

Fuzzy Soc–Semi T–ABSO Modules and Related Concepts

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Abstract— in this study, the connection between the fuzzy semi-T-ABSO module and the fuzzy socle is compared. To address this query, we provide an idea known as the fuzzy socle semi-TABSO modules. Several characteristics were found in the research that supported the new theory. Also, using simple algebraic methods, the connection between the fuzzy socle-semi T-ABSO module and the fuzzy socle-T-ABSO module was found. Also, we looked into the structure of the fuzzy socle semi-TABSO module for the fuzzy direct sum. The connections between the fuzzy socle semi-TABSO module and other fuzzy module types were also discussed as a divisible module and a comultiplication module. The definition of the new fuzzy system semi-TABS concepts will benefit from the research's conclusions.

Keywords: Fuzzy Soc-T–ABSO modules; Fuzzy Soc–Semi T–ABSO modules; Fuzzy Soc–Semi T–ABSO ideal.

1 INTRODUCTION

A fuzzy subset A of X was described in 1965 by Zadeh [1]. Since then, other fields get developed the theory of fuzzy sets. Historical branches of pure mathematics and algebra presented the concept of fuzzy sets. Rosenfeld [2] introduced fuzzy subgroups and the fuzzy subgroupoid in 1971. Negoita and Ralescu first introduce the ideas of fuzzy modules and fuzzy submodules in 1975 [3]. Fuzzy quotient modules and fuzzy finitely produced submodules [4]. Therefore, Saikia and Kalita [5] constructed fuzzy essential submodules and investigated their properties. Also looked into this In this article, all rings are unitary and commutative with identity. Hadi [6] 2004 proposed the idea of a semiprime fuzzy submodule. Saad [7], discussed the fuzzy socle semiprime submodule

In (2019), Wafaa [8] proposed the idea of a semi-TABSO fuzzy submodule (2019) In 2022, Saad [7] introduced the fuzzy Soc-T-ABSO submodule concept. Mustafa and Wafaa [9], presented the concept of socle semi-T-ABSO submodule

In this article, the notion of semi T-ABSO fuzzy module is generalized to a fuzzy Socle semi-T-Absorbing module. Two sections make up this article. We give a few basic definitions and attributes that we will need in the following section in section one. The fuzzy Socle semi-T-ABSO module's many fundamental characteristics, outputs, and findings are examined in section two.

Note: o.w., F – set, F – submo , F – ideal , F – module, F – singleton, and F – Socle T – ABSO submodule are abbreviations for otherwise, fuzzy sets, submodules, ideals, modules, and singletons.

2 Basic concept

There are many fundamental ideas, and this part outlines them along with any traits they require for the next section.

Definition 2.1. [1]: let $x_1: D \rightarrow I$ be an F-set in D, where $x \in D$, $t \in I$ defined by:

$$x_t(y) = \begin{cases} t & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases}$$

For all $y \in D$. x_t is said to be an F-singleton or F-point in D.

If $x=0$ and $v=1$, then $0_1(y) = \begin{cases} 1 & \text{if } y = 0 \\ 0 & \text{if } y \neq 0 \end{cases}$, [9]

Definition 2.2 [10]: let B be an F-set in D, for all $t \in I$ the set $B_t = \{x \in D; B(x) \geq t\}$ is said to be a level subset of B. Keep in mind that, B_t is a subset of D in the ordinary sense.

Definition 2.3. [11]: If C is an F-module of a W-module G, then the submod C_t of M is called the level submod of G where $t \in I$

Remark 2.4. [12]: Let A and B be two F sets in S, then:

- 1- $A = B$ iff $A(x) = B(x)$.
- 2- $A \subseteq B$ iff $A(x) \leq B(x)$.
- 3- $A = B$ iff $A_t = B_t$

if $A < B$ and there exist $x \in S$ such that $A(x) < B(x)$, then A is a proper F-subset of B and written as $A < B$.

By part (2), we can deduce that $x_t \subseteq A$ iff $A(x) \geq t$

Definition 2.5. [12]: If G is a W-module. An F-set X of M is called an F-module of a W-module G if:

- 1- $X(x - y) \geq \min(X(x), X(y))$, for all $x, y \in G$.
- 2- $X(rx) \geq X(x)$ for all $x \in G$ and $r \in W$.
- 3- $X(0) = 1$.

