \leq kG$$

and

$$|\mathbb{R}^{n+1}f(z) - \mathbb{R}^n f(z)| < Gz,$$

then $|\mathbb{R}^n f(z)| < G$.

Proof; Let $\check{\theta}(a, b, \varsigma, d; z) = b - a$, **and** $\Omega = \mathbb{M}(U)$, **where**

$$\mathbb{M}(z) = Gz \quad (G > 0).$$

Use corollary (2.3), we need to show that $\check{\theta} \in \mathbb{H}_t[\Omega, G]$, that is the admissibility condition (2.14) is satisfied. This follows readily, since it is seen that

$$|\check{\theta}(a, b, \varsigma, d; z)| = |k - 1| \geq G,$$

where $z \in U$, $\forall \theta \in \mathbb{R}$, **and** $k \geq 0$. The required result now follows from corollary (2.3). This completes the proof.

Definition(2.3): Let Ω be a set in $\mathbb{C}$, $q \in Q_1 \cap A_1$. The class $\mathbb{H}_{t,1}[\Omega, q]$ of admissible functions consists of those function $\check{\theta} : \mathbb{C}^4 \times U \rightarrow \mathbb{C}$, which satisfy the following admissibility conditions:

$$\check{\theta}(a, b, \varsigma, d; z) \notin \Omega,$$

whenever $a = q(\varepsilon)$, $b = \delta \varepsilon q'(\varepsilon) + q(\varepsilon)$,

$$\operatorname{Re} \left(\frac{\varsigma - 2b - (1 - 2\delta)a}{\delta(b - a)} \right) \geq k \operatorname{Re} \left(\frac{\varepsilon q''(\varepsilon)}{q'(\varepsilon)} + 1 \right)$$

and

$$\operatorname{Re} \left(\frac{d - (3 + 3\delta)\varsigma - ([\delta(3 + \delta) + 3] + (3 + 3\delta)(2 - \delta))b}{\delta^2(b - a)} - \frac{[(3 + 3\delta)(1 - \delta) - \delta(3 + \delta) - 2]a}{\delta^2(b - a)} \right) \geq k^2 \operatorname{Re} \left(\frac{\varepsilon^2 q'''(\varepsilon)}{q'(\varepsilon)} \right),$$

(where $z \in U, \varepsilon \in \partial U | E(q), k \geq 2$).

Theorem(2.4): Let $\phi \in H_{t,1}[\Omega, q]$. If the function $f \in A$, and $q \in Q_1$, satisfy the following conditions :

$$\operatorname{Re} \left(\frac{\varepsilon q''(\varepsilon)}{q'(\varepsilon)} \right) \geq 0, \quad \left| \frac{R^n f(z)}{z q'(z)} \right| \leq k \quad (2.15)$$

and

$$\left\{ \phi \left(\frac{R^n f(z)}{z}, \frac{R^{n+1} f(z)}{z}, \frac{R^{n+2} f(z)}{z}, \frac{R^{n+3} f(z)}{z}; z \right) : z \in \Omega \right\} \subset \Omega, \quad (2.16)$$

then

$$\frac{R^n f(z)}{z} < q(z), \quad (z \in U).$$

Proof: Define the analytic function $p(z)$ in U by

$$p(z) = \frac{R^n f(z)}{z}. \quad (2.17)$$

From equation (1.6) and (2.3), we have

$$\frac{R^{n+1} f(z)}{z} = \delta z p'(z) + p(z). \quad (2.18)$$

By a similar argument, we get

$$\frac{R^{n+2} f(z)}{z} = \delta^2 z^2 p''(z) + \delta(2 + \delta) z p'(z) + p(z), \quad (2.19)$$

and

$$\frac{R^{n+3}f(z)}{z} = \delta^3 z^3 p'''(z) + \delta^2(3 + 3\delta)z^2 p''(z) + \delta[(3 + \delta) + 3]z p'(z) + p(z). \quad (2.20)$$

Define the transformation from $\mathbb{C}^4$ to $\mathbb{C}$ by

$$a(e, s, r, t) = e, \quad b(e, s, r, t) = \delta s + e$$

$$c(e, s, r, t) = \delta^2 r + \delta(2 + \delta)s + e, \quad (2.21)$$

and

$$d(e, s, r, t) = \delta^3 t + \delta^2(3 + 3\delta)r + \delta[\delta(3 + \delta) + 3]s + e. \quad (2.22)$$

$$\text{Let } K(e, s, r, t) = \emptyset(a, b, c, d)$$

$$= \emptyset(e, \delta s + e, \delta^2 r + \delta(2 + \delta)s + e, \delta^3 t + \delta^2(3 + 3\delta)r + \delta[\delta(3 + \delta) + 3]s + e). \quad (2.23)$$