Definition 2.6. [13]: Let X and A be two F -modules of W -module G . A is said to be an F -submod of X if $A \subseteq X$.

Proposition 2.7. [10]: Let A be an F -set of a W -module G . Then the level subset $A_t, t \in I$ is a submod of G if A is an F -submod of X where X is an F -module of a W -module G .

Definition 2.8. [12]: Let A and B be two F -submods of an F -module X . The residual quotient of A and B denoted by $(A : B)$ is the F -subset of W defined by:

$$(A : B)(r) = \sup\{t \in [0: 1] : r_t B \subseteq A\}, \text{ for all } r \in R.$$

That $(A : B) = \{r_t : r_t B \subseteq A; r_t \text{ is a } F\text{-singleton of } R\}$. If $B = \langle x_k \rangle$, then $(A : \langle x_k \rangle) = \{r_t : r_t x_k \subseteq A; r_t \text{ is a } F\text{-singleton of } R\}$.

Lemma 2.9. [14]: Let A be an F -submod of F -module X , $(A_t : X_t) \supseteq (A : X)_t$, For all $t \in I$.

It follows that if, $X = A \oplus B$, for $A, B \leq X$ then $X_t = (A \oplus B)_t = A_t \oplus B_t$.

Definition 2.10. [13]: An F -subset K of a ring W is called an F -ideal of W , if $\forall x, y \in R$:

- 1) $K(x - y) \geq \min\{K(x), K(y)\}$.
- 2) $K(xy) \geq \max\{K(x), K(y)\}$.

Proposition 2.11. [13]: Let A and B be two F -submods of an F -module X of a W -module G . Then $(A : B)$ is an F -ideal of W .

Definition 2.12. [15]: Let X be an F -module of a W -module M , X is called F -simple iff X has no proper F -submods (in fact X is F -simple iff X has only itself and 0_1).

Moreover, the F -Socle of X is sum of simple F -submod of X and denoted by $F\text{-Soc}(X)$. That is X is called Semi-simple if $X = F\text{-Soc}(X)$.

Lemma 2.13. [9]: $(F\text{-Soc}(X))_t = \text{Soc}(X_t)$ for any F -module X for each $t \in I$ with $(F\text{-Soc}(X))_t \neq X_t$

Definition 2.14. [16]: Let X be an F -module of a W -module G . X is called a multiplication F -module iff for each F -submod A of X , there exists an F -ideal K of W such that $A = KX$.

Definition 2.15. [12]: An F – module X of a W -module G is called a cyclic F -module if there exists $x_v \subseteq X$ such that $y_k \subseteq X$ written as $y_k = r_l x_v$ for some F -singleton r_l of R , where $k, l, v \in L$ in this case, write $X = \langle x_v \rangle$ to denote the cyclic F -module generated by x_v .

Definition 2.16 [17]: A fuzzy submodule A of fuzzy module X is called a divisible fuzzy if for each F -singleton $x_v \subseteq A$ there exists a fuzzy singleton $y_h \subseteq A$ and for each $r \in R$, $r \neq 0$, $x_v = ry_h$ where $(ry)_h = ry_h$, X is called a divisible fuzzy module if X is a fuzzy divisible submodule of itself

Proposition 2.17. [18]: Let X be an F – module of a W -module G and P, H be a F – submod of X . Then P is F -Soc-semi T -ABSO submod iff $r_s^2 H \subseteq P$ for F -singleton r_s of W , implies $r_s H \subseteq P + F - Soc(X)$ or $r_s^2 \subseteq (P + F - Soc(X): X)$

Definition 2.18. [19]:

let X be a F -module of a W -module G . then X is called a semiprime F -module if for each non-empty F -submod A of X , F -ann A is a semiprime F -ideal of W

3 F-Soc-Semi T-ABSO Modules

Definition 3.1: let X be an F – module of a W – module G . Then X is called an $F - Soc - T - ABSO$ (F -Soc-prime) module if 0_1 is an $F - Soc - T - ABSO$ (F -Soc-prime) submod of X