The proof will make use of lemma (1.5) using the equations (2.17), and (2.19), and from the equation (2.23), we have

$$K(p(z), z p'(z), z^2 p''(z), z^3 p'''(z); z) = \emptyset\left(\frac{R^n f(z)}{z}, \frac{R^{n+1} f(z)}{z}, \frac{R^{n+2} f(z)}{z}, \frac{R^{n+3} f(z)}{z}; z\right). \quad (2.24)$$

Hence, clearly (2.16), becomes

$$K(p(z), z p'(z), z^2 p''(z), z^3 p'''(z); z) \in \Omega,$$

we note that

$$\frac{r}{s} + 1 = \frac{\zeta - 2b + (1 - 2\delta)a}{\delta(b - a)},$$

and

$$\frac{t}{s} = \frac{d - (3 + 3\delta)\zeta - ([\delta(3 + \delta) + 3] + (3 + 3\delta)(2 - \delta))b}{\delta^2(b - a)} - \frac{[(3 + 3\delta)(1 - \delta) - \delta(3 + \delta) - 2]a}{\delta^2(b - a)}.$$

Thus clearly, the admissibility condition for $\phi \in H_{i,1}[\Omega, q]$, in definition (2-3) is equivalent to admissibility condition $K \in H_2[\Omega, q]$ as given in definition (1.4), with $n = 2$. Therefore, by using (2.15), and lemma (1.5), we have

$$\frac{R^n f(z)}{z} < q(z).$$

If $\Omega \neq \mathbb{C}$, is simply connected domain, the $\Omega = M(U)$, for some conformal mapping $M(z)$ of U on to Ω . In this case the class $H_{i,1}[M(U), q]$ is written as $H_{i,1}[M, q]$.

This leads to the following immediate consequence of theorem (2.4), is stated below.

Theorem(2.5): Let $\phi \in H_{i,1}[M, q]$. If the function $f \in A$, and $q \in Q_1$, satisfy the following conditions :

$$R\left(\frac{\varepsilon q''(\varepsilon)}{q'(\varepsilon)}\right) \geq 0, \quad \left| \frac{R^{n+1}f(z)}{z q'(\varepsilon)} \right| \leq k, \quad (2.25)$$

and

$$\phi\left(\frac{R^n f(z)}{z}, \frac{R^{n+1}f(z)}{z}, \frac{R^{n+2}f(z)}{z}, \frac{R^{n+3}f(z)}{z}; z\right) < M(z), \quad (2.26)$$

then

$$\frac{R^n f(z)}{z} < q(z) \quad (z \in U).$$

In view of definition (2.3), and in special case when $q(z) = Gz$ ($G > 0$), the class $H_{i,1}[\Omega, q]$, of admissible functions, denoted by $H_{i,1}[\Omega, G]$ is expressed follows.

Definition(2.4): Let Ω be set in $\mathbb{C}$, and $G > 0$. The class $H_{t,1}[\Omega, G]$, of admissible functions consists of those functions $\phi: \mathbb{C}^4 \times U \rightarrow \mathbb{C}$ such that:

$$\phi(Ge^{i\theta}, [\delta D + 1]Ge^{i\theta}, \delta^2 E + [\delta(2 + \delta)D + 1]Ge^{i\theta}, \delta^3 L + \delta^2(3 + 3\delta)E + (\delta[\delta(3 + \delta) + 3]D + 1)Ge^{i\theta}) \notin \Omega, \quad (2.27)$$

where $(z \in U), \operatorname{Re}(Ee^{-i\theta}) \geq (k - 1)kG$

and $\operatorname{Re}(Le^{-i\theta}) \geq 0, \quad \forall \theta \in R, k \geq 2.$

Corollary(2.6): Let $\phi \in H_{t,1}[\Omega, G]$. If the function $f \in A$, satisfies the following conditions:

$$\left| \frac{R^{n+1}f(z)}{z} \right| \leq kG, \quad (z \in U; k \geq 2; G > 0),$$

and

$$\phi\left(\frac{R^n f(z)}{z}, \frac{R^{n+1}f(z)}{z}, \frac{R^{n+2}f(z)}{z}, \frac{R^{n+3}f(z)}{z}; z\right) \in \Omega,$$

then

$$\left| \frac{R^n f(z)}{z} \right| < G.$$

In special case, when $\Omega = q(U) = \{\omega: |\omega| < G\}$, the class $H_{t,1}[\Omega, G]$ is simply denoted by $H_{t,1}[G]$. Corollary (2.6) can be rewritten in the following form.