Example 3.2: let $X = Z_4 \rightarrow I$ such that $X(y) = \begin{cases} 1 & \text{if } y \in Z_4 \\ 0 & \text{o.w} \end{cases}$

It's clear that X is an F -module of a Z -module Z_4 .

$$0_{\frac{1}{4}}(y) = \begin{cases} \frac{1}{4} & \text{if } y = 0 \\ 0 & \text{if } y \neq 0 \end{cases}$$

$$F - Soc(X)(y) = \begin{cases} \frac{1}{2} & \text{if } y \in (\bar{2}) \\ 0 & \text{if } y \notin (\bar{2}) \end{cases}$$

$$0_{\frac{1}{4}} + F - Soc(X)(y) = \begin{cases} \frac{1}{2} & \text{if } y \in (\bar{2}) \\ 0 & \text{if } y \notin (\bar{2}) \end{cases}$$

$$(0_1 + F - Soc(X):X)(y) = \begin{cases} 1 & \text{if } y \in 2Z \\ 0 & \text{if } y \notin 2Z \end{cases}$$

X is an F-Soc-T-ABSO (F-Soc-prime) since $2_{\frac{1}{2}} \cdot$

$$2_{\frac{1}{2}} \left(\bar{1}_{\frac{1}{4}} \right) = 0_{\frac{1}{4}} (4_{\frac{1}{4}} \cdot \left(\bar{1}_{\frac{1}{2}} \right)) = 0_{\frac{1}{4}}, \text{ then } 2_{\frac{1}{2}} \left(\bar{1}_{\frac{1}{4}} \right) = 2_{\frac{1}{4}} \subseteq 0_{\frac{1}{4}} + F - Soc(X), \text{ where } 0_{\frac{1}{4}} + F - Soc(X)(2) = \frac{1}{2} > \frac{1}{4} (4_{\frac{1}{4}} = 0_{\frac{1}{4}} \subseteq 0_{\frac{1}{4}} + F - Soc(X) \text{ where } 0_{\frac{1}{4}} + F - Soc(X)(4) = \frac{1}{2} > \frac{1}{4}) \text{ and } 2_{\frac{1}{2}} \cdot 2_{\frac{1}{2}} = 4_{\frac{1}{2}} \subseteq (0_1 + F - Soc(X):X), \text{ where } (0_1 + F - Soc(X):X)(4) = 1 > \frac{1}{2}.$$

Definition 3.3: let X be an F - module of a W - module G . Then X is called an F-Soc-quasi module if 0_1 is an F-Soc-T-ABSO submod of X .

Remark and Example 3.4

- 1) Every F-semiprime module is an F-Soc-semi T-ABSO module, but the converse is not true in general, for example:

$$\text{Let } X: Z_{49} \rightarrow I \text{ like that } X(y) = \begin{cases} 1 & \text{if } y \in Z_{49} \\ 0 & \text{o.w} \end{cases}$$

It's clear that X is an F-module of a Z -module Z_{49}

$$0_{\frac{1}{3}}(y) = \begin{cases} \frac{1}{3} & \text{if } y = 0 \\ 0 & \text{if } y \neq 0 \end{cases}$$

$$F - Soc(x)(y) = \begin{cases} \frac{1}{2} & \text{if } y \in (\bar{7}) \\ 0 & \text{o.w} \end{cases}$$

$$0_{\frac{1}{3}} + F - Soc(x)(y) = \begin{cases} \frac{1}{2} & \text{if } y \in (\bar{7}) \\ 0 & \text{if } y \notin (\bar{7}) \end{cases}$$

$$(F - Soc(X):X)(y) = \begin{cases} 1 & \text{if } y \in 7Z \\ 0 & \text{if } y \notin 7Z \end{cases}$$

X is the F-Soc-semi T-ABSO module since $7_{\frac{1}{2}}^2 \cdot$

$$\left(\bar{1}_{\frac{1}{3}} \right) \subseteq 0_{\frac{1}{3}} \text{ then } 7_{\frac{1}{2}} \cdot 1_{\frac{1}{3}} = 7_{\frac{1}{3}} \subseteq 0_{\frac{1}{3}} + F - Soc(X), 0_{\frac{1}{3}} + F - Soc(X)(7) = \frac{1}{2} > \frac{1}{3} \text{ or } 7_{\frac{1}{2}}^2 \subseteq (F - Soc(X):X),$$

since $F - Soc(X):X \left(49\frac{1}{2}\right) = 1 > 1/2$, but X is not

F-semiprime submod, since $7\frac{1}{2} \cdot \left(1\frac{1}{2}\right) = 7\frac{1}{2} \not\subseteq 0\frac{1}{3}$

where $0\frac{1}{3}(7) = 0 \not\supseteq \frac{1}{2}$

- 2) Every (F-T-ABSO) F-Soc-T-ABSO module is an F-Soc-semi T-ABSO module, but the converse is not true in general, for instance:

Let $X: Z_{12} \rightarrow I$ such that $X(y) = \begin{cases} 1 & \text{if } y \in Z_{12} \\ 0 & \text{o.w} \end{cases}$

It is clear that X is an F-module of a Z -module Z_{12}

$$0\frac{1}{3}(y) = \begin{cases} \frac{1}{3} & \text{if } y = 0 \\ 0 & \text{if } y \neq 0 \end{cases}$$

$$F - Soc(X)(y) = \begin{cases} \frac{1}{2} & \text{if } y \in (\bar{6}) \\ 0 & \text{o.w} \end{cases}$$

$$0\frac{1}{3} + F - Soc(X)(y) = \begin{cases} \frac{1}{2} & \text{if } y \in (\bar{6}) \\ 0 & \text{if } y \notin (\bar{6}) \end{cases}$$

$$(F - Soc(X):X)(y) = \begin{cases} 1 & \text{if } y \in 6Z \\ 0 & \text{if } y \notin 6Z \end{cases}$$

$$(0_1:X)(y) = \begin{cases} 1 & \text{if } y \in 12Z \\ 0 & \text{if } y \notin 12Z \end{cases}$$

X is $F - Soc - semi\ T - ABSO$ module since $4\frac{1}{2} \cdot$

$(\bar{3}\frac{1}{3}) = 0\frac{1}{3}$, then $4\frac{1}{2} \cdot 3\frac{1}{3} = 12\frac{1}{3} \subseteq 0\frac{1}{3} + F - Soc(X)$,

since $\left(0\frac{1}{3} + F - Soc(X)\right)(12) = \frac{1}{2} > \frac{1}{3}$, but X is not

an F-T-ABSO module since $2\frac{1}{2} \cdot 2\frac{1}{2} \cdot \left(\bar{3}\frac{1}{3}\right) = 0\frac{1}{3}$, then

$2\frac{1}{2} \cdot 3\frac{1}{3} = 6\frac{1}{3} \neq 0\frac{1}{3}$ where $0\frac{1}{3}(6) = 0 \not\supseteq \frac{1}{3}$ and $2\frac{1}{2} \cdot 2\frac{1}{2} = 4\frac{1}{2} \notin (0_1:X)$ where $(0_1:X)(4) = 0 \not\supseteq \frac{1}{2}$

- 3) Every F-quasi-prime module is an F-Soc-semi T-ABSO module, but the counter wise not true see the instance in part (1) where $X = Z_{49}$ as Z -module is an F-Soc-semi T-ABSO module, but X is not F-quasi-

prime module since $7_{\frac{1}{2}} \cdot 7_{\frac{1}{2}} \cdot \left(\bar{1}_{\frac{1}{3}}\right) = 0_{\frac{1}{3}}$ and $7_{\frac{1}{2}} \cdot \left(\bar{1}_{\frac{1}{4}}\right) \neq 0_{\frac{1}{3}}$.

- 4) Every F-prime module is an $F - Soc - semi\ T - ABSO$ module, but the converse is not true to see the example in part (2) where X is an $F - Soc - semi\ T - ABSO$ module, but X is not an F-prime module since $4_{\frac{1}{2}} \cdot \left(\bar{3}_{\frac{1}{3}}\right) = 0_{\frac{1}{3}}$, then $3_{\frac{1}{3}} \neq 0_{\frac{1}{3}}$ and $4_{\frac{1}{2}} \notin (0_1 : X)$

Proposition 3.5:

If an F-module X of a W -module G is an $F - Soc - semi\ T - ABSO$ module, then $F - ann\ X$ is an $F - Soc - semi\ T - ABSO$ ideal of W .