Corollary (2.7): Let $\phi \in H_{t,1}[\Omega, G]$. If the function $f \in A$, satisfies the following conditions:

$$\left| \frac{R^n f(z)}{z} \right| \leq kG, \quad (z \in U; k \geq 2; G > 0),$$

and

$$\left| \phi\left(\frac{R^n f(z)}{z}, \frac{R^{n+1}f(z)}{z}, \frac{R^{n+2}f(z)}{z}, \frac{R^{n+3}f(z)}{z}; z\right) \right| < G,$$

then $\left| \frac{R^n f(z)}{z} \right| < G.$

Definition(2.5): Let Ω be a set in $\mathbb{C}$, $q \in Q_1 \cap A_1$. The class $H_{i,2}[\Omega, q]$ of admissible functions consists of those functions $\phi : \mathbb{C}^4 \times U \rightarrow \mathbb{C}$, which satisfy the following admissibility conditions: $\phi(a, b, c, d; z) \notin \Omega$,

whenever $a = q(\epsilon)$, $b = \frac{\delta \epsilon q'(\epsilon) + (q(\epsilon))^2}{q(\epsilon)}$,

$$Re \left(\frac{\zeta b + 2a^2 - 3ab}{\delta(b-a)} \right) \geq k Re \left(\frac{\epsilon q''(\epsilon)}{q'(\epsilon)} + 1 \right),$$

and

$$Re([(\zeta b - ab + a^2 + \delta a)[bd - b\zeta + ab - a^2 - \delta a] - b^3(\zeta - b) - [b(\zeta - \delta - 3a) + a(\delta + 2a)][b(5b - 5a - \zeta + 2\delta) + a(\delta + 2a) + \frac{a^2}{\delta^3}[(b(\zeta - 3a - \delta) + a(\delta + 2a))^2 - (b-a)[ab(\delta^2 - a + 4b) + b(\delta^2 + \delta b - 2b^2) - a(\delta + \delta a + \delta^2 a - a^2)]] * [\delta^2(b-a)^{-1}] \geq k^2 Re(\frac{\epsilon^2 q'''(\epsilon)}{q'(\epsilon)}),$$

(where $z \in U$, $\epsilon \in \partial U \setminus E(q)$, $k \geq 2$).

Theorem(2.6): Let $\phi \in H_{i,2}[\Omega, q]$. If the function $f \in A$, and $q \in Q_1 \cap A_1$, satisfy the following conditions :

$$Re \left(\frac{\epsilon q''(\epsilon)}{q'(\epsilon)} \right) \geq 0, \quad \left| \frac{R^{n+2} f(z)}{R^{n+1} f(z)} \right| \leq k, \quad (2.28)$$

and

$$\left\{ \phi \left(\frac{R^{n+1} f(z)}{R^n f(z)}, \frac{R^{n+2} f(z)}{R^{n+1} f(z)}, \frac{R^{n+3} f(z)}{R^{n+2} f(z)}, \frac{R^{n+4} f(z)}{R^{n+3} f(z)}; z \right) : z \in U \right\} \subset \Omega, \quad (2.29)$$

then

$$\frac{R^{n+1}f(z)}{R^n f(z)} < q(z), \quad (z \in U).$$

Proof: Define the analytic function $p(z)$ in U by

$$p(z) = \frac{R^{n+1}f(z)}{R^n f(z)}. \quad (2.30)$$

From equation (1.6) and (2.30), we have

$$\frac{R^{n+2}f(z)}{R^{n+1}f(z)} = \frac{\delta_z p'(z) + (p(z))^2}{p(z)} = V. \quad (2.31)$$

By a similar argument, we get

$$\frac{R^{n+3}f(z)}{R^{n+2}f(z)} = B, \quad (2.32)$$

and

$$\frac{R^{n+4}f(z)}{R^{n+3}f(z)} = B + B^{-1}[E + V^{-1}D - V^{-2}E^2], \quad (2.33)$$

where

$$B = \frac{\delta_z p'(z) + (p(z))^2}{p(z)} + \frac{\frac{\delta^2 z^2 p''(z) + \delta^2 z p'(z)}{p(z)} - \left(\frac{\delta_z p'(z)}{p(z)}\right)^2 + \delta_z p(z)}{\frac{\delta_z p'(z) + (p(z))^2}{p(z)}},$$

$$E = \frac{\delta^2 z^2 p''(z) + \delta^2 z p'(z)}{p(z)} - \left(\frac{\delta_z p'(z)}{p(z)}\right)^2 + \delta_z p'(z),$$

and

$$D = \frac{\delta^3 z^3 p'''(z) + 3\delta^2 z^2 p''(z) + \delta^3 z p'(z)}{p(z)} - \frac{3\delta^3 z^3 p'(z) p''(z) + 3(\delta z p'(z))^2}{(p(z))^2} \\ + 2 \left(\frac{\delta z p'(z)}{p(z)} \right)^2 + \delta^2 z^2 p''(z) + \delta^2 z p'(z).$$