Proof: let $n_s^2 m_r \subseteq F - ann\ X$ for F-singletons n_s, m_r of W , then $n_s^2 m_r X \subseteq 0_1$, so that $n_s^2 (m_r x_v) \subseteq 0_1$ for all $x_v \in X$, but X is the F-Soc-semi T-ABSO module, then we have $n_s (m_r x_v) \subseteq 0_1 + F - Soc(X)$ or $n_s^2 \subseteq (0_1 + F - Soc(X) : X)$, hence $n_s m_r \subseteq (0_1 + F - Soc(X) : X)$ or $n_s^2 \subseteq (0_1 + F - Soc(X) : X)$ so that $n_s m_r \subseteq (0_1 : X) + F - Soc(W)$ or $n_s^2 \subseteq (0_1 : X) + F - Soc(W)$

Thus, $F - annX$ is an $F - Soc - semi\ T - ABSO$ ideal

Proposition 3.6:

An F-multiplication module X of a W -module G is an F-Soc-semi T-ABSO module iff $F - annX$ is F-Soc-semi T-ABSO ideal of W .

Proof: $\Rightarrow$) by proposition 3.5, we have the result.

$\Leftarrow$) let $n_s^2 x_v \subseteq 0_1$ for n_s F-singleton n_s of W and $x_v \in X$, but X is an F-multiplication module, so that $\langle x_v \rangle = HX$ for some F-ideal H of W , hence $n_s^2 HX \subseteq 0_1$, so that $n_s^2 H \subseteq (0_1 : X) = F - annX$, then either $n_s H \subseteq F - annX + F - soc(W)$ or $n_s^2 \subseteq F - annX + F - Soc(W)$ since $F - annX$ is an F-Soc-semi T-ABSO ideal.

So that either $n_s H \subseteq (0_1 + F - Soc(X) : X)$ or $n_s^2 \subseteq (0_1 + F - Soc(X) : X)$, hence $n_s HX \subseteq 0_1 + F - Soc(X)$

or $n_s^2 \subseteq (0_1 + F - Soc(X):X)$, then either $n_s x_v \subseteq 0_1 + F - Soc(X)$ or $n_s^2 \subseteq (0_1 + F - Soc(X):X)$.

Thus, X is an $F - Soc - semi\ T - ABSO$ module

Proposition 3.7:

Let W be a ring that has no F -zero divisors and $X \neq 0_1$ an F -divisible module of a W -module G such that $F - Soc(W) \subseteq F - ann(X)$. Then X is an $F - Soc - semi\ T - ABSO$ module iff X is an F -Soc-quasi prime module.

Proof: $\Rightarrow$) let $r_n s_m x_v \subseteq 0_1$ for F -singletons r_n, s_m of W and $x_v \subseteq X$. If $r_n s_m \subseteq 0_1$, then $r_n \subseteq 0_1$ or $s_m \subseteq 0_1$ so that $r_n x \subseteq 0_1$ or $s_m x_v \subseteq 0_1$, hence $r_n x \subseteq 0_1 + F - soc(X)$ or $s_m x_v \subseteq 0_1 + F - Soc(X)$. If $r_n s_m \not\subseteq 0_1$, then $r_n \not\subseteq 0_1$ or $s_m \not\subseteq 0_1$, since W that has no F -zero divisors

If $r_n x_v \subseteq 0_1 + F - Soc(X)$, then the proof is perfect.

If $r_n x_v \not\subseteq 0_1 + F - Soc(X)$, $r_n \not\subseteq 0_1$ and X is an $F - divisible$ module, hence $r_n X = X$, then $x_v = r_n h_y$ for F -singletons $h_y \subseteq X$, thus $r_n s_m x_v = r_n s_m r_n h_y = r_n^2 s_m h_y \subseteq 0_1$, but 0_1 is an $F - Soc - semi\ T - ABSO$ submod, then either $r_n s_m h_y \subseteq 0_1 + F - Soc(X)$ or $r_n^2 \subseteq F - annX + F - Soc(W)$, hence $r_n^2 \subseteq F - annX$ since $F - Soc(W) \subseteq F - ann(X)$, but $r_n \not\subseteq 0_1$, then $r_n^2 \not\subseteq 0_1$, so that $r_n^2 X = X \subseteq 0_1$

This is contradiction. Therefore $r_n^2 \not\subseteq F - ann(X)$, hence $r_n^2 \not\subseteq F - ann(X) + F - Soc(W)$, then only we have $r_n s_m h_y \subseteq 0_1 + F - Soc(X)$, so that $s_m x_v \subseteq 0_1 + F - Soc(X)$. Thus, 0_1 is an $F - Soc - quasi - prime$ submod, that is X is an F -Soc-quasi-prime module.