We now define the transformation from $\mathbb{C}^4$ to $\mathbb{C}$ by

$$a(e, s, r, t) = e, \quad b(e, s, r, t) = \frac{\delta s + e^2}{e} = S,$$

$$c(e, s, r, t) = \frac{\delta s + e^2}{e} + \frac{\frac{\delta^2 r + \delta^2 s}{e} - \left(\frac{\delta s}{e} \right)^2 + \delta s}{\frac{\delta s + e^2}{e}} = C, \quad (2.34)$$

and

$$d(e, s, r, t) = C + C^{-1}[L + G^{-1}S - S^{-2}L^2], \quad (2.35)$$

$$\text{where } L = \frac{\delta^2 r + \delta^2 s}{e} - \left(\frac{\delta s}{e} \right)^2 + \delta s$$

and

$$G = \frac{\delta^3 t + 3\delta^3 r + \delta^3 s}{e} - \frac{3\delta^3 r s + 3\delta^3 s^2}{e} + 2 \left(\frac{\delta s}{e} \right)^3 + \delta^2 r + \delta^2 s$$

$$\text{Let } K(e, s, r, t) = \emptyset(a, b, c, d)$$

$$= \emptyset(e, S, C, C + C^{-1}[L + G^{-1}S - S^{-2}L^2]). \quad (2.36)$$

The proof will make use of lemma (1.5) using the equations (2.30), and (2.33), and from the equation (2.36), we have

$$\begin{aligned} & \mathbb{K}(\mathbb{p}(z), z \mathbb{p}'(z), z^2 \mathbb{p}''(z), z^3 \mathbb{p}'''(z); z) \\ &= \mathbb{O} \left(\frac{\mathbb{R}^{n+1} f(z)}{\mathbb{R}^n f(z)}, \frac{\mathbb{R}^{n+2} f(z)}{\mathbb{R}^{n+1} f(z)}, \frac{\mathbb{R}^{n+3} f(z)}{\mathbb{R}^{n+2} f(z)}, \frac{\mathbb{R}^{n+4} f(z)}{\mathbb{R}^{n+3} f(z)}; z \right). \quad (2.37) \end{aligned}$$

Hence ,clearly (2.29) ,becomes

$$\mathbb{K}(\mathbb{p}(z), z \mathbb{p}'(z), z^2 \mathbb{p}''(z), z^3 \mathbb{p}'''(z); z) \in \Omega.$$

We note that

$$\frac{\mathbb{r}}{\mathbb{s}} + 1 = \frac{\zeta \mathbb{b} + 2\mathbb{a}^2 - 3\mathbb{b}\mathbb{a}}{\delta(\mathbb{b} - \mathbb{a})},$$

and

$$\begin{aligned} \frac{\mathbb{t}}{\mathbb{s}} &= [(\mathbb{b}\zeta - \mathbb{a}\mathbb{b} + \mathbb{a}^2 + \delta\mathbb{a})[\mathbb{b}\delta - \mathbb{b}\zeta + \mathbb{a}\mathbb{b} - \mathbb{a}^2 - \delta\mathbb{a}] - \mathbb{b}^3(\zeta - \mathbb{b}) - [\mathbb{b}(\zeta - \delta - 3\mathbb{a}) \\ &+ \mathbb{a}(\delta + 2\mathbb{a})][\mathbb{b}(5\mathbb{b} - 5\mathbb{a} - \zeta + 2\delta) + \mathbb{a}(\delta + 2\mathbb{a}) \\ &+ \frac{\mathbb{a}^2}{\delta^3}[(\mathbb{b}(\zeta - 3\mathbb{a} - \delta) + \mathbb{a}(\delta + 2\mathbb{a}))^2 - (\mathbb{b} - \mathbb{a})[\mathbb{a}\mathbb{b}(\delta^2 - \mathbb{a} + 4\mathbb{b}) \\ &+ \mathbb{b}(\delta^2 + \delta\mathbb{b} - 2\mathbb{b}^2) - \mathbb{a}(\delta + \delta\mathbb{a} + \delta^2\mathbb{a} - \mathbb{a}^2)]] * [\delta^2(\mathbb{b} - \mathbb{a})]^{-1}. \end{aligned}$$

Thus clearly ,the admissibility condition for $\mathbb{O} \in \mathbb{H}_{i,2}[\Omega, \mathbf{q}]$,in definition (2.5) is equivalent to admissibility condition $\mathbb{K} \in \mathbb{H}_2[\Omega, \mathbf{q}]$ as given in definition (1.4) , with $\mathbf{n} = 2$. Therefore ,by using (2.30) ,and lemma (1.5) , we have

$$\frac{\mathbb{R}^{n+1} f(z)}{\mathbb{R}^n f(z)} < \mathbf{q}(z).$$

This completes the proof of theorem (2.6).

If $\Omega \neq \mathbb{C}$,is simply connected domain ,the $\Omega = \mathbb{M}(\mathbf{U})$,for some conformal mapping $\mathbb{M}(z)$ of $\mathbf{U}$ on to Ω .In this case, the class $\mathbb{H}_{i,2}[\mathbb{M}(\mathbf{U}), \mathbf{q}]$ is written as $\mathbb{H}_{i,2}[\mathbb{M}, \mathbf{q}]$.

This leads to the following immediate consequence of theorem (2.6) ,is stated below

Theorem(2.7): Let $\phi \in H_{i,1}[\Omega, q]$. If the functions $f \in A$, and $q \in Q_1$, satisfy the following conditions (2.29), and

$$\phi \left(\frac{R^{n+1}f(z)}{R^n f(z)}, \frac{R^{n+2}f(z)}{R^{n+1}f(z)}, \frac{R^{n+3}f(z)}{R^{n+2}f(z)}, \frac{R^{n+4}f(z)}{R^{n+3}f(z)}; z \right) < M(z),$$

then

$$\frac{R^{n+1}f(z)}{R^n f(z)} < q(z), \quad (z \in U).$$

Definition(2.6): Let Ω be a set in $\mathbb{C}$, $q \in Q_0 \cap A_0$ with $q'(z) \neq 0$. The class $H'_i[\Omega, q]$ of admissible functions consists of those functions

$$\phi : \mathbb{C}^4 \times \underline{U} \rightarrow \mathbb{C}$$

That satisfy the following admissibility conditions:

$$\phi(a, b, c, d) \in \Omega,$$

whenever $a = q(z)$, $b = \delta z q'(z)$

$$\operatorname{Re} \left(\frac{c - (1 - \delta)[2b - (1 - \delta)a]}{\delta^2 b - \delta^2(1 - \delta)a} \right) \geq \frac{1}{h} \operatorname{Re} \left(\frac{\xi q''(\xi)}{q'(\xi)} + 1 \right)$$

and

$$\operatorname{Re} \left(\frac{d - 3c - (1 - \delta)[(1 - \delta)b + (1 + \delta)a] + 2(2 - \delta)b}{\delta^2 b - \delta^2(1 - \delta)a} \right) \geq \frac{1}{h^2} \operatorname{Re} \left(\frac{\xi^2 q'''(\xi)}{q'(\xi)} \right),$$

(where $z \in U$, $\xi \in \partial U|E(q)$, $h \geq 2$).

Theorem(2.8): Let $\phi \in H'_i[\Omega, q]$. If the function $f \in A$, and $R^n f(z) \in Q_0$, and $q \in \hat{W}_0$, with $q'(z) \neq 0$, satisfy the following conditions :

$$\operatorname{Re} \left(\frac{z q''(z)}{q'(z)} \right) \geq 0, \quad \left| \frac{R^n f(z)}{q'(z)} \right| \leq h, \quad (2.38)$$

and

$$\mathcal{O}(R^n f(z), R^{n+1} f(z), R^{n+2} f(z), R^{n+3} f(z); z),$$

is univalent in U , then

$$\Omega \subset \{\mathcal{O}(R^n f(z), R^{n+1} f(z), R^{n+2} f(z), R^{n+3} f(z); z): z \in U\}, \quad (2.39)$$

implies that $q(z) < R^n f(z), \quad (z \in U).$

Proof: Let the function $p(z)$ be defined by (2.3) and $\mathcal{O}$ by (2.9). Since $H'_i[\Omega, q]$. From (2.10) and (2.39), we have

$$\Omega \subset \{K(p(z), z p'(z), z^2 p''(z), z^3 p'''(z); z): z \in U\}.$$

From (2.9), we see that the admissibility condition for $K \in H'_i[\Omega, q]$ in definition (2.6) is equivalent to the admissibility for $\mathcal{O} \in H'_n[\Omega, q]$ as given in definition (1.4) with $n = 2$.

Hence $\mathcal{O} \in H'_2[\Omega, q]$ and by using (2.39) and lemma (1.5), we find that

$$q(z) < R^n f(z) \quad (z \in U).$$

This completes the proof of the theorem (2.8).

If $\Omega \neq \mathbb{C}$, is simply connected domain, then $\Omega = M(U)$, for some conformal mapping $M(z)$ of U on to Ω . In this case the class $H'_i[M(U), q]$ is written as $H'_i[M, q]$.

This leads to the following immediate consequence of theorem (2.8) is stated below.

Theorem(2.9): Let $\phi \in H'_i[M, q]$ and If the function $f \in A$, and $R^n f(z) \in Q_0$, and $q \in \hat{W}_0$, with $q'(z) \neq 0$, satisfy the following conditions (2.38) and the function

$$\phi(R^n f(z), R^{n+1} f(z), R^{n+2} f(z), R^{n+3} f(z); z)$$

is univalent in U , then

$$M(z) < \phi(R^n f(z), R^{n+1} f(z), R^{n+2} f(z), R^{n+3} f(z); z), \quad (2.40)$$

implies that $q(z) < R^n f(z) \quad (z \in U)$.

Theorem (2.8) and (2.9), can only be used to obtain subordination for the third-order differential superordination of the form (2.39) or (2.40). The following theorem given the existence of the best subordinant of (2.40) for suitable ϕ .