$\Leftarrow$) It is obvious

Proposition 3.8:

An $F - module\ X$ of a $W - module\ G$ is an $F - Soc - semi\ T - ABSO$ module iff either $F - ann\ r_s x_v + F - Soc(W) = F - ann\ r_k^2 x_v + F - Soc(W)$ for any F -

singletons r_s of W and $x_v \subseteq X$ such that $r_s x_v \not\subseteq 0_1 + F - Soc(X)$ or $r_s^2 X \subseteq 0_1 + F - Soc(X)$

Proof:

($\Rightarrow$) let $y_b \subseteq F - ann\ r_s^2 x_v + F - Soc(W)$, $r_s x_v \not\subseteq 0_1$ then $r_s^2 y_b x_v \subseteq 0_1 + F - Soc(X)$, but X is an $F - Soc - semi\ T - ABSO$ module and let $r_s^2 \not\subseteq (0_1 + F - Soc(X):X)$ hence $r_s y_b x_v \subseteq (0_1 + F - Soc(X):r_s x_v)$, $y_b \subseteq (0_1 + F - Soc(X):r_s x_v) = (0_1:r_s x_v) + F - Soc(W)$, $y_b \subseteq F - ann\ r_s x_v + F - Soc(W)$.

($\Leftarrow$) It is obvious

Proposition 3.9:

Let an $F - module\ X$ of a $W - module\ G$ is an $F - Soc - semi\ T - ABSO$ module, then $F - ann_x J$ is an $F - Soc - semi\ T - ABSO$ submod, for every F -ideal J of W with $J \not\subseteq F - ann X$;

Proof: since $JX \not\subseteq 0_1$ hence $(0_1:_X J) \neq X$ let $s_a^2 x_v \subseteq (0_1:_X J)$ for F -singleton s_a of R and $x_v \subseteq X$, then $s_a^2 J x_v \subseteq 0_1$ since X is an $F - Soc - semi\ T - ABSO$ module and by proposition (2.24), we get either $s_a x_v \subseteq 0_1 + F - Soc(X)$ or $s_a^2 \subseteq (0_1 + F - Soc(X):X)$, hence $s_a x_v \subseteq (0_1 + F - Soc(X):_X J)$ or $s_a^2 \subseteq ((0_1 + F - Soc(X):_X J):X)$. Thus $F - ann_x J$ is an $F - Soc - semi\ T - ABSO$ submod of X .

Proposition 3.10:

Let X be an $F - module$ of a $W - module\ G$ such that $(F - Soc(X):X)$ is an $F - Soc - semiprime$ ideal of W . Then X is an $F - Soc - semi\ T - ABSO$ module iff X an F -Soc-semiprime module.

Proof: ($\Rightarrow$) let $r_s^2 x_v \subseteq 0_1$ for F -singletons r_s of W and $x_v \subseteq X$ since X is an $F - Soc - semi\ T - ABSO$ module, then $r_s x_v \subseteq 0_1 + F - Soc(X)$ or $r_s^2 \subseteq (0_1 + F - Soc(X):_W X)$. Hence $r_s x_v \subseteq 0 + F - Soc(X)$ or $r_s^2 x_v \subseteq F - Soc(X):X$, since $(F - Soc(X):X)$ is an F -Soc-semiprime ideal of W . Thus $r_s x_v \subseteq 0_1 + F - Soc(X)$, $\forall x_v \subseteq X$.

Then 0_1 is an F-Soc-semiprime submod, so that X is an F-Soc-semiprime module.

($\Leftarrow$) It's evident

Proposition 3.11:

let X be an F – module of a W – module G . if X is an F – Soc – semi T – ABSO module, then F – $\text{ann}K$ is an F – Soc – semi T – ABSO ideal for each non-empty F-submod K of X .