Theorem(2.10): Let the function M be univalent in U . Also $K: \mathbb{C}^4 \times U \rightarrow \mathbb{C}$, and ϕ given by (2.9). Suppose that following differential equation:

$$K(p(z), z p'(z), z^2 p''(z), z^3 p'''(z); z) = M(z), \quad (2.41)$$

has a solution $q(z) \in \phi_0$. If $f \in A$, and $R^n f(z) \in Q_0$, and if $q \in \hat{W}_0$, with $q'(z) \neq 0$, satisfying the condition (2.38), and if the function

$$\phi(R^n f(z), R^{n+1} f(z), R^{n+2} f(z), R^{n+3} f(z); z) \text{ is analytic in } U,$$

then

$$M(z) < \phi(R^n f(z), R^{n+1} f(z), R^{n+2} f(z), R^{n+3} f(z); z),$$

implies that $q(z) < R^n f(z), \quad (z \in U)$

and $q(z)$ is the best subordinant.

Proof : By applying theorem (2.8) and (2.9) , we deduce that q is a subordinant of (2.40) .Since q satisfies (2.41) ,it is also a solution of (2.40) and therefore, q will be subordinant by all subordinants .Hence q is the best subordinant .This completes the proof of the theorem (2.10) .

Definition(2.7): Let Ω be a set in $\mathbb{C}$, $q \in \hat{W}_1$ with $q'(z) \neq 0$.The class $H'_{t,1}[\Omega, q]$ of admissible function consists of those functions $\phi : \mathbb{C}^4 \times \underline{U} \rightarrow \mathbb{C}$,which satisfy the following admissibility conditions:

$$\phi(a, b, \varsigma, d; z) \in \Omega,$$

whenever $a = q(z)$, $b = \delta z q'(z) + (1 - \delta)q(z)$,

$$Re \left(\frac{\varsigma - 2b - (1 - 2\delta)a}{\delta(b - a)} \right) \geq \frac{1}{h} Re \left(\frac{\xi q''(\xi)}{q'(\xi)} + 1 \right),$$

and

$$Re \left(\frac{d - (3 + 3\delta)\varsigma - ([\delta(3 + \delta) + 3] + (3 + 3\delta)(2 - \delta))b}{\delta^2(b - a)} - \frac{[(3 + 3\delta)(1 - \delta) - \delta(3 + \delta) - 2]a}{\delta^2(b - a)} \right) \geq \frac{1}{h^2} Re \left(\frac{\xi^2 q'''(\xi)}{q'(\xi)} \right),$$

(where $z \in U$, $\xi \in \partial U \setminus E(q)$, $h \geq 2$).

Theorem(2.11): Let $\phi \in H_{t,1}[\Omega, q]$.If the function $f \in A$, and $\frac{R^{n+1}f(z)}{z} \in Q_1$, and $q \in \hat{W}_1$, with $q'(z) \neq 0$, satisfy the following conditions :

$$Re \left(\frac{\xi q''(\xi)}{q'(\xi)} \right) \geq 0, \quad \left| \frac{R^{n+1}f(z)}{z q'(z)} \right| \leq h, \quad (2.42)$$

and

the function $\phi \left(\frac{R^n f(z)}{z}, \frac{R^{n+1}f(z)}{z}, \frac{R^{n+2}f(z)}{z}, \frac{R^{n+3}f(z)}{z}; z \right)$

is univalent in U , then

$$\Omega \subset \left\{ \phi \left(\frac{R^n f(z)}{z}, \frac{R^{n+1} f(z)}{z}, \frac{R^{n+2} f(z)}{z}, \frac{R^{n+3} f(z)}{z}; z \right) : z \in U \right\}, \quad (2.43)$$

implies that $\frac{R^n f(z)}{z} < q(z), (z \in U).$

Proof: Let the function $p(z)$ be defined by (2.17) and K by (2.23). Since $\phi \in H'_{i,1}[\Omega, q]$. We find from (2.24) and (2.43), we have

$$\Omega \subset \{K(p(z), z p'(z), z^2 p''(z), z^3 p'''(z); z) : z \in U\}.$$

From the equations (2.21) and (2.22), we see that the admissibility condition for $\phi \in H'_{i,1}[\Omega, q]$ in definition (2.7) is equivalent to the admissibility for $\phi \in H'_n[\Omega, q]$ as given in definition (1.4) with $n=2$.

Hence $\phi \in H'_2[\Omega, q]$ and by using (2.43) and lemma (1.5), we find that

$$q(z) < \frac{R^{n+1} f(z)}{z} \quad (z \in U).$$

This completes the proof of the theorem (2.11).

If $\Omega \neq \mathbb{C}$, is simply connected domain, then $\Omega = M(U)$, for some conformal mapping $M(z)$ of U on to Ω . In this case the class $H'_{i,1}[M(U), q]$ is written as $H'_{i,1}[M, q]$.

This leads to the following immediate consequence of theorem (2.11) is:

Theorem(2.12): Let $\phi \in H'_{i,1}[M, q]$ and .If the function $f \in A$, and $\frac{R^n f(z)}{z} \in Q_1$, and $q \in \hat{W}_1$, with $q'(z) \neq 0$, satisfying the following conditions (2.42) and the function

$$\phi \left(\frac{R^n f(z)}{z}, \frac{R^{n+1} f(z)}{z}, \frac{R^{n+2} f(z)}{z}, \frac{R^{n+3} f(z)}{z}; z \right)$$

is univalent in U , then

$$M(z) < \phi \left(\frac{R^n f(z)}{z}, \frac{R^{n+1} f(z)}{z}, \frac{R^{n+2} f(z)}{z}, \frac{R^{n+3} f(z)}{z}; z \right),$$

implies that $q(z) < \frac{R^n f(z)}{z}, \quad (z \in U).$

Definition(2.8): Let Ω be a set in $\mathbb{C}$ and $q \in \hat{W}_1$ with $q'(z) \neq 0$. The class $H'_{i,1}[\Omega, q]$ of admissible functions consists of those functions $\phi : \mathbb{C}^4 \times U \rightarrow \mathbb{C}$, which satisfy the following admissibility conditions:

$$\phi(a, b, c, d; z) \in \Omega,$$

whenever $a = q(z), \quad b = \frac{\delta z q'(z) + (q(z))^2}{q(z)},$

$$R \left(\frac{c b + 2 a^2 - 3 a b}{\delta(b-a)} \right) \geq \frac{1}{h} R \left(\frac{\varepsilon q''(\varepsilon)}{q'(\varepsilon)} + 1 \right),$$

and

$$\begin{aligned} & Re \left([(b c - a b + a^2 + \delta a) [b d - b c + a b - a^2 - \delta a] - b^3 (c - b) \right. \\ & \quad - [b(c - \delta - 3a) + a(\delta + 2a)] \left[b(5b - 5a - c + 2\delta) \right. \\ & \quad + a(\delta + 2a) + \frac{a^2}{\delta^3} [(b(c - 3a - \delta) + a(\delta + 2a))^2 \\ & \quad - (b - a)[a b(\delta^2 - a + 4b) + b(\delta^2 + \delta b - 2b^2) \\ & \quad \left. \left. - a(\delta + \delta a + \delta^2 a - a^2)] \right] * [\delta^2(b - a)]^{-1} \right) \geq \frac{1}{h^2} Re \left(\frac{\varepsilon^2 q'''(\varepsilon)}{q'(\varepsilon)} \right), \end{aligned}$$

(where $z \in U, \varepsilon \in \partial U | E(q), h \geq 2$).

Theorem(2.13): Let $\phi \in H_{i,2}[\Omega, q]$. If the function $f \in A$, and $\frac{R^{n+1} f(z)}{R^n f(z)} \in$

Q_1 , and $q \in \hat{W}_1$, with $q'(z) \neq 0$, satisfy the following conditions :

$$\operatorname{Re} \left(\frac{\xi q''(\xi)}{q'(\xi)} \right) \geq 0, \quad \left| \frac{R^{n+1}f(z)}{R^n f(z)} \right| \leq h, \quad (2.44)$$

and

$$\phi \left(\frac{R^{n+1}f(z)}{R^n f(z)}, \frac{R^{n+2}f(z)}{R^{n+1}f(z)}, \frac{R^{n+3}f(z)}{R^{n+2}f(z)}, \frac{R^{n+4}f(z)}{R^{n+3}f(z)}; z \right)$$

is univalent in U , then

$$\Omega \subset \left\{ \phi \left(\frac{R^{n+1}f(z)}{R^n f(z)}, \frac{R^{n+2}f(z)}{R^{n+1}f(z)}, \frac{R^{n+3}f(z)}{R^{n+2}f(z)}, \frac{R^{n+4}f(z)}{R^{n+3}f(z)}; z \right) : z \in U \right\}, \quad (2.45)$$

implies that $q(z) < \frac{R^{n+1}f(z)}{R^n f(z)}, \quad (z \in U).$

Proof: Let the function $p(z)$ be defined by (2.30) and K by (2.36). Since $\phi \in H'_{i,2}[\Omega, q]$. We find from (2.37) and (2.45), we have

$$\Omega \subset \{K(p(z), z p'(z), z^2 p''(z), z^3 p'''(z); z) : z \in U\}.$$

From the equations (2.34) and (2.35), we see that the admissibility condition for $\phi \in H'_{i,2}[\Omega, q]$ in definition (2.8) is equivalent to the admissibility for $K \in H'_n[\Omega, q]$ as given in definition (1.4) with $n = 2$.

Hence $K \in H'_2[\Omega, q]$ and by using (2.44) and lemma (1.5), we find that

$$q(z) < \frac{R^{n+1}f(z)}{R^n f(z)} \quad (z \in U).$$

This completes the proof of the theorem (2.13).

If $\Omega \neq \mathbb{C}$, is simply connected domain, then $\Omega = M(U)$, for some conformal mapping M of U on to Ω . In this case the class $H'_{i,2}[M(U), q]$ is written as $H'_{i,2}[M, q]$.