Proof: let K be a non-empty F-submodule of X and F – $\text{ann}_W K \neq \lambda_W$ where $\lambda_W(r) = 1 \forall r \in W$ because if F – $\text{ann}_W K = \lambda_W$ then $K = 0_1$ which is discrepancy. Now, assume that $r_s^2 b_y \subseteq F$ – $\text{ann}_W K$ for F-singletons r_s, b_y of W . Hence $r_s^2 b_y K \subseteq 0_1$. Since X is an F – Soc – semi T – ABSO module, then either $r_s b_y K \subseteq 0_1 + F$ – $\text{Soc}(X)$ or $r_s^2 \subseteq (0_1 + F$ – $\text{Soc}(X) :_W X)$ by proposition (2.24). Hence $r_s b_y \subseteq (0_1 : K) + F$ – $\text{Soc}(W)$ or $r_s^2 \subseteq (0_1 : X) + F$ – $\text{Soc}(W) \subseteq (0_1 : K) + F$ – $\text{Soc}(W)$. Thus, F – $\text{ann}K$ is an F-Soc-semi T-ABSO ideal.

Proposition 3.12:

Let $X = X_1 \oplus X_2$ be an F – module of a W – module $G = G_1 \oplus G_2$ then X is an F – Soc – semi T – ABSO module iff X_1 and X_2 are F-Soc-semi T-ABSO modules

Proof: ($\Rightarrow$) let $r_n^2 x_v \subseteq 0_1$ and $r_n^2 y_h \subseteq 0_1$ for F-singleton r_n of W and $x_v \subseteq X_1, y_h \subseteq X_2$, hence $r_n^2(x_v, y_h) \subseteq (0_1, 0_1)$ but X is an F-Soc-semi T-ABSO module, then either $r_n(x_v, y_h) \subseteq (0_1, 0_1) + F$ – $\text{Soc}(X) = X_1 \oplus X_2$ or $r_n^2 \subseteq F$ – $\text{ann}X + F$ – $\text{Soc}(W) = F$ – $\text{ann}(X_1) \cap F$ – $\text{ann}(X_2) + F$ – $\text{Soc}(W)$ so that $r_n x_v \subseteq 0_1 + F$ – $\text{Soc}(X_1)$ or $r_n^2 \subseteq F$ – $\text{ann}(X_1) + F$ – $\text{Soc}(W)$ and $r_n y_h \subseteq 0_1 + F$ – $\text{Soc}(X_2)$ or $r_n^2 \subseteq F$ – $\text{ann}(X_2) + F$ – $\text{Soc}(W)$. Thus X_1 and X_2 are F-Soc-semi T-ABSO modules.

($\Leftarrow$) let $r_n^2(x_v, y_h) \subseteq (0_1, 0_1)$ for F-singleton r_n of W and $(x_v, y_h) \subseteq X = X_1 \oplus X_2$, hence $r_n^2 x_v \subseteq 0_1$ and $r_n^2 y_h \subseteq 0_1$ but X_1 and X_2 are F-Soc-semi T-ABSO modules, then either $r_n x_v \subseteq 0_1 + F$ – $\text{Soc}(X_1)$ or $r_n^2 \subseteq F$ – $\text{ann}(X_1) +$

$F - Soc(W)$ and $r_n y_h \subseteq 0_1 + F - Soc(X_2)$ or $r_n^2 \subseteq F - ann(X_2) + F - Soc(W)$, so that $r_n(x_v, y_h) \subseteq (0_1 + F - Soc(X_1), 0_1 + F - Soc(X_2))$ or $r_n^2 \subseteq F - ann(X_1) + F - Soc(W) \cap F - ann(X_2) + F - Soc(W)$, hence $r_n(x_v, y_h) \subseteq (0_1, 0_1) + F - Soc(X_1 \oplus X_2) = (0_1, 0_1) + F - Soc(X)$ or $r_n^2 \subseteq F - ann(X_1) \cap F - ann(X_2) + F - Soc(W) = F - annX + F - Soc(W)$.

Thus X is an $F - Soc - semi\ T - ABSO$ module.

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