This lead s to the following immediate consequence of theorem (2.13) is:

Theorem(2.14): Let $\phi \in H'_{i,2}[M, q]$ and .If the function $f \in A$, and $\frac{R^{n+1}f(z)}{R^n f(z)} \in Q_1$, and $q \in \hat{W}_1$, with $q'(z) \neq 0$, , satisfying the following conditions (2.44) and the function

$$\phi \left(\frac{R^{n+1}f(z)}{R^n f(z)}, \frac{R^{n+2}f(z)}{R^{n+1}f(z)}, \frac{R^{n+3}f(z)}{R^{n+2}f(z)}, \frac{R^{n+4}f(z)}{R^{n+3}f(z)}; z \right)$$

is univalent in U , then

$$M(z) < \phi \left(\frac{R^{n+1}f(z)}{R^n f(z)}, \frac{R^{n+2}f(z)}{R^{n+1}f(z)}, \frac{R^{n+3}f(z)}{R^{n+2}f(z)}, \frac{R^{n+4}f(z)}{R^{n+3}f(z)}; z \right),$$

implies that $q(z) < \frac{R^{n+1}f(z)}{R^n f(z)}, \quad (z \in U).$

By combining theorem (2.2) and (2.9) , we obtain following sandwich-type theorem .

Theorem (2.15): Let M_1 and q_1 be analytic functions in U .Als o let M_2 b e univalent function in U ,and $q_2 \in Q_1$ with $q_1(0) = q_2(0) = 1$ and $\phi \in H'_{i,2}[M_1, q_1] \cap H'_{i,2}[M_2, q_2]$.If the function $f \in A$, with $R^n f(z) \in Q_0 \cap A_0$, and the function

$$\phi(R^n f(z), R^{n+1}f(z), R^{n+2}f(z), R^{n+3}f(z); z),$$

is univalent in U ,and if the condition (2.1) and (2.38) are satisfied,

then

$$M_1(z) < \phi(R^n f(z), R^{n+1}f(z), R^{n+2}f(z), R^{n+3}f(z); z) < M_2(z),$$

implies that $q_1(z) < R^n f(z) < q_2(z), \quad (z \in U). \quad (2.46)$

On the other hand ,we combine theorem (2.5) and (2.12) ,we obtain the following Sandwich-type theorem .

Theorem (2.16): Let M_1 and q_1 be analytic functions in U . Also let M_2 be univalent function in U , and $q_2 \in Q_1$ with $q_1(0) = q_2(0) = 1$ and $\emptyset \in H'_{i,1}[M_1, q_1] \cap H'_{i,1}[M_2, q_2]$. If the function $f \in A$, with $\frac{R^n f(z)}{z} \in Q_1 \cap A_1$, and the function

$$\emptyset \left(\frac{R^n f(z)}{z}, \frac{R^{n+1} f(z)}{z}, \frac{R^{n+2} f(z)}{z}, \frac{R^{n+3} f(z)}{z}; z \right),$$

is univalent in U , and if the conditions (2.15) and (2.42) are satisfied,

then

$$M_1(z) < \emptyset \left(\frac{R^n f(z)}{z}, \frac{R^{n+1} f(z)}{z}, \frac{R^{n+2} f(z)}{z}, \frac{R^{n+3} f(z)}{z}; z \right) < M_2(z),$$

implies that $q_1(z) < \frac{R^n f(z)}{z} < q_2(z)$, ($z \in U$). (2.47)

Theorem (2.17): Let M_1 and q_1 be analytic functions in U . Also let M_2 be univalent function in U , and $q_2 \in Q_1$ with $q_1(0) = q_2(0) = 1$ and $\emptyset \in H'_{i,2}[M_1, q_1] \cap H'_{i,2}[M_2, q_2]$. If the function $f \in A$, with $\frac{R^{n+1} f(z)}{R^n f(z)} \in Q_1 \cap A_{p1}$, and the function

$$\emptyset \left(\frac{R^{n+1} f(z)}{R^n f(z)}, \frac{R^{n+2} f(z)}{R^{n+1} f(z)}, \frac{R^{n+3} f(z)}{R^{n+2} f(z)}, \frac{R^{n+4} f(z)}{R^{n+3} f(z)}; z \right)$$

is univalent in U , and if the condition (2.28) and (2.44) are satisfied,

then

$$M_1(z) < \emptyset \left(\frac{R^{n+1} f(z)}{R^n f(z)}, \frac{R^{n+2} f(z)}{R^{n+1} f(z)}, \frac{R^{n+3} f(z)}{R^{n+2} f(z)}, \frac{R^{n+4} f(z)}{R^{n+3} f(z)}; z \right) < M_2(z),$$

implies that $q_1(z) < \frac{R^{n+1} f(z)}{R^n f(z)} < q_2(z)$, ($z \in U$). (2.48)

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